Derived Algebraic Geometry
IMT-SCU

October 7–25, 2019, Chengdu, China

General Information

This is the second joint annual program between the Institut de Mathématiques de Toulouse (IMT, Toulouse, France) and the School of Mathematics, Sichuan University (SCU, Chengdu, China).

The 2019 program is carried out in Chengdu, China. It consists of a school (five mini-courses), followed by a workshop (nine talks), thematically oriented towards Derived Algebraic Geometry.

Scientific Committee

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Part 1. Minicourses (October 8–25, 2019)

I. Introduction to model categories

Lecturer: Farkhod Eshmatov (Sichuan Unviersity)

Time: 2:00-5:00pm, October 8–9.

Venue: W303 School of Mathematics, SCU.

This mini-course is to give a brief introduction to Quillen's theory of closed model categories. We start with the definition of a model category, and then give several examples:

(1) the category of CW complexes;

(2) the category of modules of a ring;

(3) the category of differential graded commutative algebras;

(4) the category of differential graded algebras and

(5) the category of differential graded Lie algebras.

After that we introduce the homotopy category of a model category, and derived functors between them. As an application, we briefly recall the André-Quillen homology of (differential graded) commutative algebras, as well as their noncommutative analogue. If time permits, we study the derived representation schemes of DG associative algebras, and some derived noncommutative geometry on them.

Reference

1. D. Quillen, Rational homotopy theory. Annals of Math. 90, 1969, 205–295.

2. S.I. Gelfand and Y.I. Manin, *Methods of Homological Algebra*. Springer 2003.

3. Y. Berest, G. Khachatryan and A. Ramadoss, Derived representation schemes and cyclic homology. Advances in Mathematics Volume 245, October 2013, 625–689.

II. Graded algebras related to noncommutative projective space

Lecturer: Quanshui Wu (Fudan University)

Jiwei He (Hangzhou Normal University)

Time: 9:00am-12:00pm, 3:00-5:00pm, October 10–11.

Venue: W303 School of Mathematics, SCU

This mini-course will cover the following topics:

1. Artin-Schelter regular algebras and Calabi-Yau algebras
   (1) Introduction to A_∞-algebras, and the proof of the Frobenius property of the Ext-algebras of Artin-Schelter regular algebras.
   (2) Homological integrals of Artin-Schelter Gorenstein algebras and the Nakayama automorphisms.
   (3) Calabi-Yau algebras.
2. Koszul duality and BGG correspondence
   (1) Koszul dualities induce BGG correspondence (DG version).
   (2) PBW deformations of Koszul algebras and Nonhomogeneous Koszul dualities.
3. Noncommutative Auslander Theorem and McKay correspondence
   (1) Hopf actions on Artin-Schelter regular algebras.
   (2) Noncommutative Auslander Theorem and Noncommutative resolutions.
   (3) Noncommutative McKay correspondence.
4. Noncommutative matrix factorizations and hypersurfaces
   (1) Noncommutative matrix factorizations and their relations with singularities of hypersurfaces.
   (2) Deformations of Koszul Frobenius algebras and noncommutative quadrics.

III. Shifted symplectic and Poisson structures

Lecturer: Damien Calaque (University of Montpellier)

Time: 2:00-5:00pm, October 14–16.

Venue: W303 School of Mathematics, SCU

The aim of this lecture series is to introduce shifted symplectic structures, and give several examples of these. We will connect these new notions to more classical concepts of symplectic geometry and geometric representation theory. We may use as prerequisites the basics of derived algebraic geometry, from the first two lecture of Bertrand Toen's mini-course. If time permits, we will also talk about shifted Poisson structures. We will try to follow the following plan, that may nevertheless be subject to changes:

• **Lecture 1.** Shifted symplectic geometry I: homotopical symplectic linear algebra, de Rham graded mixed complex, (closed) forms on a derived stack, examples (smooth scheme, BG, Perf).

• **Lecture 2.** Shifted symplectic geometry II: shifted symplectic and lagrangian structures on derived stacks, hamiltonian reduction in the derived setting, examples (shifted cotangent stacks, derived critical loci, BG, Perf). Optional: shifted symplectic reduction of derived critical loci.

