Canonical metrics on reflexive sheaves

Jiayu Li

USTC

April 2019, Chengdu

Outline

Stability of Bundles

Let M be a Kähler manifold (C^n), with the Kähler form ω.

Let E be a holomorphic vector bundle on M (we simply call it bundle in the following).

At each point $x \in M$, $H_x = C^r$.

Suppose $\{U_\alpha, \alpha \in I\}$ is a covering of M, locally $E = U_\alpha \times C^r$, if $U_\alpha \cap U_\beta \neq$, $\phi_{\alpha\beta} : U_\alpha \times C^r \to U_\beta \times C^r$ is holomorphic.

Let H be a Hermitian metric on E. It mean at each point $x \in M$, H_x is a Hermitian metric on E_x. and H is smooth.

Associated to H there is a Chern connection A_H, and its curvature is defined by $F_H = DA_H$.

The degree of E is defined by

$$deg(E) = \frac{1}{2\pi} \int_M TrF_H \wedge \omega^{n-1}.$$

The stability of bundles was a well established concept in algebraic geometry.

A holomorphic vector bundle E is called **stable** (semi-stable), if for every subbundle $E' \hookrightarrow E$ of lower rank, it holds:

$$\mu(E') = \frac{deg(E')}{rankE'} < (\leq) \mu(E) = \frac{deg(E)}{rankE}. \tag{1.1}$$

Here the degree of E is defined as follow

$$deg(E) = \int_M C_1(E) \wedge \omega^{n-1},$$

where $C_1(E)$ is the first chern class of E

The Harder-Narasimhan filtration

- Let E be a bundle. Then there is a filtration of E by subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E,$$

called the Harder-Narasimhan filtration of bundle E (abbr, HN-filtration), such that $Q_i = E_i/E_{i-1}$ is semistable. Moreover, $\mu(Q_i) > \mu(Q_{i+1})$, and the associated graded object $Gr^{hn}(E) = \oplus_{i=1}^{l} Q_i$ is uniquely determined by the isomorphism class of E.

Seshadri filtration

- Let V be a semistable bundle , then there is a filtration of V by subbundles
$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V,$$
called the Seshadri filtration of V, such that V_i/V_{i-1} stable. Moreover, $\mu(V_i/V_{i-1}) = \mu(V)$ for each i, and the associated graded object $Gr^s V = \oplus_{i=1}^{l} V_i/V_{i-1}$ is uniquely determined by the isomorphism class of V.

The Harder-Narasimhan-Seshadri filtration

- Let E be a bundle. Then there is a double filtration , called a Harder-Narasimhan-Seshadri filtration of bundle E (abbr, HNS-filtration), with the following properties: if $\{E_i\}_{i=1}^{l}$ is the HN filtration of E, then

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,l_i} = E_i$$

and the successive quotient $Q_{i,j} = E_{i,j}/E_{i,j-1}$ are stable subbundles. Moreover, $\mu(Q_{i,j}) = \mu(Q_{i,j+1})$ and $\mu(Q_{i,j}) > \mu(Q_{i+1,j})$, the associated graded object:

$$Gr^{hns}(E, \overline{\partial}_E) = \oplus_{i=1}^{l} \oplus_{j=1}^{l_i} Q_{i,j}$$

is uniquely determined by the isomorphism class of E.

Harder-Narasimhan-Seshadri filtration

Let E be a bundle. There is a filtration of E by subbundles

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E},$$

called the Harder-Narasimhan-Seshadri filtration of E (abbr, HNS-filtration), such that the quotient sheaf $Q_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ is stable. Moreover, $\mu(Q_i) \geq \mu(Q_{i+1})$, and the associated graded object

$$Gr^{hns}(\mathscr{E}) = \oplus_{i=1}^{l} Q_i$$

is uniquely determined by the isomorphism class of $\mathscr{E}$.

