Canonical metrics on reflexive sheaves

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Outline

1. Stability and the Harder-Narasimhan-Seshadri filtration

2. Hermitian-Einstein metric and Donaldson-Uhlenbeck-Yau theorem

3. Bando-Siu Conjecture
   - Main idea of the proof
Stability of Bundles

Let $M$ be a Kähler manifold ($\mathbb{C}^n$), with the Kähler form $\omega$.

Let $E$ be a holomorphic vector bundle on $M$ (we simply call it bundle in the following).

At each point $x \in M$, $H_x = \mathbb{C}^r$.

Suppose $\{U_\alpha, \alpha \in I\}$ is a covering of $M$, locally $E = U_\alpha \times \mathbb{C}^r$, if $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_{\alpha \beta} : U_\alpha \times \mathbb{C}^r \to U_\beta \times \mathbb{C}^r$ is holomorphic.
Let $H$ be a Hermitian metric on $E$. It means at each point $x \in M$, $H_x$ is a Hermitian metric on $E_x$. and $H$ is smooth.

Associated to $H$ there is a Chern connection $A_H$, and its curvature is defined by $F_H = DA_H$.

The degree of $E$ is defined by

$$\text{deg}(E) = \frac{1}{2\pi} \int_M \text{Tr} F_H \wedge \omega^{n-1}.$$
The stability of bundles was a well established concept in algebraic geometry.

A holomorphic vector bundle $E$ is called **stable** (semi-stable), if for every subbundle $E' \hookrightarrow E$ of lower rank, it holds:

$$
\mu(E') = \frac{\deg(E')}{{\text{rank}}E'} < (\leq) \mu(E) = \frac{\deg(E)}{{\text{rank}}E}.
$$

(1.1)

Here the degree of $E$ is defined as follow

$$
\deg(E) = \int_M C_1(E) \wedge \omega^{n-1},
$$

where $C_1(E)$ is the first chern class of $E$
Let $E$ be a bundle. Then there is a filtration of $E$ by subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E,$$

called the Harder-Narasimhan filtration of bundle $E$ (abbr, HN-filtration), such that $Q_i = E_i / E_{i-1}$ is semistable. Moreover, $\mu(Q_i) > \mu(Q_{i+1})$, and the associated graded object $Gr^{hn}(E) = \bigoplus_{i=1}^{l} Q_i$ is uniquely determined by the isomorphism class of $E$. 
Let $V$ be a semistable bundle, then there is a filtration of $V$ by subbundles

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V,$$

called the Seshadri filtration of $V$, such that $V_i/V_{i-1}$ stable. Moreover, $\mu(V_i/V_{i-1}) = \mu(V)$ for each $i$, and the associated graded object $Gr^s V = \bigoplus_{i=1}^l V_i/V_{i-1}$ is uniquely determined by the isomorphism class of $V$. 
The Harder-Narasimhan-Seshadri filtration

Let $E$ be a bundle. Then there is a double filtration, called a Harder-Narasimhan-Seshadri filtration of bundle $E$ (abbr, HNS-filtration), with the following properties: if $\{E_i\}_{i=1}^l$ is the HN filtration of $E$, then

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,l_i} = E_i$$

and the successive quotient $Q_{i,j} = E_{i,j}/E_{i,j-1}$ are stable subbundles. Moreover, $\mu(Q_{i,j}) = \mu(Q_{i,j+1})$ and $\mu(Q_{i,j}) > \mu(Q_{i+1,j})$, the associated graded object:

$$\text{Gr}^{hns}(E, \bar{\partial}E) = \bigoplus_{i=1}^l \bigoplus_{j=1}^{l_i} Q_{i,j}$$

is uniquely determined by the isomorphism class of $E$. 
Let $E$ be a bundle. There is a filtration of $E$ by subbundles

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E},$$

called the Harder-Narasimhan-Seshadri filtration of $E$ (abbr, HNS-filtration), such that the quotient sheaf $Q_i = \mathcal{E}_i / \mathcal{E}_{i-1}$ is stable. Moreover, $\mu(Q_i) \geq \mu(Q_{i+1})$, and the associated graded object

$$Gr^{hns}(\mathcal{E}) = \bigoplus_{i=1}^l Q_i$$

is uniquely determined by the isomorphism class of $\mathcal{E}$. 
We then have $R$-tuple of numbers

$$
\vec{\mu}(E) = (\mu_1, \cdots, \mu_R)
$$

(1.2)

from the HNS-filtration by setting: $\mu_i = \mu_{\omega}(Q_j)$, for $rank(E_{j-1}) + 1 \leq i \leq rank(E_j)$. We call $\vec{\mu}(E)$ the Harder-Narasimhan type of $E$. 
Chern connection

