Introduction to Delay-Differential Equations

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but in general there is no explicit formula for the solution.

Delay-differential equations:

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Delay-differential equations (delay equations) arise as mathematical models in many areas of science and engineering.

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  (finite speed of signal transmission)

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Nonautonomous systems, variable delays ($r = r(t)$ or $r = r(x(t))$), infinite delays,...

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One can find solutions of the form $x(t) = e^{\lambda t} v$ where

$$(\lambda I - A - Be^{-\lambda r})v = 0, \quad \text{and}$$

$$\Delta(\lambda) = \det(\lambda I - A - Be^{-\lambda r}) = 0.$$

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In this sense, a delay equation is infinite-dimensional.

The nonlinear **Mackey-Glass equation** arises as a model of blood cell population:

$$\dot{x}(t) = -ax(t) + g(x(t-r)), \quad g(x) = \frac{bx}{1+x^k}$$

where $a, b, r > 0$ and $k > 1$.

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Note: The same equation, with a different g, arises in nonlinear optics.

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Wright's Equation

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)]$$

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one specifies an initial condition as

$$x_{t_0} = \varphi \in C$$

where

$$C = C([-r, 0], \mathbf{R}^n),$$

$x_t \in C$ is given by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$.

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Then for every $t \geq t_0$ one has $x_t \in C$ for the solution. Existence is for forward time only, and uniqueness is problematic for variable delays.

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This program has been carried out quite successfully for problems with constant delay, but is still (actively!) underway for problems with variable delays.

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Similar result for the case of a variable delay $r = r(x(t))$, but little is known for multiple delays.

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$\|x\| = \sup_{t\in\mathbf{R}} |x(t)| \geq K > 0$ as $\beta \to 0$ provided that either

- $r \equiv 1$ (constant delay);
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Each case requires a different argument, and the asymptotic shapes of the solutions as $\beta \to 0$ in the four cases are radically different.

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The solution need not be unique, and need not be stable. The vertical parts of the wave have thickness $O(\beta)$ and are described by transition layer equations.

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In contrast to the constant delay case, for small β the periodic solution is **unique** and **superstable** with asymptotic period

$$p = k + 1 + \frac{\beta|\log\beta|}{k-1} + O(\beta).$$

Very detailed asymptotics are known for this solution as $\beta \to 0$.

TENSOR PRODUCTS, POSITIVE OPERATORS,

AND DELAY-DIFFERENTIAL EQUATIONS

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A linear ODE

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namely a family of linear maps satisfying

$$T(t_0, t_0) = I, \qquad T(t, t_1)T(t_1, t_0) = T(t, t_0)$$

for all t, t_0, t_1.

Letting $\wedge$ denote the **exterior product**, define

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They obtained information about the nonlinear equation $\dot{y} = f(y)$.

Tensor Products

Let V and W be vector spaces. Then $V \otimes W$ is the vector space generated by all elements $v \otimes w$ (with $v \in V$ and $w \in W$) under the relations

$$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w),$$

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Tensor product $A \otimes B$ of linear maps $A : V \to V$ and $B : W \to W$.

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is the subspace generated by all elements

$$v_1 \wedge v_2 \wedge \cdots \wedge v_m = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(m)},$$

for $v_i \in V$ where $\mathcal{S}_m$ is the symmetric group on m elements.

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(There are $\binom{n}{m}$ such products.)

Infinite-Dimensional Tensor Products

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The Injective Tensor Product

The **injective norm** $\|\cdot\|_\varepsilon$ on $X \otimes Y$ is defined to be

$$\left\| \sum_{j=1}^{m} x_i \otimes y_i \right\|_\varepsilon = \sup_{\|\xi\|=\|\eta\|=1} \sum_{j=1}^{m} \xi(x_i)\eta(y_i),$$

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The resulting space is not in general complete, so we takes its completion and obtain a new Banach space denoted $X \otimes_\varepsilon Y$.

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This extends to any (finite) number of factors, so

$$(C[-1, 0])^{\otimes m} = C([-1, 0]^m).$$

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$$\varphi(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \ldots, \theta_{\sigma(m)}) \equiv (-1)^{\sigma} \varphi(\theta_1, \theta_2, \ldots \theta_m)$$

holds identically for every $\sigma \in \mathcal{S}_m$.

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$$[T(1)\varphi](\theta) = \varphi(0) - \int_{-1}^{\theta} b(s+1)\varphi(s)\, ds$$

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Begin with $\varphi \in (C[-1,0])^{\wedge 2}$ and do the same. In this case **there will be cancellations in the final formula.**

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\begin{aligned}
&[T(1)^{\otimes 2}\varphi](\theta_1, \theta_2) \\
&\qquad = \varphi(0,0) - \int_{-1}^{\theta_2} b(s+1)\varphi(0,s)\,ds \\
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If m is **even** the minus signs disappear.