Harder-Narasimhan type

We then have R-tuple of numbers

$$\vec{\mu}(E) = (\mu_1, \cdots, \mu_R) \tag{1.2}$$

from the HNS-filtration by setting: $\mu_i = \mu_\omega(Q_j)$, for $rank(\mathscr{E}_{j-1}) + 1 \leq i \leq rank(\mathscr{E}_j)$. We call $\vec{\mu}(E)$ the Harder-Narasimhan type of E.

Chern connection

Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle over a complex manifold M, and H be a Hermitian metric on E, then there exists a unique connection D_H, which is called the **Chern connection** which satisfies the following two conditions:

- D_H is compatible with the Hermitian metric H;
- Its $(0,1)$-part coincides to the holomorphic structure, i.e. $D_H^{0,1} = \overline{\partial}_E$.

Hermitian-Einstein metric

A Hermitian metric H in E is called a Hermitian-Einstein metric, if the curvature F_H of the chern connection A_H satisfies the Einstein condition:

$$\sqrt{-1}\Lambda_{\omega} F_H = \lambda Id_E, \tag{2.1}$$

where Λ_{ω} denotes the contraction of differential forms by Kähler form ω, and the real constant λ is given by $\lambda = \frac{2\pi}{Vol(M)rank(E)} deg(E)$.

Donaldson-Uhlenbeck-Yau theorem

D-U-Y theorem states that a holomorphic bundle is poly-stable if and only if it admits a Hermitian-Einstein metric.

A short history of the theorem:

- Narasimhan and Seshadri (1965) for Riemann surface.
- Donaldson (1985) for algebraic surfaces, and (1986) for algebraic manifolds.
- Uhlenbeck and Yau (1986) for Kähler manifolds.

Atiyah and Bott (1982) conjectured that the asymptotic behavior of the Yang-Mills flow on a Riemann surface at infinity is decided by the HNS filtration.

Some interesting generalizations

- Li and Yau (1987) for Hermitian manifolds with Gauduchon metric (i.e. $\partial\bar{\partial}(\eta^{n-1}) = 0$).
- Hitchin (1987), Simpson (1988) for Higgs bundle.
- Bando and Siu (1994) for reflexive sheaf.
- Bartolomeis and Tian (1996) for complex vector bundle on almost Hermitian manifold.
- Biquard (1996), Li-Narasimhan (1999), Li (2000) for parabolic bundle.
- Jost-Zuo, harmonic map and representation.
- Zuo, Sheng, Xu, algebraic version.

Donaldson's heat flow

Donaldson's heat flow for Hermitian metrics on the bundle E with initial metric H_0:

$$H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega F_H - \lambda Id_E). \tag{2.2}$$

In the following, we denote $\Phi(H) = \sqrt{-1}\Lambda_\omega F_H - \lambda Id_E$ for simplicity.

Donaldson's heat flow

- The flow exists globally;
- If the bundle is stable, the flow converges to a H-E metric, proved by Donaldson.
- Li-Zhang-Zhang (2017) (for Higgs sheaves) show that under semi-stability assumption, we have $\|\Phi(H(t))\|_{L^\infty} \to 0$ along the heat flow, solve a conjecture by Kobayashi (1986).

sheaves

A sheaf $\mathscr{F}$ on M associates to each open set $U \cap M$ a module (or group, other algebraic structures) $\mathscr{F}(U)$, called the sections of $\mathscr{F}$ over U, and to each pair $U \subset V$ of open sets a map $r_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$, called the restriction map, satisfying

- If $U \subset V \subset W$, we have

$$r_{W,U} = r_{W,V} \cdot r_{V,U}.$$

- For any open sets $U, \ V \subset M$ and sections $\sigma \in \mathscr{F}(U)$, $\tau \in \mathscr{F}(V)$ such that

$$\sigma|_{U \cap V} = \tau|_{U \cap V},$$

  then there is a section $\rho \in \mathscr{F}(U \cup V)$ with

$$\rho|_U = \sigma, \quad \rho|_V = \tau.$$

- If $\rho \in \mathscr{F}(U \cup V)$ and

$$\rho|_U = \rho|_V = 0,$$

  then $\rho = 0$.