Let \((E, \bar{\partial}_E)\) be a holomorphic vector bundle over a complex manifold \(M\), and \(H\) be a Hermitian metric on \(E\), then there exists a unique connection \(D_H\), which is called the **Chern connection** which satisfies the following two conditions:

1. \(D_H\) is compatible with the Hermitian metric \(H\);
2. Its \((0, 1)\)-part coincides to the holomorphic structure, i.e. \(D_H^{0,1} = \bar{\partial}_E\).
A Hermitian metric $H$ in $E$ is called a Hermitian-Einstein metric, if the curvature $F_H$ of the chern connection $A_H$ satisfies the Einstein condition:

$$\sqrt{-1}\Lambda_\omega F_H = \lambda Id_E,$$

(2.1)

where $\Lambda_\omega$ denotes the contraction of differential forms by Kähler form $\omega$, and the real constant $\lambda$ is given by $\lambda = \frac{2\pi}{Vol(M) \text{rank}(E)} \text{deg}(E)$. 
Donaldson-Uhlenbeck-Yau theorem

D-U-Y theorem states that a holomorphic bundle is poly-stable if and only if it admits a Hermitian-Einstein metric.

A short history of the theorem:

- Narasimhan and Seshadri (1965) for Riemann surface.
- Donaldson (1985) for algebraic surfaces, and (1986) for algebraic manifolds.
- Uhlenbeck and Yau (1986) for Kähler manifolds.

Atiyah and Bott (1982) conjectured that the asymptotic behavior of the Yang-Mills flow on a Riemann surface at infinity is decided by the HNS filtration.
Some interesting generalizations

- Li and Yau (1987) for Hermitian manifolds with Gauduchon metric (i.e. $\bar{\partial} \bar{\partial} (\eta^{n-1}) = 0$).
- Bando and Siu (1994) for reflexive sheaf.
- Bartolomeis and Tian (1996) for complex vector bundle on almost Hermitian manifold.
- Jost-Zuo, harmonic map and representation.
- Zuo, Sheng, Xu, algebraic version.
Donaldson’s heat flow for Hermitian metrics on the bundle $E$ with initial metric $H_0$:

$$H^{-1} \frac{\partial H}{\partial t} = -2(\sqrt{-1} \Lambda_\omega F_H - \lambda I_{d_E}).$$  \hspace{1cm} (2.2)

In the following, we denote $\Phi(H) = \sqrt{-1} \Lambda_\omega F_H - \lambda I_{d_E}$ for simplicity.
Donaldson’s heat flow

- The flow exists globally;
- If the bundle is stable, the flow converges to a H-E metric, proved by Donaldson.
- Li-Zhang-Zhang (2017) (for Higgs sheaves) show that under semi-stability assumption, we have \( \| \Phi(H(t)) \|_{L^\infty} \to 0 \) along the heat flow, solve a conjecture by Kobayashi (1986).
A sheaf $\mathcal{F}$ on $M$ associates to each open set $U \cap M$ a module (or group, other algebraic structures) $\mathcal{F}(U)$, called the sections of $\mathcal{F}$ over $U$, and to each pair $U \subset V$ of open sets a map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, called the restriction map, satisfying
If $U \subset V \subset W$, we have

$$r_{W,U} = r_{W,V} \cdot r_{V,U}.$$

For any open sets $U$, $V \subset M$ and sections $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that

$$\sigma|_{U \cap V} = \tau|_{U \cap V},$$

then there is a section $\rho \in \mathcal{F}(U \cup V)$ with

$$\rho|_U = \sigma, \quad \rho|_V = \tau.$$

If $\rho \in \mathcal{F}(U \cup V)$ and

$$\rho|_U = \rho|_V = 0,$$

then $\rho = 0$.

