Iteration Theories for Periodic Orbits, Old and New

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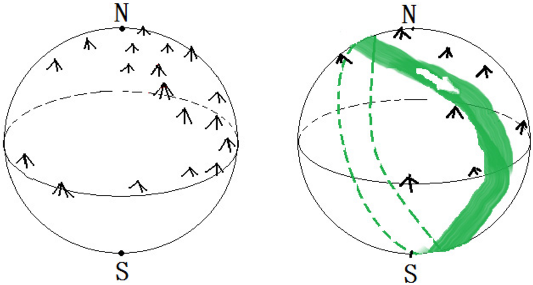


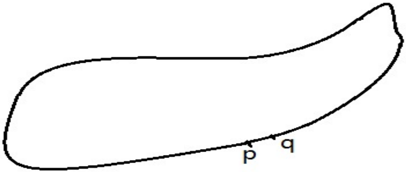
Figure: Riemannian and Finsler metrics on S^2.

A reversible (including Riemannian) or irreversible compact Finsler manifold (M, F).

$c : \mathbf{R} \to (M, F)$ is a closed geodesic, if

(i) it is a closed curve, and

(ii) it is locally the shortest curve connecting any 2 nearby points.



4.18, 2019, "google scholar" found 131000 papers/books related to "closed geodesics" in 0.05 seconds.

Existence of at least one closed geodesic on (M, F):

1898, J.Hadamard, 1905 H.Poincaré.

1917-1927, G.D.Birkhoff: $^{\#}\mathrm{CG}(S^d, g) \geq 1$, $\forall$ Riemannian g on S^d.

1951, L.Lyusternik-A.Fet: $^{\#}\mathrm{CG}(M, g) \geq 1$, $\forall$ Riemannian g on any compact M.

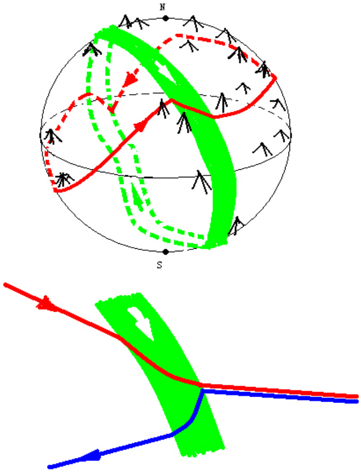
Proved by variational method (specially, minimax method) $\Longrightarrow$ $^{\#}\mathrm{CG}(M, F) \geq 1$, $\forall$ Finsler metric F on a compact manifold M.

cf. V. Bangert, (1985, JDMV);
J. Jost, (2002, "Riem. Geom. and Geom. Anal.")

Variational methods (letting $S^1 = \mathbf{R}/\mathbf{Z}$),

$$E(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))^2 dt, \qquad \text{for } \ \gamma \in \Lambda M = W^{1,2}(S^1, M).$$

Then $E'(c) = 0$ and $E(c) > 0 \Longleftrightarrow c$ is a CG on (M, F).



Closed geodesics—Global questions on their
Existence, Multiplicity, Stability.

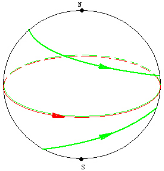
Multiplicity ?   Stability ?

Multiplicity question: Estimate ${}^{\#}\mathrm{CG}(M, F)$ or ${}^{\#}\mathrm{CG}(M, g)$?

A long standing conjecture:

${}^{\#}\mathrm{CG}(M, g) = +\infty$, $\forall$ Rieman. g on compact M with dim $M \geq 2$?

Yes, when dim $M = 2$, by V. Bangert (1993) and J. Franks (1990).

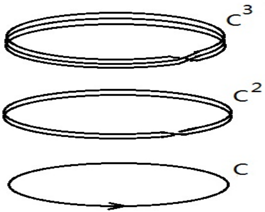


A. Katok metrics (1973):

$${}^{\#}\mathrm{CG}(S^d, F_K) = 2[\tfrac{d+1}{2}] \text{ with } d \geq 2.$$

A. Katok – D. Anosov conjecture (1973-74):

$${}^{\#}\mathrm{CG}(S^d, F) \geq 2[\tfrac{d+1}{2}] \qquad \forall \text{ Finsler } F \text{ on } S^d \text{ with } d \geq 2.$$



The major difficulty for multiplicity!