For examples, the sheaf of holomorphic functions on the complex manifold M denoted by $\mathscr{O}_M$, the sheaf of holomorphic sections on bundles.

A $\mathscr{O}_M$-module sheaf is called an analytic sheaf.
An analytic sheaf $\mathscr{F}$ is said to be **Coherent** if for each point $x \in M$, there is a neighborhood U of x such that there is an exact sequence of sheaves over U,

$$\mathscr{O}^p|_U \to \mathscr{O}^q|_U \to \mathscr{F}|_U \to 0 \tag{3.1}$$

for some $p < q$.

In other words, if for any point $x_0 \in M$, there is a neighborhood U_{x_0} of x_0 and finite many sections $\{f_1, \cdots, f_K\}$ of $\mathscr{F}|_{U_{x_0}}$ that generate each $\mathscr{O}_x$-module $\mathscr{F}_x$ ($x \in U_{x_0}$), and if moreover the relations among these generators are also finitely generated over U_{x_0}.

Coherent sheaves can be seen as a generalization of holomorphic vector bundles.

- A coherent sheaf $\mathscr{F}$ is said to **locally free** if for each point $x \in M$ there exists a neighborhood U such that

$$\mathscr{F}|_U \cong \mathscr{O}^p. \tag{3.2}$$

- There is a one-to-one correspondence between (isomorphism classes of) holomorphic bundles over M and locally free coherent sheaves.

The dual of a coherent sheaf $\mathscr{F}$ is defined to be the coherent sheaf

$$\mathscr{F}^* = Hom(\mathscr{F}, \mathscr{O}). \tag{3.3}$$

There is a natural homomorphism σ of $\mathscr{F}$ into its double dual $\mathscr{F}^{**}$

$$\sigma : \mathscr{F} \to \mathscr{F}^{**}. \tag{3.4}$$

- If σ is injective, then $\mathscr{F}$ will be called **torsion-free**.
- If σ is bijective, then $\mathscr{F}$ will be called **reflexive**.

One can check that every locally-free sheaf must be reflexive. The converse is not right. In the following, we denote the set of **singularities** where $\mathscr{F}$ is not locally free by Σ (i.e. outside Σ, $\mathscr{F}$ can be seen as a bundle). It is well known:

- If $\mathscr{F}$ is torsion-free, the singular set Σ is an analytic subset of codimension at least 2.
- If $\mathscr{F}$ is reflexive, the singular set Σ is an analytic subset of codimension at least 3.

Regularization

A **regularization on the reflexive sheaf** $\mathscr{E}$ (by Hironaka's flattening theorem): take blowing up finitely many times $\pi_i : M_i \to M_{i-1}$, where $i = 1, \cdots, k$ and $M_0 = M$, so that the pull-back of $\mathscr{E}$ to M_k is locally free and

$$\pi = \pi_1 \circ \cdots \circ \pi_k : M_k \to M \tag{3.5}$$

is biholomorphic outside $\Sigma \subset M$.

We denote M_k by $\tilde{M}$, the exceptional divisors $\pi^{-1}\Sigma$ by $\tilde{\Sigma}$, and the holomorphic vector bundle $\pi^*\mathscr{E}$ by E.

Since $\mathscr{E}$ is locally free outside Σ, the holomorphic bundle E is isomorphic to $\mathscr{E}$ on $\tilde{M} \setminus \tilde{\Sigma}$.

Stability of coherent sheaves

Let (M,ω) be a compact Kähler manifold, and $\mathscr{E}$ be a torsion-free coherent sheaves over M.