For examples, the sheaf of holomorphic functions on the complex manifold $M$ denoted by $\mathcal{O}_M$, the sheaf of holomorphic sections on bundles.
A $\mathcal{O}_M$-module sheaf is called an analytic sheaf. An analytic sheaf $\mathcal{F}$ is said to be **Coherent** if for each point $x \in M$, there is a neighborhood $U$ of $x$ such that there is an exact sequence of sheaves over $U$,

$$\mathcal{O}^p|_U \rightarrow \mathcal{O}^q|_U \rightarrow \mathcal{F}|_U \rightarrow 0$$

for some $p < q$.

In other words, if for any point $x_0 \in M$, there is a neighborhood $U_{x_0}$ of $x_0$ and finite many sections $\{f_1, \cdots, f_K\}$ of $\mathcal{F}|_{U_{x_0}}$ that generate each $\mathcal{O}_x$-module $\mathcal{F}_x$ ($x \in U_{x_0}$), and if moreover the relations among these generators are also finitely generated over $U_{x_0}$. 

**Main idea of the proof**

**Bando-Siu Conjecture**

**Hermitian-Einstein metric and Donaldson-Uhlenbeck-Yau theorem**

**Stability and the Harder-Narasimhan-Seshadri filtration**
Coherent sheaves can be seen as a generalization of holomorphic vector bundles.

- A coherent sheaf $\mathcal{F}$ is said to be **locally free** if for each point $x \in M$ there exists a neighborhood $U$ such that

  $$\mathcal{F}|_U \cong \mathcal{O}^p. \quad (3.2)$$

- There is a one-to-one correspondence between (isomorphism classes of) holomorphic bundles over $M$ and locally free coherent sheaves.
The dual of a coherent sheaf $\mathcal{F}$ is defined to be the coherent sheaf

$$\mathcal{F}^* = \text{Hom}(\mathcal{F}, \mathcal{O}).$$  \hspace{1cm} (3.3)

There is a natural homomorphism $\sigma$ of $\mathcal{F}$ into its double dual $\mathcal{F}^{**}$

$$\sigma : \mathcal{F} \rightarrow \mathcal{F}^{**}. \hspace{1cm} (3.4)$$

- If $\sigma$ is injective, then $\mathcal{F}$ will be called **torsion-free**.
- If $\sigma$ is bijective, then $\mathcal{F}$ will be called **reflexive**.
One can check that every locally-free sheaf must be reflexive. The converse is not right. In the following, we denote the set of singularities where $\mathcal{F}$ is not locally free by $\Sigma$ (i.e. outside $\Sigma$, $\mathcal{F}$ can be seen as a bundle). It is well known:

- If $\mathcal{F}$ is torsion-free, the singular set $\Sigma$ is an analytic subset of codimension at least 2.
- If $\mathcal{F}$ is reflexive, the singular set $\Sigma$ is an analytic subset of codimension at least 3.
Regularization

A regularization on the reflexive sheaf $\mathcal{E}$ (by Hironaka’s flattening theorem): take blowing up finitely many times $\pi_i : M_i \to M_{i-1}$, where $i = 1, \cdots, k$ and $M_0 = M$, so that the pull-back of $\mathcal{E}$ to $M_k$ is locally free and

$$\pi = \pi_1 \circ \cdots \circ \pi_k : M_k \to M$$

(3.5)

is biholomorphic outside $\Sigma \subset M$.

We denote $M_k$ by $\tilde{M}$, the exceptional divisors $\pi^{-1}\Sigma$ by $\tilde{\Sigma}$, and the holomorphic vector bundle $\pi^* \mathcal{E}$ by $E$.

Since $\mathcal{E}$ is locally free outside $\Sigma$, the holomorphic bundle $E$ is isomorphic to $\mathcal{E}$ on $\tilde{M} \setminus \tilde{\Sigma}$.
Let \((M, \omega)\) be a compact Kähler manifold, and \(\mathcal{E}\) be a torsion-free coherent sheaves over \(M\).