Let $c : S^1 = \mathbf{R}/\mathbf{Z} \to (M, F)$ be a closed curve.

$$c^m(t) \equiv \psi^m(c)(t) \equiv c(mt), \quad \forall\ t \in \mathbf{R},\ m \in \mathbf{N}.$$

$E'(c) = 0$ and $E'(c^m) = 0$.

$$\kappa_m \equiv E(c^m) = m^2 E(c) \to +\infty, \qquad \text{as} \quad m \to +\infty,$$

But c and c^m give the same geometric orbit on (M, F)!

A brief review on ω-index theory of symplectic matrix paths

Consider the Hamiltonian system:

$$\begin{cases} \dot{x}(t) = JH'(t, x(t)), & \forall t \in \mathbf{R}, \\ x(\tau) = x(0), \end{cases} \qquad \text{(HS)}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $H \in C^2(\mathbf{R}/(\tau\mathbf{Z}) \times \mathbf{R}^{2n}, \mathbf{R})$,

$$x : \mathbf{R}/(\tau\mathbf{Z}) \to \mathbf{R}^{2n}.$$

Consider the linearized Hamiltonian system at x:

$$\begin{cases} \dot{y}(t) = JH''(t, x(t))y(t) & \forall t \in \mathbf{R}, \\ y(\tau) = y(0). \end{cases} \qquad (LHS)$$

Its fundamental solution $\gamma(t) = \gamma_x(t)$ is defined by

$$\begin{cases} \dot{\gamma}(t) = JH''(t, x(t))\gamma(t) & \forall t \geq 0, \\ \gamma(0) = I. \end{cases} \qquad (LHS)$$

Then γ is a path in $\mathrm{Sp}(2n) = \{M \in GL(\mathbf{R}^{2n}) \,|\, M^t JM = J\}$ with $\gamma(0) = I$.

(LHS) has a solution $y \neq 0 \iff 1 \in \sigma(\gamma(\tau))$
$\iff \det(\gamma(\tau) - I) = 0.$

Thus we consider the following degenerate hypersurface in $\mathrm{Sp}(2n)$:

$$\mathrm{Sp}(2n)_1^0 = \{M \in \mathrm{Sp}(2n) \,|\, \det(M - I) = 0\}.$$

For $\omega \in \mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$, we let

$$D_\omega(M) = (-1)^{n-1}\omega^{-n} \det(M - \omega I),$$

and define the degenerate hypersurface for ω by

$$\mathrm{Sp}(2n)_\omega^0 = \{M \in \mathrm{Sp}(2n) \,|\, D_\omega(M) = 0\}.$$

An intuitive model (Long, *Science in China*, 1990)

For each $M \in \mathrm{Sp}(2)$, we have:

$$M = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \leftrightarrow (r, \theta, z) \in \mathbf{R}^3 \backslash \{z\mathrm{-axis}\}.$$

Matrices in $\mathrm{Sp}(2)$ are one-to-one correspondent to points in $\mathbf{R}^3 \setminus \{z - \mathrm{axis}\}$ in cylindrical coordinates.

$$\det(M - I) = 0 \Longleftrightarrow (r^2 + z^2 + 1)\cos\theta = 2r.$$

$$\begin{aligned} \mathrm{Sp}(2)_1^0 &= \{M \in \mathrm{Sp}(2) \mid 1 \in \sigma(M)\} \\ &= \{(r, \theta, z) \in \mathbf{R}^3 \setminus \{z - \mathrm{axis}\} \mid (r^2 + z^2 + 1)\cos\theta = 2r\}. \end{aligned}$$

$\mathrm{Sp}(2)_1^0$ forms a singular surface in $\mathrm{Sp}(2)$ as shown below in the cylindrical coordinates of $\mathbf{R}^3$, together with the path γ.