The stability was a well established concept in algebraic geometry. A torsion-free coherent sheaves $\mathscr{E}$ is called **stable** (semi-stable) (in the sense of Mumford-Takemoto), if for every saturated coherent sub-sheaf $\mathscr{F} \hookrightarrow \mathscr{E}$ of lower rank, it holds:

$$\mu(\mathscr{F}) = \frac{deg(\mathscr{F})}{rank\mathscr{F}} < (\leq) \mu(\mathscr{E}) = \frac{deg(\mathscr{E})}{rank\mathscr{E}}. \tag{3.6}$$

Harder-Narasimhan-Seshadri filtration

Let $\mathscr{E}$ be a torsion-free coherent sheaf on a compact Kähler manifold (M,ω). There is a filtration of $\mathscr{E}$ by coherent sub-sheaves

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E},$$

called the Harder-Narasimhan-Seshadri filtration of the reflexive sheaf $\mathscr{E}$ (abbr, HNS-filtration), such that the quotient sheaf $Q_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ is torsion-free and stable. Moreover, $\mu(Q_i) \geq \mu(Q_{i+1})$, and the associated graded object

$$Gr^{hns}(\mathscr{E}) = \oplus_{i=1}^{l} Q_i$$

is uniquely determined by the isomorphism class of $\mathscr{E}$.

Harder-Narasimhan type

For a reflexive sheaf $\mathscr{E}$ of rank R over a compact Kähler manifold (M,ω), construct a nonincreasing R-tuple of numbers

$$\vec{\mu}(\mathscr{E}) = (\mu_{1,\omega},\cdots,\mu_{R,\omega}) \tag{3.7}$$

from the HN-filtration by setting: $\mu_{i,\omega} = \mu_\omega(Q_j)$, for $rank(\mathscr{E}_{j-1})+1 \leq i \leq rank(\mathscr{E}_j)$. We call $\vec{\mu}(\mathscr{E})$ the Harder-Narasimhan type of $\mathscr{E}$.

Admissible Hermitian metric

- The admissible Hermitian metric on a coherent sheaf was introduced by Bando and Siu (1994).

Let Σ be the set of singularities where $\mathscr{E}$ is not locally free. A Hermitian metric H on the holomorphic bundle $\mathscr{E}|_{M\setminus\Sigma}$ is called **admissible** if
(1) $|F_H|_{H,\omega}$ is square integrable;
(2) $|\Lambda_\omega F_H|_H$ is uniformly bounded.

Bando and Siu's theorem

Bando and Siu extended the Donaldson-Uhlenbeck-Yau theorem to reflexive sheaves:

- If a reflexive sheaf $\mathscr{E}$ on a compact Kähler manifold (M, ω) is stable, then it admits an admissible Hermitian-Einstein metric.

Choosing an initial Hermitian metric

It is well known that $\tilde{M}$ is also Kähler. Fix a Kähler metric η on $\tilde{M}$ and set

$$\omega_\varepsilon = \pi^*\omega + \varepsilon\eta \tag{3.8}$$

for any small $0 < \varepsilon \leq 1$.

Given a smooth Hermitian metric $\hat{H}$ on the bundle E, it is easy to see that there exists a constant $\hat{C}_0$ such that

$$\int_{\tilde{M}} (|\Lambda_{\omega_\varepsilon} F_{\hat{H}}|_{\hat{H}}) \frac{\omega_\varepsilon^n}{n!} \leq \hat{C}_0, \tag{3.9}$$

for all $0 < \varepsilon \leq 1$.

Uniform bound of heat kernels

Let $K_\varepsilon(t,x,y)$ be the heat kernel with respect to the Kähler metric ω_ε. Bando and Siu obtained a uniform Sobolev inequality for $(\tilde{M},\omega_\varepsilon)$, using Cheng and Li's estimate , they got a uniform upper bound of the heat kernels $K_\varepsilon(t,x,y)$. Furthermore, combining Grigor'yan's result, we have

- For any $\tau > 0$, there exists a constant $C_K(\tau)$ which is independent of ε, such that

$$0 \leq K_\varepsilon(x,y,t) \leq C_K(\tau)(t^{-n}\exp(-\frac{(d_{\omega_\varepsilon}(x,y))^2}{(4+\tau)t})+1) \qquad (3.10)$$

  for every $x,y \in \tilde{M}$ and $0 < t < +\infty$, where $d_{\omega_\varepsilon}(x,y)$ is the distance between x and y with respect to the metric ω_ε.