The stability was a well established concept in algebraic geometry. A torsion-free coherent sheaves \(\mathcal{E}\) is called **stable** (semi-stable) (in the sense of Mumford-Takemoto), if for every saturated coherent sub-sheaf \(\mathcal{F} \hookrightarrow \mathcal{E}\) of lower rank, it holds:

\[
\mu(\mathcal{F}) = \frac{\text{deg}(\mathcal{F})}{\text{rank}\mathcal{F}} < (\leq) \mu(\mathcal{E}) = \frac{\text{deg}(\mathcal{E})}{\text{rank}\mathcal{E}}.
\]
Let $\mathcal{E}$ be a torsion-free coherent sheaf on a compact Kähler manifold $(M, \omega)$. There is a filtration of $\mathcal{E}$ by coherent sub-sheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E},$$

called the Harder-Narasimhan-Seshadri filtration of the reflexive sheaf $\mathcal{E}$ (abbr, HNS-filtration), such that the quotient sheaf $Q_i = \mathcal{E}_i / \mathcal{E}_{i-1}$ is torsion-free and stable. Moreover, $\mu(Q_i) \geq \mu(Q_{i+1})$, and the associated graded object

$$Gr^{\text{hns}}(\mathcal{E}) = \bigoplus_{i=1}^{l} Q_i$$

is uniquely determined by the isomorphism class of $\mathcal{E}$. 
For a reflexive sheaf $\mathcal{E}$ of rank $R$ over a compact Kähler manifold $(M, \omega)$, construct a nonincreasing $R$-tuple of numbers

$$\bar{\mu}(\mathcal{E}) = (\mu_{1,\omega}, \cdots, \mu_{R,\omega})$$

(3.7)

from the HN-filtration by setting: $\mu_{i,\omega} = \mu_{\omega}(Q_j)$, for $\text{rank}(\mathcal{E}_{j-1}) + 1 \leq i \leq \text{rank}(\mathcal{E}_j)$. We call $\bar{\mu}(\mathcal{E})$ the Harder-Narasimhan type of $\mathcal{E}$.
The admissible Hermitian metric on a coherent sheaf was introduced by Bando and Siu (1994).

Let $\Sigma$ be the set of singularities where $E$ is not locally free. A Hermitian metric $H$ on the holomorphic bundle $E|_{M\setminus \Sigma}$ is called admissible if

1. $|F_H|_{H,\omega}$ is square integrable;
2. $|\Lambda_\omega F_H|_H$ is uniformly bounded.
Bando and Siu’s theorem

Bando and Siu extended the Donaldson-Uhlenbeck-Yau theorem to reflexive sheaves:

- If a reflexive sheaf $\mathcal{E}$ on a compact Kähler manifold $(M, \omega)$ is stable, then it admits an admissible Hermitian-Einstein metric.
Choosing an initial Hermitian metric

It is well known that $\tilde{M}$ is also Kähler. Fix a Kähler metric $\eta$ on $\tilde{M}$ and set

$$\omega_\epsilon = \pi^* \omega + \epsilon \eta$$

(3.8)

for any small $0 < \epsilon \leq 1$.

Given a smooth Hermitian metric $\hat{H}$ on the bundle $E$, it is easy to see that there exists a constant $\hat{C}_0$ such that

$$\int_{\tilde{M}} (|\Lambda \omega_\epsilon F_{\hat{H}}|_{\hat{H}}) \frac{\omega_\epsilon^n}{n!} \leq \hat{C}_0,$$

(3.9)

for all $0 < \epsilon \leq 1$. 

Jiayu Li

Canonical metrics on reflexive sheaves
Uniform bound of heat kernels

Let $K_{\epsilon}(t, x, y)$ be the heat kernel with respect to the Kähler metric $\omega_{\epsilon}$. Bando and Siu obtained a uniform Sobolev inequality for $(\tilde{M}, \omega_{\epsilon})$, using Cheng and Li’s estimate, they got a uniform upper bound of the heat kernels $K_{\epsilon}(t, x, y)$. Furthermore, combining Grigor’yan’s result, we have

- For any $\tau > 0$, there exists a constant $C_K(\tau)$ which is independent of $\epsilon$, such that

$$0 \leq K_{\epsilon}(x, y, t) \leq C_K(\tau)(t^{-n} \exp \left( -\frac{(d_{\omega_{\epsilon}}(x, y))^2}{(4 + \tau) t} \right) + 1) \quad (3.10)$$

for every $x, y \in \tilde{M}$ and $0 < t < +\infty$, where $d_{\omega_{\epsilon}}(x, y)$ is the distance between $x$ and $y$ with respect to the metric $\omega_{\epsilon}$.
We consider the following Hermitian-Yang-Mills flow on holomorphic bundle $E$ with the fixed initial metric $\hat{H}$ and with respect to the Kähler metric $\omega_\varepsilon$,

$$
\begin{cases}
H_\varepsilon(t)^{-1} \frac{\partial H_\varepsilon(t)}{\partial t} = -2(\sqrt{-1}\Lambda_{\omega_\varepsilon} (F_{H_\varepsilon(t)}) - \lambda_\varepsilon \text{Id}_E), \\
H_\varepsilon(0) = \hat{H},
\end{cases}
$$