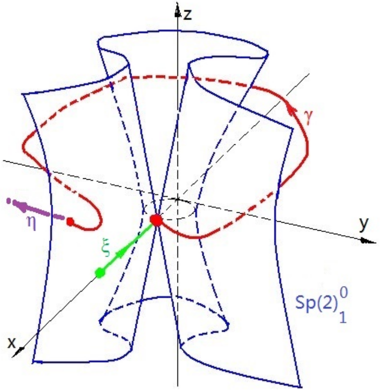


Figure: Graph of γ and $\mathrm{Sp}(2)^0_1$

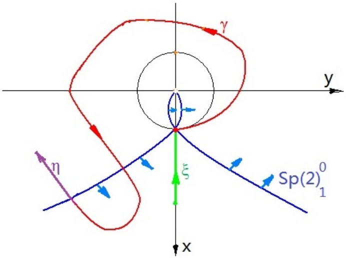


Figure: Illustrations on the graphs of γ and $\mathrm{Sp}(2)_1^0$ when $z = 0$

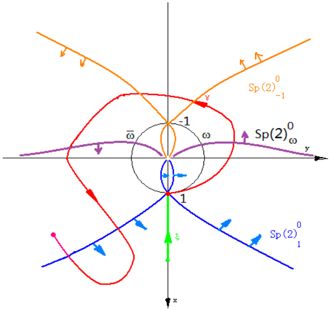


Figure: Illustrations on the graphs of γ and $\mathrm{Sp}(2)^0_\omega$ when $z = 0$

Let ξ be the segment path connecting $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{\diamond n}$ to I_{2n}.
Let $\eta(t) = \gamma(\tau)e^{-t\epsilon J}$ with $t \in [0, \tau]$ and $\epsilon > 0$ small.
We define the orientation of $\mathrm{Sp}(2n)^0_\omega$ as shown in the Figure.

Definition For $\gamma \in C([0, \tau], \mathrm{Sp}(2n))$ with $\gamma(0) = I$, we define

$$\begin{aligned} \nu_1(\gamma) &= \dim \ker(\gamma(\tau) - I), \\ i_1(\gamma) &= [\eta * \gamma * \xi : \mathrm{Sp}(2n)^0_1]. \end{aligned}$$

Similarly, for every $\omega \in \mathbf{U}$ we define

$$\begin{aligned} \nu_\omega(\gamma) &= \dim_{\mathbf{C}} \ker_{\mathbf{C}}(\gamma(\tau) - \omega I), \\ i_\omega(\gamma) &= [\eta * \gamma * \xi : \mathrm{Sp}(2n)^0_\omega]. \end{aligned}$$

Then $\qquad (i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}, \qquad \forall\, \omega \in \mathbf{U}.$

ω-index theory for symplectic paths in $\mathrm{Sp}(2n)$

1960, V. P. Maslov; 1964, V. I. Arnold: 1-index theory for loops of Lagrangian subspaces.

1984, C. Conley-E. Zehnder, non-degenerate paths in $\mathrm{Sp}(2n)$ with $n \geq 2$ and $\omega = 1$,

1990, Y. Long-E. Zehnder, non-degenerate paths in $\mathrm{Sp}(2)$ with $\omega = 1$,

1990, Y. Long, C. Viterbo, degenerate Hamiltonian systems with $\omega = 1$,

1997, Y. Long, degenerate symplectic paths in $\mathrm{Sp}(2n)$ with $\omega = 1$,

1999, Y. Long, ω-index theory for symplectic paths in $\mathrm{Sp}(2n)$ for every $\omega \in \mathbf{U}$.

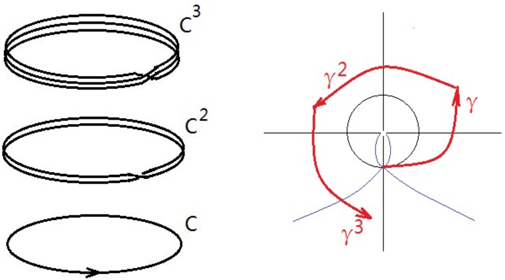


Figure: Illustrations on the iterates of a CG c and a symplectic path γ

For $m \in \mathbf{N}$, the definitions of the m-th iterates

$$
\begin{aligned}
c^m(t) &= c(mt) \qquad \forall\, t \in \mathbf{R}, \\
\gamma^m(t) &= \gamma(t - j\tau)\gamma(\tau)^j \qquad \forall\, j\tau \le t \le (j+1)\tau,\ 0 \le j \le m-1.
\end{aligned}
$$

$$
(i_1(\gamma^m), \nu_1(\gamma^m)) \in \mathbf{Z} \times [0, 2n], \qquad (i(c^m), \nu(c^m)) \in \mathbf{Z} \times [0, 2n].
$$

R. Bott's index iteration theory (1956)

For a given CG c on a Finsler manifold (M, F), let $(i(c^m), \nu(c^m))$ the Morse index and nullity of the energy functional E at c^m with $m \in \mathbf{N}$.