The Hermitian-Yang-Mills flow

We consider the following Hermitian-Yang-Mills flow on holomorphic bundle E with the fixed initial metric $\hat{H}$ and with respect to the Kähler metric ω_ε,

$$\begin{cases} H_\varepsilon(t)^{-1}\dfrac{\partial H_\varepsilon(t)}{\partial t} = -2(\sqrt{-1}\Lambda_{\omega_\varepsilon}(F_{H_\varepsilon(t)}) - \lambda_\varepsilon Id_E), \\ H_\varepsilon(0) = \hat{H}, \end{cases} \tag{3.11}$$

where $\lambda_\varepsilon = \frac{2\pi}{Vol(\tilde{M},\omega_\varepsilon)}\mu_{\omega_\varepsilon}(E)$.

Taking the limit $\varepsilon \to 0$

- We obtain uniform local C^∞-estimates for $H_\varepsilon(t)$.
- Taking the limit as $\varepsilon \to 0$, we have a long time solution $H(t)$ of the following evolution equation on $M \setminus \Sigma \times [0, +\infty)$, i.e. $H(t)$ satisfies:

$$\begin{cases} H(t)^{-1}\dfrac{\partial H(t)}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_{H(t)}) - \lambda Id_{\mathscr{E}}), \\ H(0) = \hat{H}. \end{cases} \tag{3.12}$$

Here $H(t)$ can be seen as a Hermitian metric defined on the locally free part of $\mathscr{E}$, i.e. on $M \setminus \Sigma$.

Taking the limit $t \to +\infty$

Following Simpson's argument for the non-compact manifolds case, Bando-Siu proved:

- If the reflexive sheaf $\mathscr{E}$ is stable, by choosing a sequence, $H(t)$ converges to an admissible Hermitian-Einsten metric H_∞.

Bando and Siu's question

If the reflexive sheaf $\mathscr{E}$ is ω-stable, it is well known that the pulling back holomorphic bundle E is also ω_ε-stable for sufficiently small ε.

- By Donaldson-Uhlenbeck-Yau theorem, there exists an ω_ε-Hermitian-Einstein metric H_ε for every small ε.
- Bando and Siu (1994) point out whether it is possible to get an ω-Hermitian-Einstein metric H on the reflexive sheaf $\mathscr{E}$ as a limit of ω_ε-Hermitian-Einstein metric H_ε of bundle E on $\tilde{M}$ as $\varepsilon \to 0$.

Our result

We (Li-Zhang-Z, 2016) solve this problem and generalize it to the **Higgs sheaf** case. We proved that:

Theorem 1

Let $(\mathscr{E}, \phi)$ be a stable Higgs sheaf on a compact Kähler manifold (M, ω), and H_ε be an ω_ε-Hermitian-Einstein metric on the Higgs bundle (E, ϕ), by choosing a subsequence and rescaling it, H_ε must converge to an ω-Hermitian-Einstein metric H in local C^∞-topology outside the exceptional divisor $\tilde{\Sigma}$ as $\varepsilon \to 0$.

The Hermitian-Yang-Mills flow on non-stable reflexive sheaves

Bando and Siu have proved the existence of long time solution of the Hermitian-Yang-Mills flow on a reflexive sheaf $\mathscr{E}$, i.e. there is a family of metrics $H(t)$ on $M\setminus\Sigma\times[0,+\infty)$ such that $H(t)$ satisfies:

$$\begin{cases} H(t)^{-1}\dfrac{\partial H(t)}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_{H(t)}) - \lambda Id_{\mathscr{E}}), \\ H(0) = H_0. \end{cases} \tag{3.13}$$

Non-stable reflexive sheaves

Bando and Siu (1994) proved that: there exists a subsequence $H(t_i)$ such that $\int_M |\nabla \Lambda_\omega F_{H(t_i)}| \to 0$. By Uhlenbeck's theorem, taking suitable gauge transformations one can take a subsequence so that Chern connections

$$A(t_i) \to A_\infty$$

weakly in $W^{1,2}$-topology outside a closed subset $\Sigma_{an} \subset M$ of Hausdorff codimension at least 4.