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If $b(t) \geq 0$ then the operator $T(1)^{\wedge 2}$ (and more generally $T(t,t_0)^{\wedge m}$ if m is **even**) is a positive operator with respect to a certain cone.

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$$K_* = \{\varphi \in (C[-1, 0])^{\wedge m} \mid \varphi(\theta_1, \theta_2, \ldots, \theta_m) \geq 0$$

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The same conclusion holds if $b(t) \leq 0$ and m is odd.

$$[T(1)^{\wedge m}\varphi](\theta_1, \ldots, \theta_m)$$

$$= \int_{\theta_1}^{\theta_2} \cdots \int_{\theta_{m-1}}^{\theta_m} b^1(s_1) \cdots b^1(s_{m-1}) \varphi(s_1, \cdots, s_{m-1}, 0)\, ds_{m-1} \cdots ds_1$$

$$+(-1)^m \int_{-1}^{\theta_1} \cdots \int_{\theta_{m-1}}^{\theta_m} b^1(s_0) \cdots b^1(s_{m-1}) \varphi(s_0, \cdots, s_{m-1})\, ds_{m-1} \cdots ds_0,$$

where $b^1(s) = b(s+1)$.

We are taking $a(t) \equiv 0$ for convenience.

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$$x \in K \implies Lx \in K.$$

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namely

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Then the Floquet multipliers μ_j (spectrum of $T(\gamma, 0)$) are infinite in number and satisfy

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Further, the Floquet solutions corresponding to μ_{2k-1} and μ_{2k} have **lap number**

$$V(x_t) \equiv 2k - 1.$$

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We wish to obtain a **computable lower bound** for this quantity, and hence for the individual multipliers $|\mu_k|$.

Lemma. Let L_1 and L_2 be linear operators for which

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with respect to a cone K, where K is both reproducing and normal. Then

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Applying this lemma to $M^{\wedge m}$, in the case that $b(t) \geq b_0 > 0$ and m is even, gives a computable lower bound

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for the magnitude of each characteristic multiplier. In fact we conclude that there are infinitely many characteristic multipliers.

Now consider the homotopy

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At the beginning of the homotopy ($\kappa = 0$) we have a constant coefficient problem, and so the gap structure

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These features are continued throughout the homotopy to $\kappa = 1$, using "pseudo-continuity" properties of the lap-number $V(\cdot)$.

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- Simplicity/uniqueness is related to "irreducibility" of the operator L.

$$\dot{x}(t) = -b(t)x(t-1), \qquad b(t) \geq b_0 > 0,$$

We take m even, say $m = 2$ for convenience. Recall and define

$$\begin{aligned}[T(1)^{\wedge 2}\varphi](\theta_1, \theta_2) \quad &= \quad \int_{\theta_1}^{\theta_2} b(s+1)\varphi(s,0)\, ds \\ &\quad + \int_{-1}^{\theta_1} \int_{\theta_1}^{\theta_2} b(s+1)b(r+1)\varphi(s,r)\, dr\, ds,\end{aligned}$$

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[T(1)^{\wedge 2}\varphi](\theta_1, \theta_2) &= \int_{\theta_1}^{\theta_2} b(s+1)\varphi(s,0)\, ds \\
&\quad + \int_{-1}^{\theta_1} \int_{\theta_1}^{\theta_2} b(s+1)b(r+1)\varphi(s,r)\, dr\, ds,
\end{aligned}$$

$$[T_0^{\wedge 2}\varphi](\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \varphi(s,0)\, ds + \int_{-1}^{\theta_1} \int_{\theta_1}^{\theta_2} \varphi(s,r)\, dr\, ds.$$

Then $T(1)^{\wedge 2} \geq c T_0^{\wedge 2}$ for some $c > 0$, and so

$$r(T(1)^{\wedge 2}) \geq c r(T_0^{\wedge 2})$$

for the spectral radii of these operators.

Kreĭn-Rutman Setting: Total cone $K \subseteq X$, operator $L : K \to K$.

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We say L is **$\mathbf{u_0}$-positive** if for some $n \geq 1$ and $u_0 \in K$, it is the case that

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Theorem. In the setting of the Kreĭn-Rutman Theorem, if additionally L is u_0-positive, then the eigenvalue r (the spectral radius) is algebraically simple.