(3.11)

where $\lambda_\varepsilon = \frac{2\pi}{Vol(M, \omega_\varepsilon)} \mu_{\omega_\varepsilon}(E)$. 
Taking the limit $\varepsilon \to 0$

- We obtain uniform local $C^\infty$-estimates for $H_\varepsilon(t)$.

- Taking the limit as $\varepsilon \to 0$, we have a long time solution $H(t)$ of the following evolution equation on $M \setminus \Sigma \times [0, +\infty)$, i.e. $H(t)$ satisfies:

\[
\begin{aligned}
H(t)^{-1} \frac{\partial H(t)}{\partial t} &= -2(\sqrt{-1}\Lambda_\omega(F_{H(t)}) - \lambda I_{H}) \\
H(0) &= \hat{H}.
\end{aligned}
\] (3.12)

Here $H(t)$ can be seen as a Hermitian metric defined on the locally free part of $E$, i.e. on $M \setminus \Sigma$. 
Taking the limit $t \to +\infty$

Following Simpson’s argument for the non-compact manifolds case, Bando-Siu proved:

- If the reflexive sheaf $\mathcal{E}$ is stable, by choosing a sequence, $H(t)$ converges to an admissible Hermitian-Einstein metric $H_\infty$. 
If the reflexive sheaf $\mathcal{E}$ is $\omega$-stable, it is well known that the pulling back holomorphic bundle $E$ is also $\omega_\varepsilon$-stable for sufficiently small $\varepsilon$.

- By Donaldson-Uhlenbeck-Yau theorem, there exists an $\omega_\varepsilon$-Hermitian-Einstein metric $H_\varepsilon$ for every small $\varepsilon$.
- Bando and Siu (1994) point out whether it is possible to get an $\omega$-Hermitian-Einstein metric $H$ on the reflexive sheaf $\mathcal{E}$ as a limit of $\omega_\varepsilon$-Hermitian-Einstein metric $H_\varepsilon$ of bundle $E$ on $\tilde{M}$ as $\varepsilon \to 0$. 
Our result

We (Li-Zhang-Z, 2016) solve this problem and generalize it to the **Higgs sheaf** case. We proved that:

**Theorem 1**

Let \((E, \phi)\) be a stable Higgs sheaf on a compact Kähler manifold \((M, \omega)\), and \(H_\varepsilon\) be an \(\omega_\varepsilon\)-Hermitian-Einstein metric on the Higgs bundle \((E, \phi)\), by choosing a subsequence and rescaling it, \(H_\varepsilon\) must converge to an \(\omega\)-Hermitian-Einstein metric \(H\) in local \(C^\infty\)-topology outside the exceptional divisor \(\tilde{\Sigma}\) as \(\varepsilon \to 0\).
The Hermitian-Yang-Mills flow on non-stable reflexive sheaves

Bando and Siu have proved the existence of long time solution of the Hermitian-Yang-Mills flow on a reflexive sheaf $\mathcal{E}$, i.e. there is a family of metrics $H(t)$ on $M \setminus \Sigma \times [0, +\infty)$ such that $H(t)$ satisfies:

$$\begin{cases} H(t)^{-1} \frac{\partial H(t)}{\partial t} = -2(\sqrt{-1} \Lambda_{\omega}(F_{H(t)}) - \lambda Id_{\mathcal{E}}), \\
H(0) = H_0. \end{cases}$$

(3.13)
Bando and Siu (1994) proved that: there exists a subsequence $H(t_i)$ such that $\int_M |\nabla A_\omega F_{H(t_i)}| \to 0$. By Uhlenbeck’s theorem, taking suitable gauge transformations one can take a subsequence so that Chern connections

$$A(t_i) \to A_\infty$$

weakly in $W^{1,2}$-topology outside a closed subset $\Sigma_{an} \subset M$ of Hausdorff codimension at least 4.
Non-stable reflexive sheaves