Non-stable reflexive sheaves

One can show that $\sqrt{-1}\Lambda_\omega F_{H(t_\infty)} = \sqrt{-1}\theta_\infty$ is parallel, we can decompose E_∞ according to the eigenvalues of $\sqrt{-1}\theta_\infty$ on $M \setminus (\Sigma_{\mathscr{E}} \cup \Sigma_{an})$, and obtain a nonincreasing R-tuple of numbers

$$\vec{\lambda}(\sqrt{-1}\theta_\infty) = (\lambda_1, \cdots, \lambda_R). \tag{3.14}$$

We also obtain a holomorphic orthogonal decomposition

$$E_\infty = \bigoplus_{i=1}^{l} E_\infty^i, \tag{3.15}$$

every E_∞^i admits Hermitian-Einstein metric.

Bando-Siu conjecture

- Bando and Siu (1994) asked: does

$$\bigoplus_{i=1}^{l} E_{\infty}^{i} \cong = \bigoplus_{i=1}^{l} Q_i \ ? \tag{3.16}$$

$$\vec{\lambda}(\sqrt{-1}\theta_{\infty}) = \vec{\mu}(E) = (\mu_1, \cdots, \mu_R)$$

- Atiyah and Bott (1982) raised the question for Riemann surface case, and this was proved by Daskalopoulos (1992).
- Li-Zhang-Zhang (2018) proved Bando-Siu conjecture.

Our result

Theorem 2

(Li-Zhang-Zhang) Bando-Siu conjecture is true.

Outline

The Yang-Mills flow

Donaldson has shown that the Hermitian-Yang-Mills flow (3.13) is formally gauge-equivalent to the Yang-Mills flow, i.e. we have:

- There is a family of complex gauge transformations $\sigma(t) \in \mathbf{G}^{\mathbb{C}}$ satisfying $\sigma^{*\hat{H}}(t)\sigma(t) = h(t) = \hat{H}^{-1}H(t)$, where $H(t)$ is the long time solution of the Hermitian-Yang-Mills flow (3.12) with the initial metric $\hat{H}$, such that $A(t) = \sigma(t)(\hat{A})$ is a long time solution of the Yang-Mills flow with the initial connection $\hat{A}$ on the Hermitian vector bundle $(\mathscr{E}|_{M\setminus\Sigma_{\mathscr{E}}}, \hat{H})$, i.e. it satisfies:

$$\begin{cases} \dfrac{\partial A(t)}{\partial t} = -D^*_{A(t)}F_{A(t)}, \\ A(0) = \hat{A}. \end{cases} \tag{3.17}$$

Key estimates

For simplicity, set

$$\theta(A(t),\omega)=\sqrt{-1}\Lambda_{\omega}F_{A(t)}-\lambda_{\mathscr{E},\omega}Id, \tag{3.18}$$

and

$$I(t)=\int_M |D_{A(t)}\theta(A(t),\omega)|^2_{\hat{H}}\frac{\omega^n}{n!}=\int_M |D_{H(t)}\theta(H(t),\omega)|^2_{H(t)}\frac{\omega^n}{n!}. \tag{3.19}$$

We prove:

- Let $H(t)$ be the long time solution of the Hermitian-Yang-Mills flow (3.13) with the initial metric $\hat{H}$, then $I(t)\to 0$ as $t\to+\infty$.

When $\mathscr{E}$ is locally free, i.e. $\Sigma_{\mathscr{E}}=\emptyset$, this was prove by Donaldson and Kronheimer . In our case that $\mathscr{E}$ is only reflexive, we need new arguments because the base manifold $M\setminus\Sigma_{\mathscr{E}}$ is non-compact.

blow-up set

Using Hong-Tian's ε-regularity theorem on the Yang-Mills flow (3.17), and modifying Hong-Tian's argument to the non-compact case, we have