Theorem. R. Bott (1956), For every $m \in \mathbf{N}$,

$$i(c^m) = i_1(c^m) = \sum_{\omega^m=1} \Lambda_\omega(c), \qquad \nu(c^m) = \nu_1(c^m) = \sum_{\omega^m=1} N_\omega(c),$$

where $\Lambda_\omega(c)$ and $N_\omega(c)$ are the ω-Bott-Morse index and nullity of E at c in the space $\{x \in W^{1,2}(S^1, M) \mid x(1) = \omega x(0)\}$.

Application of R. Bott's index iteration formula

Bott's formula yields: $\hat{i}(c) \equiv \lim_{m\to\infty} \frac{i(c^m)}{m} > 0 \Longrightarrow$ there exist $\epsilon > 0$ and $a > 0$ such that

$$i(c^{m+k}) - i(c^m) \geq \epsilon k - a, \qquad \forall\ m, k \in \mathbf{N}.$$

Theorem. D. Gromoll - W. Meyer, (1969)
That $\{b_k(\Lambda M)\}_{k\in\mathbf{N}}$ is unbounded $\Longrightarrow$ $^{\#}\mathrm{CG}(M,F) = +\infty$.

Theorem. M. Vigué-Poirrier and D. Sullivan (1976): (M,g) is a cpt. simply conn. Riem. mfd. Then
$\{b_j(\Lambda M)\}_{j\in\mathbf{N}}$ is bounded $\Longleftrightarrow$ $H^*(M;\mathbf{Q})$ has only one generator.
$\Longleftrightarrow$ $H^*(M;\mathbf{Q}) \cong T_{d,n+1}(x) = \mathbf{Q}[x]/(x^{n+1} = 0)$
with a generator x of degree $d \geq 2$ and hight $n+1 \geq 2$.

Further index iteration theories

I. Ekeland's index theory. (1984) for periodic solutions x of convex Hamiltonian systems on $\mathbf{R}^{2n}$.

$$i_1^E(x^m) = \sum_{\omega^m=1} i_\omega^E(x), \qquad \nu_1^E(c^m) = \sum_{\omega^m=1} \nu_\omega^E(c), \qquad \forall\, m \in \mathbf{N},$$

where $i_\omega^E(x)$ and $\nu_\omega^E(x)$ are the Ekeland index and nullity of the action functional $I(x)$ at x in $\{x \in W^{1,2}(S^1, \mathbf{R}^{2n}) \mid x(1) = \omega x(0)\}$.

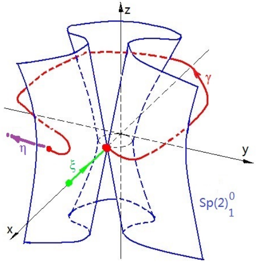
W. Klingenberg (1980s), C. Viterbo (non-degenerate, star-shaped, 1989), V. Brousseau (1986 ?) . . .

Theorem. I. Ekeland - H. Hofer (1987)

$$^{\#}\mathrm{CC}(\Sigma) \geq 2 \qquad \forall \text{ compact convex hypersurface } \Sigma \text{ in } \mathbf{R}^{2n},$$

where $\mathrm{CC}(\Sigma)$ is the set of geometrically distinct periodic solutions of the corresponding Hamiltonian system on Σ.

The Maslov-type index theory for symplectic matrix paths



Let $\mathrm{Sp}(2n) = \{M \in GL(2n) \mid M^T JM = J\}$ and
$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n) \mid \gamma(0) = I\}$ with $\tau > 0$.
The Maslov-type index theory
(C. Conley-E. Zehnder 1984, Y. Long 1990)

$$(i_1(\gamma), \nu_1(\gamma)) \in \mathbf{Z} \times [0, 2n], \qquad \forall\, \gamma \in \mathcal{P}_\tau(2n).$$

Note that $\nu_1(\gamma) = \dim\ker(\gamma(\tau) - I)$.