• **Lecture 3.** The AKSZ-PTVV construction I: d-oriented stacks, trangression of (closed) forms, examples (various derived moduli stacks). Optional: a noncommutative version.

• **Lecture 4.** The AKSZ-PTVV construction II: relative orientations, category of lagrangian correspondences, shifted symplectic groupoids, oriented co-groupoids. Optional: the AKSZPTVV construction as a fully extended TFT.

• **Lecture 5.** Shifted Poisson structures on derived schemes: dg Lie algebra of polyvector fields, definition and examples of shifted Poisson structures on (affine) derived schemes, Melani's theorem.

• **Lecture 6.** Shifted Poisson structures on derived stacks: formal geometry in the derived setting, shifted Poisson structures on derived stacks, examples.

IV. Derived loop spaces

Lecturer: Bertrand Toën (IMT)

Time: 2:00-5:00pm, October 17–18.

Venue: W303 School of Mathematics, SCU

The purpose of the present series of lectures is to introduce notions from derived algebraic geometry in order to define the derived loop space of an algebraic variety (or a scheme or a stack). I will explain the relations between the derived loop spaces and classical notions such as de Rham cohomology and characteristic classes. One of the major result of this course is a highly structured version of the Hochschild-Kostant-Rosenberg theorem, and its applications to the construction of symplectic structures. In the final lecture, I will also present some new results concerning positive or mixed characteristic situations.

• **Lecture 1.** Derived schemes I: simplicial commutative rings and their homotopy theory, examples.

• **Lecture 2.** Derived schemes II: derived schemes as ringed spaces, derived stacks as functors, cotangent complexes and formal comple- tion. Example : the derived moduli space of representations.

• **Lecture 3.** The derived loop space: definition and examples, the HKR theorem in characteristic zero, shifted symplectic structures.

• **Lecture 4.** The filtered loop space: the filtered circle and the HKR- filtration, applications to symplectic structures and the theory of singular supports in non-zero characteristic.

Reference

Bertrand Toën, Derived Algebraic Geometry. EMS Surv. Math. Sci. 1 (2014), no. 2, 153-245.

V. Representation theory, algebraic homotopy theory and link invariants

Lecturer: Yuri Berest (Cornell University)

Time: 2:00-5:00pm, October 21–22, 24–25.

Venue: W303 School of Mathematics, SCU

Deep connections between representation theory and low-dimensional topology became apparent in the late 80s with the discovery of Jones' polynomial and its generalizations related to quantum groups. Since then, much work has been done to clarify the relation of these new 'quantum' invariants to classical geometric invariants of 3-manifolds, such as representation varieties of fundamental groups.

In recent years, new types of connections and, in fact, an entirely new paradigm of the interactions between representation theory and topology have emerged. On the algebraic side, the focus has shifted to developing a higher (categorical) version of representation theory, where classical linear representations are replaced by additive (abelian or triangulated) categories and the principal objects of study are 'categorified' quantum groups and related structures (such as affine and double affine Hecke algebras). On the topological side, the main interest is in understanding various new homology theories associated to knots and links (such as Khovanov homology and triply graded link homology) and their relation to classical structures in homotopy theory and geometric topology (such as braids and mapping class groups).

Yet another new, very interesting class of homology theories of links originates from symplectic topology. This includes the Legendrian Floer homology of Ozsváth, Szabó and the so-called knot contact homology (introduced by L. Ng and studied by Etnyre, Ekholm, Sullivan and others in recent years). Despite efforts of many mathematicians, a precise relation between these geometric homology theories and those coming from representation theory and classical topology remains quite a mystery (although there are some very intriguing conjectures in this direction).