Regularity of Solutions of Delay-Differential Equations: Analyticity versus C^∞

John Mallet-Paret
Division of Applied Mathematics
Brown University

Regularity of Solutions

$$\dot{x}(t) = f(t, x(t), x(t - r_1), \ldots, x(t - r_N))$$

Suppose $x(t)$ is a bounded solution defined for all $t \in \mathbf{R}$ (e.g., a periodic solution or more generally a solution on the attractor).

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Theorem (Nussbaum). If each $r_k > 0$ is a constant, and f is analytic and independent of t, then $x(t)$ is analytic in t.

But in general the answer is not so clear.

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It is known that $x(t)$ has an analytic extension to the lower half-plane $\operatorname{Im} t < 0$.

We believe the real axis $\operatorname{Im} t = 0$ is the boundary of the region where $x(t)$ is analytic (holomorphic).

$$\dot{x}(t) = f(x(t), x(t-r)), \qquad r = r(x(t)),$$

If $f(u, v)$ and $r(u)$ are analytic in u and v, then $x(t)$ is C^∞ in t.

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If $f(u, v)$ and $r(u)$ are analytic in u and v, then $x(t)$ is C^∞ in t.

Analyticity is **unknown** in general, and there is reason to believe $x(t)$ is analytic for some t but not in general for all t.

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The dynamics of the "history" map

$$t \to \eta(t) = t - r(t)$$

plays a role in determining for which t the solution $x(t)$ is analytic.

For a given C^∞ solution $x(t)$ we distinguish two sets:

$$\mathcal{A} = \{t_0 \mid x(t) \text{ is analytic for } t \text{ in some neighborhood of } t_0\},$$

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Note that $\mathcal{A} \subseteq \mathbf{R}$ is open and $\mathcal{N} \subseteq \mathbf{R}$ is closed.

Theorem. Consider the equation

$$\dot{x}(t) = \alpha(t)x(t) + \beta(t)x(\eta(t))$$

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$$\eta(\mathcal{N}) \subseteq \mathcal{N}, \qquad \eta(\mathcal{A} \setminus \mathcal{M}) \subseteq \mathcal{A},$$

where $\mathcal{M} \subseteq \mathbf{R}$ is the set of local minima and maxima of $\eta(t)$.

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Suppose $x(t)$ is analytic in a neighborhood of $t = t_1$. Then $x(\eta(t))$ is analytic near $t = t_0$. Regarding $\beta(t)x(\eta(t))$ as a known forcing term in the differential equation, we conclude that $x(t)$ is analytic near $t = t_0$.

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$$x(t) = \sum_{k=0}^{\infty} x_{mk} (t - t_1)^k,$$

and so $x(t)$ is analytic near t_1. ///

Theorem. Suppose for some t_0 that $\eta(t_0) = t_0$ and $|\dot{\eta}(t_0)| < 1$.

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Idea of Proof. Write the equation in integrated form and apply a standard contraction mapping argument in the space of functions analytic in a disc about t_0.

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Consider the equation

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This solution also satisfies the differential equation

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Analytic Hartman-Grobman Theorem: By means of an analytic change of variables $t = \sigma(\tau)$, we replace $\eta(t)$ by $\mu\tau$ near $t = 0$.

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This leads to the equation

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where the series for $\alpha(\tau)$ converges, but the series for $y(\tau)$ need not. The coefficients y_n are uniquely determined by the recursion

$$\lambda(n+1)y_{n+1} = (1 - \mu^{n+1}) \sum_{k=0}^{n} \alpha_{n-k} y_k.$$

Define quantities w_n by

$$y_n = \left(\frac{(-1)^n \alpha_0^n M_n}{\lambda^n n!} \right) w_n, \qquad M_n = \prod_{k=1}^{n} (\mu^k - 1) \sim \mu^{n(n+1)/2}.$$

Then

$$w_{n+1} = w_n + \sum_{k=0}^{n-1} (-1)^{n-k} \left(\frac{\lambda^n n!}{\alpha_0^n M_n} \right) \left(\frac{\alpha_0^k M_k}{\lambda^k k!} \right) \left(\frac{\alpha_{n-k}}{\alpha_0} \right) w_k$$

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Theorem. The limit

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If $w_\infty = 0$ then there exists a unique analytic solution through $\tau = 0$ (although there may exist other non-analytic solutions).

Examples with $w_\infty \neq 0$

Consider

$$\lambda x(t) = \int_{t+(\mu-1)\sin t - 2\pi m}^{t} x(s)\, ds$$

where $1 < \mu < 2\pi m + 1$.

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One also must show that the Hartman-Grobman conjugacy $\sigma(\tau) = \sigma(\tau, \mu)$ is well-behaved for large μ.