The Maslov-type index iteration theories for symplectic matrix paths

The ω-index theory (Long 1999)

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times [0, 2n], \qquad \forall\, \gamma \in \mathcal{P}_\tau(2n).$$

Theorem (Bott type iteration formula). Y. Long (1999)

$$i_1(\gamma^m) = \sum_{\omega^m=1} i_\omega(\gamma), \qquad \nu_1(\gamma^m) = \sum_{\omega^m=1} \nu_\omega(\gamma), \qquad \forall\, m \in \mathbf{N},$$

Theorem (Precise iteration formula). Y. Long (2000)

$$i_1(\gamma^m) = A(\gamma)m + (\text{asympt. linear terms on } m) + (\text{bounded terms}),$$

where all terms are explicitly depending on $\gamma(\tau)$ and $(i_1(\gamma), \nu_1(\gamma))$ only. Note that we have

$$\nu_1(\gamma^m) = \dim\ker(\gamma(\tau)^m - I), \qquad \forall\, m \in \mathbf{N}.$$

(The basic problem of the index iteration theories)

Index iteration inequalities (D. Dong - Y. Long 1997,
C. Liu - Y. Long 2000, 2001, Y. Long - C. Zhu 2002,)

Common index jump theorem (Y. Long - C. Zhu 2002))

Resonance identities for Maslov-type indices
(X. Hu - Y. Long - W. Wang 2007,
H. Liu - Y. long - W. Wang 2014)

Enhanced common index jump theorem
(H. Duan - Y. Long - W. Wang 2016)

Applications of the index iteration theory

Theorem. Y. Long - C. Zhu (2002)
${}^{\#}\mathrm{CC}(\Sigma) \geq [\frac{n}{2}] + 1 \quad \forall$ compact convex hypersurface Σ in $\mathbf{R}^{2n}$.

Theorem. V. Bangert - Y. Long (2010)
${}^{\#}\mathrm{CG}(S^2, F) \geq 2 \quad \forall$ Finsler metric F on S^2,
i.e., Katok-Anosov conjecture holds for S^2.

Theorem. H. Duan - Y. Long - W. Wang (2016)
Let (M, F) be a compact simply connected Finsler manifold with dim $M \geq 2$, and
$H^*(M; \mathbf{Q}) \cong T_{d,n+1}(x) = \mathbf{Q}[x]/(x^{n+1} = 0)$ with a generator x of degree $d \geq 2$ and hight $n + 1 \geq 2$
Suppose F is bumpy and $K \geq 0$.
$\Longrightarrow$ ${}^{\#}\mathrm{CG}(M, F) \geq \frac{dn(n+1)}{2}$, if d is even,
${}^{\#}\mathrm{CG}(M, F) \geq d + 1$, if d is odd (then $n = 1$ and $M \simeq S^d$).

Specially, ${}^{\#}\mathrm{CG}(S^d, F) \geq 2[\frac{d+1}{2}] \quad \forall\, d \geq 2$.

New goal, Multiplicity in higher dim. case

Let (M, F) be a compact simply connected Finsler manifold with $\dim M \geq 2$, satisfying

$$H^*(M; \mathbf{Q}) \cong T_{d,n+1}(x) = \mathbf{Q}[x]/(x^{n+1} = 0)$$

with a generator x of degree $d \geq 2$ and hight $n + 1 \geq 2$

To prove $^{\#}\mathrm{CG}(M, F) \geq 2$, suppose that there exists only one prime CG c on (M, F). Try to find a contradiction !

Example: $(\mathbf{S}^d, F)$ with $d \geq 3$;

$\mathbf{CP}^n$ $(d = 2)$ and $\mathbf{HP}^n$ $(d = 4)$ with $n \geq 1$;

$\mathbf{CaP}^2$ $(d = 8$ and $n = 2)$.

Multiplicity in higher dim. cases

(i) For any large integer $m_0 > 0$, there exists a large enough integer $\tau \in 4n(m_0!)\mathbf{N}$ such that for every $k \in [1, m_0] \cap \mathbf{N}$ we have

$$i(c^{\tau+k}) = i(c^{\tau}) + p(c) + i(c^k) = \bar{p} + i(c^k), \quad \nu(c^{\tau+k}) = \nu(c^k).$$

(ii) When ${}^{\#}\mathrm{CG}(M, F) = 1$, the following distribution diagram of $\mathrm{rank}\overline{C}_j(E, c^m)$ holds for any $j \geq 0$ and $m \geq 1$.