The goal of these lectures is to give a survey of various connections –both old and new – between representation theory, classical homotopy theory and low-dimensional topology aimed at graduate students. Although the required background and the level of presentation will vary from lecture to lecture (some topics will be covered in a systematic way while others will be touched upon only briefly), we will try to make the overall discussion accessible for the beginning graduate students. A tentative plan is as follows:

• **Lecture 1:** The Jones polynomial and its generalizations

(Topics may include: universal link invariants associated to quantum groups, colored Jones polynomials, Habiro's cyclotomic expansion, Kauffman bracket skein modules, relations to classical invariants: Melvin-Morton conjecture, AJ conjecture and the Volume conjecture)

• **Lecture 2:** Khovanov homology and categorification

(Khovanov homology: combinatorial construction and derived functor intepretation, action of braid groups on categories (Deligne's theorem), Khovanov-Rozansky (triply graded) homology: Soergel bimodules and Rouquier's construction, higher Markov's traces)

• **Lecture 3:** Representation homology of spaces

(simplicial homotopy theory, Kan's loop group construction, representation homology as functor homology, relation to higher Hochschild homology and derived algebraic geometry (derived moduli spaces of local systems and the Toen-Vezzosi construction), representation homology of link complements)

• **Lecture 4:** Knot contact homology and abstract homotopy theory

(geometric definition of knot contact homology, Ekholm-Etnyre-Ng-Sullivan Theorem, homotopy-theoretic construction of knot contact homology, augmentation varieties, relation to representation homology, link DG categories and perverse sheaves)

Part 2. Workshop on Calabi-Yau Categories and Related Topics (October 19-20, 2019)

Speakers

Dario Beraldo (IMT)
Benjamin Hennion (Orsay)
Ming Lu (SCU)
Marco Robalo (IMJ)
Sinan Yalin (Angers)
Changjian Fu (SCU)
Zheng Hua (Hong Kong)
Fan Qin (Shanghai Jiaotong)
Fan Xu (Tsinghua)

Schedule

Saturday October 19

09:00 - 10:00	Dario Beraldo (IMT) Deligne-Lusztig duality and opers
10:30 - 11:30	Fan Xu (Tsinghua) Derived Hall algebras and categorification
14:00 - 15:00	Benjamin Hennion (Orsay) Tangent space to K-theory through formal moduli problems
15:00 - 16:00	Zheng Hua (Hong Kong) Feigin-Odesskii Poisson structures via derived geometry
16:30 - 17:30	Sinan Yalin (Angers) Integrability of derived complex spaces
17:30 - 18:30	Ming Lu (SCU) Hall algebras and quantum symmetric pairs

Sunday October 20

09:00 - 10:00	Fan Qin (Shanghai Jiaotong) Bases for upper cluster algebras and tropical points
10:30 - 11:30	Marco Robalo (IMJ) HKR theorems and elliptic curves
11:30 - 12:30	Changjian Fu (SCU) On f-vectors and the exchangeable conjecture

Venue: Room 207, East Second Teaching Building (四川大学东二教 207 室).

Titles and Abstracts

Dario Beraldo (**IMT**)

Title: Deligne-Lusztig duality and opers

Abstract: In the setting of the geometric Langlands conjecture, we argue that the phenomenon of divergence at infinity on the stack of G-bundles on a smooth complete curve is controlled by the locus of semisimple local systems (for the Langlands dual group). We prove this by first introducing the Deligne-Lusztig functors (substitutes for the Serre functors, which do not make sense in our situation), and then by describing these functors explicitly. Along the way, we obtain a global geometric version of a theorem of Lusztig on the Steinberg representation. We will also explain the relation between semisimple local systems and Griffiths-transverse filtrations (alias: extended opers).

Changjian Fu (**SCU**)

Title: On f-vectors and the exchangeable conjecture

Abstract: There are various integer vectors arising in cluster theory, such as d-vectors, g-vectors, c-vectors and f-vectors. In this talk, we will focus on f-vectors.

In particular, we will introduce f-compatibility of cluster variables, which is a generalization of the c-compatibility introduced by Fomin and Zelevinsky for cluster algebras of bipartite finite type.

We formulate the exchangeable conjecture: two cluster variables form an exchange pair iff their f-compatibilities equal 1.

Several classes of cluster algebras of skew-symmetric type provide evidences for the exchangeable conjecture. This is joint work with Yasuaki Gyoda.

