Hua Luogeng and André Weil

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Knowing Hua

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There were some problems with my approach to this dream. First of all I did poorly on the math college exam in 1980, and was admitted consequently to the Chemistry department of Zhongshan University against my wish. Luckily, I was allowed to transfer to the math department after one semester.

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Secondly, after reading part of the book "Goldbach conjecture" by Pan Chendong and Pan Chenbiao, I started to think that the Goldbach conjecture (and analytic number theory) is too hard for me. So I was thinking about an "easier problem"—Fermat's last theorem—and read as much algebra as possible in my college years.

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In 1984, for the only time in my life, I saw Hua on stage in a big meeting.

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In order to read Faltings' paper, I started to study Hartshorne's "algebraic geometry", a standard textbook for graduate students. The book starts with one chapter of classical material, and continues with three chapters of the compressed version of Grothendieck's scheme theory. The most interesting part is an appendix which gives one motivation of scheme theory: the Weil conjecture.

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Moreover I had a chance to become a visiting student at Princeton for one year, during which I took lecture notes and wrote a book with Faltings, and had lunches with André Weil at IAS. But Weil didn't remember much other than mentioning Chern to me.

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Hua Luogeng (1910-1985)

Hua is also a public figure famous for his self-learning during his childhood, and for his dedication to the population of mathematics during the culture revolution.

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André Weil (1906–1998)

Weil also had a personal experience as a prisoner and refuge, and took jobs in every continent. He is the (spiritual) leader of Bourbaki.

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Hua is more a problem solver, a doer;
Weil is more a problem interpreter, a visionary.

Waring problem over integers

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Theorem (Hilbert 1909)

For any positive integer k, there is a positive integer g such that any positive integer n can be written as a sum of g terms of kth powers of integers:

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The first few numbers of $g(k)$: 1, 4 (Lagrange), 9, 19, 37 (Chen Jinrun), 73, 143, 279, 548, 1079, 2132, 4223, 8384, 16673, 33203, 66190, 132055 ....

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This is the same as to solve the equation over the finite field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. More generally, one may consider much more general equations over any $\mathbb{F}_q$,

$$n = a_1 x_1^{k_1} + a_2 x_2^{k_2} \cdots + a_g x_g^{k_g}, \qquad n, a_i \in \mathbb{F}_q$$

and ask for the number N of solutions.

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For Weil, however this is the beginning of his Riemann Hypothesis over function field.

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Here $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ is the dual group of G. Thus for any function f on G, there is a Fourier expansion

$$f = \sum_{\chi} \langle f, \chi \rangle \chi, \qquad \langle f, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \bar{\chi}(g)$$

A reformulation

The above equation is equivalent to the following system:

$$\begin{cases} n = y_1 + \cdots y_g & \text{additional equation,} \\ y_i / a_i = x^{k_i} \quad (i = 1, \cdots g) & \text{multiplicative equations.} \end{cases}$$

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For each k and each $u \in \mathbb{F}_q^\times$, let $N_k(u)$ denote the number of solutions for the equation $u = x^k$,
and let δ_n denote the Dirac function on $\mathbb{F}_q$:

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Then we have the following expression

$$N = \sum_{\substack{y_1, \cdots, y_g \\ y_i \neq 0}} \delta_n(y_1 + y_2 + \cdots y_g) \prod_i N_{k_i}(y_i/a_i).$$

Solving $u = x^k$

Apply the Fourier analysis on the finite groups $\mathbb{F}_q^\times$ to obtain

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It follows that

$$N_k(u) = \sum_{\chi^k = 1} \chi(u)$$

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Put two Fourier expansions together to get

$$N = \sum_{\substack{n = y_1, \cdots, y_g \\ y_i \neq 0}} \frac{1}{q} \sum_{\psi} \psi(-n)\psi(y_1 + \cdots + y_g) \cdot \prod_{i=1}^{g} \sum_{\chi_i^{k_i}=1} \chi_i(y_i/a_i).$$

$$= \frac{1}{q} \sum_{\substack{\chi_1, \cdots, \chi_g, \psi \\ \chi_i^{k_i}=1}} \psi(-n) \prod_i \chi_i(a_i^{-1}) \sum_{y_i} \psi(y_i)\chi_i(y_i).$$

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This completes the proof of the Theorem of Hua-Vendiver and Weil.

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Let X be a smooth projective variety of dimension d over $\mathbb{F}_q$. Let N_m be the number of rational points on X over $\mathbb{F}_{q^m}$. Consider the power series

$$Z(T) = \exp\left(\sum_{m=1}^{\infty} N_m \frac{T^m}{m}\right).$$

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If there is a reasonable cohomology theory, then we can calculate N_m using the Lefschetz fixed point theorem for the Frobenius operator $F : X \longrightarrow X$ by raising q-th power on coordinates.:

$$N_m = \#\{x \in X, \quad F^m x = x\} = \sum_i (-1)^i \mathrm{tr}(F^m : H^i(X)).$$

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Furthermore the number N_m can be explicit bounded.

Weil conjecture

The function $Z(T)$ is a rational function, satisfying the functional equation

$$Z\left(\frac{1}{q^d T}\right) = \pm q^{d\chi/2} T^{\chi} Z(T)$$

where χ is the Euler-Poincare characteristic of X. Furthermore,

$$Z(T) = \frac{P_1(T)P_3(T)\cdots P_{2d-1}(T)}{P_0(T)P_2(T)\cdots P_{2d}(T)}$$

with $P_0(T) = 1 - T$, $P_{2d}(T) = 1 - q^d T$, and

$$P_i(T) = \prod_{j=1}^{b_i}(1 - \alpha_{ij}T), \qquad 1 \leq i \leq 2d-1$$

where α_{ij} are algebraic integers of absolute value $q^{i/2}$.

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The proof of the Weil conjecture is considered as one of the most important achievement in mathematics in the last century.

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The identification of these L-functions is a major topic in arithmetic geometry and Langlands program in the 21st century.