$\cdots$														$\cdots$	$\cdots$
$\kappa_{\tau+m_0}$											$\cdots$	$*$	$*$	$\cdots$	$\cdots$
$\cdots$										$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\kappa_{\tau+1}$										$*$	$\cdots$	$*$	$*$	$\cdots$	$\cdots$
κ_{τ}								d_0	$\cdots$	$d_{p(c)}$	$\cdots$	d_{μ}	$d_{\mu+1}$	0	$\cdots$
$\kappa_{\tau-1}$					$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0	0	$\cdots$
$\cdots$					$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdot$	$\cdot$	$\cdots$
κ_{m_0}		$\cdots$	$*$	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	$\cdot$	$\cdot$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdot$	$\cdot$	$\cdots$
κ_1	$*$	$\cdots$	$*$	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0	0	$\cdots$
κ_m	C_0	$\cdots$	C_{dh-3}	C_{dh-2}	$\cdots$	$C_{\sigma-1}$	C_{σ}	$\cdots$	$C_{\sigma+p(c)}$	$\cdots$	$C_{\sigma+\mu}$	$C_{\sigma+\mu+1}$	$C_{\sigma+\mu+2}$	$\cdots$	

where $\dim M = dh$, $\kappa_m = E(c^m)$, $\sigma = i(c^{\tau})$, $\mu = p(c) + dh - 3$, $p(c)$ is an integer determined by the linearized Poincaré map P_c of c.

Further developments

We need

$\cdots$													$\cdots$	$\cdots$
$\kappa_{\tau+m_0}$										$\cdots$	$*$	$*$	$\cdots$	$\cdots$
$\cdots$									$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\kappa_{\tau+1}$									$*$	$\cdots$	$*$	$*$	$\cdots$	$\cdots$
κ_τ							d_0	$\cdots$	$d_{p(c)}$	$\cdots$	d_μ	$d_{\mu+1}$	0	$\cdots$
$\kappa_{\tau-1}$					$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0	0	$\cdots$
$\cdots$					$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdot$	$\cdot$	$\cdots$
κ_{m_0}		$\cdots$	$*$	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	$\cdot$	$\cdot$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdot$	$\cdot$	$\cdots$
κ_1	$*$	$\cdots$	$*$	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0	0	$\cdots$
κ_m	C_0	$\cdots$	C_{dh-3}	C_{dh-2}	$\cdots$	$C_{\sigma-1}$	C_σ	$\cdots$	$C_{\sigma+p(c)}$	$\cdots$	$C_{\sigma+\mu}$	$C_{\sigma+\mu+1}$	$C_{\sigma+\mu+2}$	$\cdots$

Claim: There exist isomorphisms on homological modules (Global homological iteration theory)

$$f^{b,a}_{h*} : H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) \xrightarrow{\cong} H_{h+\bar{p}}(\overline{\Lambda}^{\kappa_{\tau+b}}, \overline{\Lambda}^{\kappa_{\tau+a}}) \quad \text{for} \quad 0 \leq a < b \leq m_0.$$

where $\bar{p} = i(c^\tau) + p(c) \in 2\mathbf{N}$, $\Lambda = W^{1,2}(S^1, M)$,
$\Lambda^\kappa = \{x \in \Lambda \mid E(x) \leq \kappa\}$, $\overline{\Lambda}^\kappa = \Lambda^\kappa / S^1$.

Difficulties:

(i) **The local homological iteration lemma.** *Suppose $k|\tau$. Then*

$$i(c^{\tau+k}) = i(c^{\tau}) + p(c) + i(c^k) = \bar{p} + i(c^k), \quad \nu(c^{\tau+k}) = \nu(c^k).$$

Let $m_k = (\tau+k)/k$. The map $\psi^{m_k} : B_\rho(c^k, \Lambda) \to B_{\rho'}(c^{\tau+k}, \Lambda)$ is well-defined, injective, and induces an isomorphism

$$\psi_*^{m_k} : \overline{C}_*(E, c^k) \to \overline{C}_{*+p}(E, c^{\tau+k}),$$

where $\overline{C}_*(E,c) = H_*\left((\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1\right)$,
$\Lambda(c) = \{x \in \Lambda \,|\, E(x) < E(c)\}$.