One can show that $\sqrt{-1} \Lambda_\omega F_{H(t_\infty)} = \sqrt{-1} \theta_\infty$ is parallel, we can decompose $E_\infty$ according to the eigenvalues of $\sqrt{-1} \theta_\infty$ on $M \setminus (\Sigma_c \cup \Sigma_{an})$, and obtain a nonincreasing $R$-tuple of numbers

$$\vec{\lambda}(\sqrt{-1} \theta_\infty) = (\lambda_1, \cdots, \lambda_R). \quad (3.14)$$

We also obtain a holomorphic orthogonal decomposition

$$E_\infty = \bigoplus_{i=1}^l E_i^\infty, \quad (3.15)$$

every $E_i^\infty$ admits Hermitian-Einstein metric.
Bando-Siu conjecture

- Bando and Siu (1994) asked: does

\[ \bigoplus_{i=1}^{l} E^i_\infty \cong \bigoplus_{i=1}^{l} Q_i ? \]  (3.16)

\[ \tilde{\lambda}(\sqrt{-1} \theta_\infty) = \bar{\mu}(E) = (\mu_1, \cdots, \mu_R) \]

- Atiyah and Bott (1982) raised the question for Riemann surface case, and this was proved by Daskalopoulos (1992).

Our result

Theorem 2

(Li-Zhang-Zhang) Bando-Siu conjecture is true.
Outline

1. Stability and the Harder-Narasimhan-Seshadri filtration

2. Hermitian-Einstein metric and Donaldson-Uhlenbeck-Yau theorem

3. Bando-Siu Conjecture
   - Main idea of the proof
The Yang-Mills flow

Donaldson has shown that the Hermitian-Yang-Mills flow (3.13) is formally gauge-equivalent to the Yang-Mills flow, i.e. we have:

- There is a family of complex gauge transformations \( \sigma(t) \in G^\mathbb{C} \) satisfying \( \sigma^* \hat{H}(t) \sigma(t) = h(t) = \hat{H}^{-1} H(t) \), where \( H(t) \) is the long time solution of the Hermitian-Yang-Mills flow (3.12) with the initial metric \( \hat{H} \), such that \( A(t) = \sigma(t)(\hat{A}) \) is a long time solution of the Yang-Mills flow with the initial connection \( \hat{A} \) on the Hermitian vector bundle \( (\mathcal{E}|_M\setminus \Sigma_{\mathcal{E}}, \hat{H}) \), i.e. it satisfies:

\[
\begin{cases}
\frac{\partial A(t)}{\partial t} = -D^*_A(t) F_A(t), \\
A(0) = \hat{A}.
\end{cases}
\]  

(3.17)
Key estimates

For simplicity, set

$$\theta(A(t), \omega) = \sqrt{-1} \Lambda_\omega F_{A(t)} - \lambda_{\mathcal{E}, \omega} \Id,$$  

(3.18)

and

$$I(t) = \int_M |D_{A(t)} \theta(A(t), \omega)|^2 \hat{H} \frac{\omega^n}{n!} = \int_M |D_{H(t)} \theta(H(t), \omega)|^2 \frac{\omega^n}{H(t) n!}.$$  

(3.19)

We prove:

- Let $H(t)$ be the long time solution of the Hermitian-Yang-Mills flow (3.13) with the initial metric $\hat{H}$, then $I(t) \to 0$ as $t \to +\infty$.

When $\mathcal{E}$ is locally free, i.e. $\Sigma_{\mathcal{E}} = \emptyset$, this was prove by Donaldson and Kronheimer. In our case that $\mathcal{E}$ is only reflexive, we need new arguments because the base manifold $M \setminus \Sigma_{\mathcal{E}}$ is non-compact.
Using Hong-Tian’s $\varepsilon$-regularity theorem on the Yang-Mills flow (3.17), and modifying Hong-Tian’s argument to the non-compact case, we have