- Let $A(t)$ be the long time solution of the Yang-Mills flow (3.17) with initial connection $\hat{A}$ on the Hermitian vector bundle $(\mathscr{E}|_{M\setminus\Sigma_{\mathscr{E}}},\hat{H})$ over $(M\setminus\Sigma_{\mathscr{E}},\omega)$. Then for every sequence $t_k\to+\infty$, there exists a subsequence $\{t_j\}$ such that as $t_j\to+\infty$, $A(t_j)$ converges, modulo gauge transformations, to a solution A_∞ of the Yang-Mills equation on a Hermitian vector bundle (E_∞,H_∞) in C^∞_{loc}-topology outside $\Sigma\subset M$, where Σ is a closed set of Hausdorff complex codimension at least 2 and $\Sigma_{\mathscr{E}}\subset\Sigma$.

blow-up set

We set

$$d_x = dist(x, \Sigma_{\mathscr{E}}), \quad U_d = \{x \in M : d_x < d\}, \tag{3.20}$$

$$\hat{\Sigma}_{k,j,i} = \{x \in M \setminus U_{r_j} : r_i^{4-2n} \int_{B_{r_i}(x)} |F_{A(t_k)}|^2_{\hat{H},\omega}(\cdot) \frac{\omega^n}{n!} \geq \varepsilon_1\}, \tag{3.21}$$

for any $k \geq 1$ and $i \geq j \geq 1$. By the standard diagonal process, we can choose a subsequence which is also denoted by $\{t_k\}$ such that for each $j \leq i$, $\hat{\Sigma}_{k,j,i}$ converges to a closed subset $\Sigma_{j,i}$ as $k \to +\infty$. Define

$$\Sigma_j = \bigcap_i \Sigma_{j,i}, \Sigma_{an} = \bigcup_j \Sigma_j, \quad \Sigma = \Sigma_{\mathscr{E}} \bigcup \Sigma_{an}. \tag{3.22}$$

blow-up set

We can prove:

- Σ is closed.
- The Hausdorff codimension of Σ is at least 4.

The limiting connection

Furthermore, we have

$$D_{A_\infty}\theta(A_\infty,\omega)=0, \tag{3.23}$$

i.e. $\theta(A_\infty,\omega)$ is parallel. We can decompose E_∞ according to the eigenvalues of $\sqrt{-1}\theta(A_\infty,\omega)$ and obtain a holomorphic orthogonal decomposition: $E_\infty=\bigoplus_{i=1}^{l}E_\infty^i$ on $M\setminus\Sigma$. Let λ_i be the eigenvalues of $\sqrt{-1}\theta(A_\infty,\omega)$, we a nonincreasing R-tuple of numbers

$$\vec{\lambda}(A_\infty)=(\lambda_1,\cdots,\lambda_R). \tag{3.24}$$

Main idea of the proof

- We prove that

$$\vec{\lambda}(A_\infty) = \vec{\mu}(\mathscr{E}), \tag{3.25}$$

i.e. the limiting sheaf has the same HN type of the initial one.

The limiting sheaf

Since the singularity set Σ is of Hausdorff codimension at least 4, $F_{A_\infty} \in L^2$ and $\Lambda_\omega F_{A_\infty} \in L^\infty$, by Bando and Siu's extension theorem, we know that every $(E^i_\infty, \overline{\partial}_{A^i_\infty})$ can be extended to the whole M as a reflexive sheaf (which is also denoted by $(E^i_\infty, \overline{\partial}_{A^i_\infty})$ for simplicity), and H^i_∞ can be smoothly extended over the place where the sheaf $(E^i_\infty, \overline{\partial}_{A^i_\infty})$ is locally free. Therefore, we proved:

- The limiting $(E_\infty, \overline{\partial}_{A_\infty})$ can be extended to the whole M as a reflexive sheaf with a holomorphic orthogonal splitting

$$(E_\infty, \overline{\partial}_{A_\infty}, H_\infty) = \bigoplus_{i=1}^{l} (E^i_\infty, \overline{\partial}_{A^i_\infty}, H^i_\infty), \tag{3.26}$$

  and H^i_∞ is an admissible Hermitian-Einstein metric on the reflexive sheaf $(E^i_\infty, \overline{\partial}_{A^i_\infty})$ for any $1 \leq i \leq l$.

Thank you for your attention!