(ii) **The natural iteration map ψ^m does not work globally !!**
When $(k+1)|\tau$ also holds, because

$$m_k = \frac{\tau}{k} + 1 \neq \frac{\tau}{k+1} + 1 = \frac{\tau+k+1}{k+1} = m_{k+1},$$

the usual iteration map ψ^{m_k} is not well-defined near c^{k+1} !!

$$\psi^{m_k} : B_\rho(c^{k+1}, \Lambda) \not\to B_{\rho'}(c^{\tau+k+1}, \Lambda).$$

(iii) **All the iteration theories so far are at the index level, and no attempts are known at the global homological level yet !!!**

Further development

The global homological iteration theorem (H. Duan-Y. Long, GHIT). *Let* (M, F) *be a compact simply connected Finsler manifold with* $\dim M \geq 2$ *possessing only one prime closed geodesic* c *and* $n = n(c)$ *be the analytical period of* c. *Let* $m_0 = m_0(c) > 0$ *be a suitably chosen large enough integer.*

Then for any integer pair $a < b$ *satisfying* $0 \leq a < b \leq m_0$ *and integer* $0 \leq h \leq \dim M - 2$, *there exists infinitely many integer* $\tau \in 4n(m_0!)\mathbf{N}$ *such that there exists an isomorphism on homological modules:*

$$f_{h*}^{b,a} : H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) \xrightarrow{\cong} H_{h+\bar{p}}(\overline{\Lambda}^{\kappa_{\tau+b}}, \overline{\Lambda}^{\kappa_{\tau+a}}),$$

where $\bar{p} = i(c^{\tau}) + p(c) \in 2\mathbf{N}$, $\Lambda = W^{1,2}(S^1, M)$, $\Lambda^{\kappa} = \{x \in \Lambda \mid E(x) \leq \kappa\}$, $\overline{\Lambda}^{\kappa} = \Lambda^{\kappa}/S^1$.

Further development

The existence of the isomorphisms on homological modules:

$$f_{h*}^{b,a}: H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) \xrightarrow{\cong} H_{h+\bar{p}}(\overline{\Lambda}^{\kappa_{\tau+b}}, \overline{\Lambda}^{\kappa_{\tau+a}}), \quad \forall\, 0 \le a < b \le m_0,$$

$\Longrightarrow$ $\exists\, \kappa > 0$ such that for $\sigma = i(c^\tau)$ and Betti numbers $b_j(\Lambda M)$,

$$\begin{aligned}
&B(d,h)(i(c^\tau)+p(c)) + (-1)^{i(c^\tau)+\mu}\kappa \\
&\quad = \sum_{j=0}^{\sigma+\mu}(-1)^j b_j(\Lambda M) - \sum_{j=0}^{\sigma+\mu}(-1)^j x_j \quad (vertical, horizontal) \\
&\quad = \sum_{j=0}^{\sigma+\mu}(-1)^j b_j(\Lambda M) - \sum_{j=0}^{\sigma+\mu}(-1)^j b_{j-\sigma-p(c)}(\Lambda M) \quad (by\ GHIT) \\
&\quad = \sum_{j=\mu-p(c)+1}^{\sigma+\mu}(-1)^j b_j(\Lambda M),
\end{aligned}$$

where $B(d,h) = -\frac{dh(h+1)}{2d(h+1)-4}$ for even d, or $\frac{d+1}{2d-2}$ for odd d.

Further development

$$B(d,h)(i(c^{\tau})+p(c))+(-1)^{i(c^{\tau})+\mu}\kappa=\sum_{j=\mu-p(c)+1}^{\sigma+\mu}(-1)^j b_j(\Lambda M).$$

$\Longrightarrow$

(below is one of the four cases, where $d \in 2\mathbf{N}$ and $dh > 4$)

$$\begin{aligned} i(c^{\tau})+p(c) &\le -\frac{1}{B(d,h)}\sum_{2j-1=\mu-p(c)+2}^{i(c^{\tau})+\mu} b_{2j-1}(\Lambda M) \\ &\le i(c^{\tau})+p(c)+dh-d-2 \\ &\quad +\frac{2D}{h(h+1)d}\left(1-\frac{dh(h-1)}{2}+\epsilon_{d,h}(i(c^{\tau})+\mu)\right), \end{aligned}$$

where setting $D=d(h+1)-2$, we have

$$\begin{aligned} \epsilon_{d,h}(k) &= \{\frac{D}{hd}\{\frac{k-(d-1)}{D}\}\}-(\frac{2}{d}+\frac{d-2}{hd})\{\frac{k-(d-1)}{D}\} \\ &\quad -h\{\frac{D}{2}\{\frac{k-(d-1)}{D}\}\}-\{\frac{D}{d}\{\frac{k-(d-1)}{D}\}\}. \end{aligned}$$