Let $A(t)$ be the long time solution of the Yang-Mills flow (3.17) with initial connection $\hat{A}$ on the Hermitian vector bundle $(\mathcal{E}|_{M \setminus \Sigma_\mathcal{E}}, \hat{H})$ over $(M \setminus \Sigma_\mathcal{E}, \omega)$. Then for every sequence $t_k \to +\infty$, there exists a subsequence $\{t_j\}$ such that as $t_j \to +\infty$, $A(t_j)$ converges, modulo gauge transformations, to a solution $A_\infty$ of the Yang-Mills equation on a Hermitian vector bundle $(E_\infty, H_\infty)$ in $C^\infty_{loc}$-topology outside $\Sigma \subset M$, where $\Sigma$ is a closed set of Hausdorff complex codimension at least 2 and $\Sigma_\mathcal{E} \subset \Sigma$. 
We set
\[ d_x = \text{dist}(x, \Sigma_\mathcal{E}), \quad U_d = \{ x \in M : d_x < d \}, \quad (3.20) \]
\[ \hat{\Sigma}_{k,j,i} = \{ x \in M \setminus U_{r_j} : r_i^{4-2n} \int_{B_{r_i}(x)} |F_{A(t_k)}|_s^2 \hat{H}, \omega(\cdot) \frac{\omega^n}{n!} \geq \epsilon_1 \}, \quad (3.21) \]
for any \( k \geq 1 \) and \( i \geq j \geq 1 \). By the standard diagonal process, we can choose a subsequence which is also denoted by \( \{ t_k \} \) such that for each \( j \leq i \), \( \hat{\Sigma}_{k,j,i} \) converges to a closed subset \( \Sigma_{j,i} \) as \( k \to +\infty \). Define
\[ \Sigma_j = \bigcap_i \Sigma_{j,i}, \Sigma_{an} = \bigcup_j \Sigma_j, \quad \Sigma = \Sigma_\mathcal{E} \cup \Sigma_{an}. \quad (3.22) \]
We can prove:

- $\Sigma$ is closed.
- The Hausdorff codimension of $\Sigma$ is at least 4.
The limiting connection

Furthermore, we have

$$D_{A_\infty} \theta(A_\infty, \omega) = 0,$$  \hspace{1cm} (3.23)

i.e. $\theta(A_\infty, \omega)$ is parallel. We can decompose $E_\infty$ according to the eigenvalues of $\sqrt{-1}\theta(A_\infty, \omega)$ and obtain a holomorphic orthogonal decomposition: $E_\infty = \bigoplus_{i=1}^l E_i^i$ on $M \setminus \Sigma$. Let $\lambda_i$ be the eigenvalues of $\sqrt{-1}\theta(A_\infty, \omega)$, we a nonincreasing $R$-tuple of numbers

$$\vec{\lambda}(A_\infty) = (\lambda_1, \cdots, \lambda_R).$$  \hspace{1cm} (3.24)
Main idea of the proof

We prove that

\[ \tilde{\lambda}(A_\infty) = \tilde{\mu}(E), \]  

(3.25)

i.e. the limiting sheaf has the same HN type of the initial one.
The limiting sheaf

Since the singularity set $\Sigma$ is of Hausdorff codimension at least 4, $F_{A_\infty} \in L^2$ and $\Lambda^\omega F_{A_\infty} \in L^\infty$, by Bando and Siu’s extension theorem, we know that every $(E^i_\infty, \overline{\partial}^{A_i}_\infty)$ can be extended to the whole $M$ as a reflexive sheaf (which is also denoted by $(E^i_\infty, \overline{\partial}^{A_i}_\infty)$ for simplicity), and $H^i_\infty$ can be smoothly extended over the place where the sheaf $(E^i_\infty, \overline{\partial}^{A_i}_\infty)$ is locally free. Therefore, we proved:

- The limiting $(E_\infty, \overline{\partial}^{A_\infty})$ can be extended to the whole $M$ as a reflexive sheaf with a holomorphic orthogonal splitting

$$
(E_\infty, \overline{\partial}^{A_\infty}, H_\infty) = \bigoplus_{i=1}^{l} (E^i_\infty, \overline{\partial}^{A_i}_\infty, H^i_\infty), \quad (3.26)
$$

and $H^i_\infty$ is an admissible Hermitian-Einstein metric on the reflexive sheaf $(E^i_\infty, \overline{\partial}^{A_i}_\infty)$ for any $1 \leq i \leq l$. 

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Thank you for your attention!