Benjamin Hennion (**Orsay**)

Title: Tangent space to K-theory through formal moduli problems

Abstract: From the work of S. Bloch in 73 to a celebrated paper of T. Goodwillie in 86, the infinitesimal behavior of K-theory has been studied, as a way to better understand K-theory itself. In this talk, we will explain how (derived) formal geometry provides a new approach to understand this infinitesimal behavior. We will show that the (suitably considered) tangent of rational K-theory is cyclic homology (also known as additive K-theory), and provide some applications.

Zheng Hua (**Hong Kong**)

Title: Feigin-Odesskii Poisson structures via derived geometry

Abstract: The Feigin-Odesskii Poisson structures are semiclassical limits of the Feigin-Odesskii elliptic algebras. These noncommutative algebras are vast generalization of Sklyanin

algebras. It is an open problem that how to classify the symplectic leaves of these Poisson structures. With Sasha Polischuk, we construct a (1-d) shifted Poisson structure on the moduli stack of bounded complexes of vector bundles on projective Calabi-Yau d-folds. When d=1, our Poisson structure descends to Feigin-Odesskii's Poisson structure on certain components of the moduli stack. The derived geometry of the moduli stack leads to a geometric description of the symplectic leaves. Using algebraic geometry, we give an explicit classification of symplectic leaves for those Poisson structures of "endomorphism" type. This is a joint work with Sasha Polishchuk.

Ming Lu (**SCU**)

Title: Hall algebras and quantum symmetric pairs

Abstract: A quantum symmetric pair consists of a quantum group and its coideal subalgebra (called an $\imath$quantum group). A quantum group can be viewed as an example of $\imath$quantum groups associated to symmetric pairs of diagonal type.

In this talk, we present a new Hall algebra construction of $\imath$quantum groups (of Kac-Moody type). This relies on the framework of modified Ringel-Hall algebras of Peng and the speaker. Our approach leads to monomial bases, PBW bases, and braid group actions for $\imath$quantum groups. In case of symmetric pairs of diagonal type, our work reduces to a reformulation of Bridgeland's Hall algebra (of the category of 2-periodic complexes) realization of a quantum group, which in turn was a generalization of earlier constructions of Ringel and Lusztig for half a quantum group. In case of the split $\imath$quantum group, it is realized by using Hall algebra of the category of 1-periodic complexes. This is joint work with Weiqiang Wang.

Fan Qin (**Shanghai Jiaotong**)

Title: Bases for upper cluster algebras and tropical points

Abstract: It is known that many (upper) cluster algebras possess very different good bases which are parametrized by the tropical points of Langlands dual cluster varieties. For any given injective reachable upper cluster algebra, we describe all of its bases parametrized by the tropical points. In addition, we obtain the existence of the generic bases for such upper cluster algebras. Our results apply to many cluster algebras arising from representation theory, including quantized enveloping algebras, quantum affine algebras, double Bruthat cells, etc.

Marco Robalo (**IMJ**)

Title: HKR theorems and elliptic curves

Abstract: In this talk I will describe a join work with T. Moulinos and B. Toen describing how to an elliptic curve we can associate a twisted form of HKR theorem and explain how

these vary in family over the moduli space of elliptic curves.

Fan Xu (**Tsinghua**)

Title: Derived Hall algebras and categorification

Abstract: In this talk, we give the geometric realization of derived Hall algebras under the context of Lusztig's construction of canonical basis. As an application, we categorify an algebra homomorphsim from some subalgebra of a derived Hall algebra to the corresponding quantum cluster algebra.

Sinan Yalin (**Angers**)

Title: Integrability of derived complex spaces

Abstract: Since the Newlander-Nirenberg integrability theorem in 1957, the description of complex manifolds through integrable almost complex structures provided many far reaching applications ranging from deformation theory to Hodge theory for example. With the rise of derived geometry during the last decade, and more recently of derived analytic geometry, comes naturally the following question: is there a fully homotopy coherent analogue of this integrability notion suitable for derived complex objects? We will explore this question through an approach inspired by operad theory. This is joint work in progress with Joan Millès.