Further development

$$\Longleftrightarrow \quad \epsilon_{d,h}(i(c^{\tau}) + \mu) \geq \frac{dh-(d-2)}{dh+(d-2)}.$$

On the other hand, we can prove

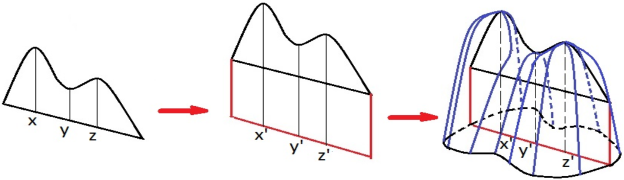
$$\epsilon_{d,h}(i(c^{\tau}) + \mu) < \frac{dh-(d-2)}{dh+(d-2)}.$$

This then produces a contradiction, which implies

Theorem (H. Duan-Y. Long). For every compact simply connected Finsler manifold (M, F) with $\dim M \geq 2$,

$$^{\#}\mathrm{CG}(M, F) \geq 2 \qquad !$$

The idea of the proof of the GHIT



Difficulties and ideas

(i) The construction of the κ_τ-mat.
(ii) Proof of the existence of the canonical representation.
(iii) Well-definedness of the homomorphism

$$f_{h*}^{b,a} : H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) \xrightarrow{\cong} H_{h+\bar{p}}(\overline{\Lambda}^{\kappa_{\tau+b}}, \overline{\Lambda}^{\kappa_{\tau+a}}), \quad \forall\, 0 \le a < b \le m_0.$$

(iv) Proof of $f_{h*}^{b,a}$ to be an isomorphism.
Behavior of the boundary maps. Commutativity in the 5-lemma. etc.

Open conjectures on closed geodesics (Duan-Long-Wang, 2016)

(I) Conjecture for reversible Finsler/Riemannian manifolds:
${}^{\#}\mathrm{CG}(M, F) = +\infty$ for every compact M with a reversible Finsler metric F (including Riemannian metric), when $\dim M \geq 3$.

(II) Conjecture for irreversible Finsler manifolds:
Let M be a compact simply connected manifold whose rational cohomology has only one generator, i.e.,

$$H^*(M; \mathbf{Q}) \cong T_{d,n+1}(x) = \mathbf{Q}[x]/(x^{n+1} = 0)$$

with the generator x of degree $d \geq 2$ and hight $n + 1 \geq 2$.
Then for every Finsler metric F on M, we have

$${}^{\#}\mathrm{CG}(M, F) = \frac{dn(n+1)}{2} \quad \text{or} \quad +\infty, \quad \text{if } d \text{ is even},$$

and

$${}^{\#}\mathrm{CG}(M, F) = d + 1, \quad \text{if } d \text{ is odd}.$$

Open conjectures on closed geodesics

Specially: For every Finsler metric on S^2, it is conjectured

$$^{\#}\mathrm{CG}(S^2, F) = 2 \quad \text{or} \quad +\infty.$$

Now $^{\#}\mathrm{CG}(S^2, F) \geq 2$ is known (Bangert-Long, 2010).

Possible considerations.

Two main ideas in the proof of the Theorem ${}^{\#}\mathrm{CG}(M, F) \geq 2$ for simply conn. cpt. Finsler mfds.

(i) The topological/homological structure.

$\cdots$													$\cdots$	$\cdots$
$\kappa_{\tau+m_0}$										$\cdots$	$*$	$*$	$\cdots$	$\cdots$
$\cdots$									$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\kappa_{\tau+1}$									$*$	$\cdots$	$*$	$*$	$\cdots$	$\cdots$
κ_{τ}							d_0	$\cdots$	$d_{p(c)}$	$\cdots$	d_μ	$d_{\mu+1}$	0	$\cdots$
$\kappa_{\tau-1}$					$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0	0	$\cdots$
$\cdots$					$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	.	.	$\cdots$
κ_{m_0}		$\cdots$	$*$	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	.	.	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	.	.	$\cdots$
κ_1	$*$	$\cdots$	$*$	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0	0	$\cdots$
κ_m	C_0	$\cdots$	C_{dh-3}	C_{dh-2}	$\cdots$	$C_{\sigma-1}$	C_σ	$\cdots$	$C_{\sigma+p(c)}$	$\cdots$	$C_{\sigma+\mu}$	$C_{\sigma+\mu+1}$	$C_{\sigma+\mu+2}$	$\cdots$

(ii) The estimates.

Thank you !