Coexistence of analyticity and non-analyticity

For the previous example with $\eta(t) = t + (\mu - 1)\sin t - 2\pi m$, for certain μ there are points t_* at which

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Here we assume that

$$r : \mathbf{R} \to \mathbf{R} \text{ is continuous (not necessarily analytic)},$$

$$r(t) \geq 0, \quad r(t + 2\pi) = r(t),$$

for all $t \in \mathbf{R}$.

Integral Operator

$$(Lx)(t) = \int_{\eta(t)}^{t} x(s)\,ds, \quad x \in X,$$

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Then $L : X \to X$ is a positive operator (with respect to the cone of nonnegative functions).

The Krein-Rutman Theorem implies there exists $\lambda > 0$ and $x \in X \setminus \{0\}$, with $x \geq 0$, such that

$$Lx = \lambda x$$

if and only if the spectral radius $\text{rad}(L)$ is positive. And if so, one can take $\lambda = \text{rad}(L)$.

Theorem. The spectral radius is positive, $\mathrm{rad}(L) > 0$, if and only if

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Remark. If $\eta(t) < t$ (that is, $r(t) > 0$) for every t, then $(*)$ holds and $\mathrm{rad}(L) > 0$. In this case the eigenfunction is unique.

Sketch of Proof. Suppose $(*)$ holds for every t. Using $(*)$ we obtain points

$$t_0 < t_1 < t_2 < \cdots < t_m \equiv t_0 (\text{mod } 2\pi)$$

such that

$$t_k \in (\eta(t_{k+1}), t_{k+1}).$$

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Taking $x \geq 0$ to be a function with small bumps at the points t_k, it follows that

$$Lx \geq cx \quad \text{for some } c > 0.$$

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This implies (upon iterating) that $\|L^n\| \geq c^n$, and thus $\text{rad}(L) \geq c > 0$.

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Then for any $\tau \geq t$ we have $t \leq \eta(\tau) \leq \tau$, and so

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Gronwall implies $x(\tau) \equiv 0$ identically, a contradiction.///

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More generally, is it possible for L to have more than one positive eigenvalue?

Consider the case that $r(t)$ has integer period m and

$$\eta(t) = k - c_k, \quad k < t < k + 1$$

where $c_k = c_{k+m} > 0$ is an integer.

This leads to a problem for $m \times m$ matrices Γ of the form

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

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Is it possible for such Γ to have more than one eigenvalue in $(1, \infty)$?

Some numerical calculations suggest not.

We now come to a main result on the analyticity set $\mathcal{A}$.

Theorem. In addition to the standing assumptions (periodicity and nonnegativity) on $r(t)$, assume that

- $r(t)$ is analytic in t,
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An example of a system satisfying the above conditions is given by

$$r(t) = \rho(1 - \cos t), \quad \rho > \rho_0.$$

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There are finitely many maximal intervals, and at least one

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- $[a - \varepsilon, a] \cap \mathcal{A}$ and $[b, b + \varepsilon] \cap \mathcal{A}$ have infinitely many connected components.

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Denote $I_k = [a + 2\pi k, a + 2\pi(k+1)]$. Then for large m we have

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Do this with $t_0 = a \in \mathcal{N}$. Then $\overline{S(a)} \subseteq \mathcal{N}$ is uncountable.

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Do this with $t_0 = a \in \mathcal{N}$. Then $\overline{S(a)} \subseteq \mathcal{N}$ is uncountable.

Iterate the points in $\overline{S(a)}$ backwards to get them in a neighborhood of $a \pmod{2\pi}$, and of b.

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Thus $\mathcal{A}$ has infinitely many components near a (and near b).

Can $\mathcal{N}$ have nonempty interior?

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Answer unknown, but if so it would be very interesting: An interval where the solution is everywhere C^∞ but nowhere analytic.

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Then there is a maximal interval $I = [0, \tau]$ for some $\tau \in (0, \frac{\pi}{2})$

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But the endpoints of I' are not points of analyticity. Thus

$$(0, \tau), (\pi - \tau, \pi) \subseteq \mathcal{A}, \qquad 0, \tau, \pi - \tau, \pi \in \mathcal{N}$$

Take any other interval (connected component) of $\mathcal{A}$, say

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But (**) is impossible due to a **stretching condition.**

In the complement (mod 2π) S of $I \cup I'$, the map η satisfies:

If $\eta^k(t) \in S$ for every $k \geq 1$ then

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A final argument shows that $\mathcal{N}$ has no isolated points, and so $\mathcal{N}$ is a generalized Cantor set.