An Analytic Grothendieck Riemann Roch Theorem

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March 26, 2019

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- This is a fundamental and beautiful problem connected to many different branches of Mathematics. We would like to take this opportunity to encourage our audience to join our journey of exploring this problem.
- This talk is based on joint work with R. Douglas, M. Jabbari, and G. Yu.

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Proposition

The kernel of the operator T_z is trivial. The cokernel of T_z, i.e. $\ker(T_z^*)$, is spanned by the constant function 1 in $L^2_a(\mathbb{D})$.

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Example

$\mathrm{ind}(T_z) = -1.$

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where $S : L^2(\mathbb{D}) \to L^2_a(\mathbb{D})$ is the orthogonal projection to the closed subspace $L^2_a(\mathbb{D}) \subset L^2(\mathbb{D})$.

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Theorem

1. *$T_f : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is Fredholm if and only if $f|_{\partial\overline{\mathbb{D}}}$ is invertible ;*
2. *When T_f is Fredholm,* $\operatorname{ind}(T_f)$ *is*

$$-\operatorname{wind}\left(f|_{\partial\overline{\mathbb{D}}} : S^1 \to \mathbb{C}^*\right).$$

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The above extension defines a K-homology class $[\mathcal{T}(\mathbb{D})]$ in $K_1(S^1)$.

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Theorem (Baum-Douglas)

In $K_1(S^1)$, $[\mathcal{T}(\mathbb{D})] = [\frac{1}{\sqrt{-1}}\frac{d}{d\theta}]$.

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Corollary (Atiyah-Singer)

$\langle [\mathcal{T}(\mathbb{D})], [f] \rangle = \langle [\frac{1}{\sqrt{-1}}\frac{d}{d\theta}], [f] \rangle = \mathrm{ind}(T_f) = -\mathrm{wind}(f|_{\partial\overline{\mathbb{D}}})$.

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The commutators

$$[T_{z^i}, T_{z^j}^*],\ i, j = 1, ..., m,$$

are compact operators on $L_a^2(\mathbb{B}^m)$.

Toeplitz index theorem for $\mathbb{B}^m$

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Let $\mathcal{T}(\mathbb{B}^n)$ be the unital C^*-algebra, norm closed $*$-subalgebra of $B(L_a^2(\mathbb{B}^m))$, generated by $T_{z^1}, \cdots, T_{z^m}$ and $\mathcal{K}(L_a^2(\mathbb{B}^m))$.

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Theorem (Venugopalkrishna, Boutet de Monvel, Baum-Douglas-Taylor)

In $K_1(\mathbb{S}^{2m-1})$, $[\mathcal{T}(\mathbb{D})] = [\not{D}]$, where $\not{D}$ is the Spin^c Dirac operator associated to the CR structure on $\mathbb{S}^{2m-1}$.

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Remark

When the above property holds, we say that $\bar{I}$ is essentially normal.

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The quotient module

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Let $\sigma_e(Q_I)$ be the essential spectrum space associated to $(T_{z^1}, \cdots, T_{z^m})$, and $\mathcal{T}(Q_I)$ be the unital C^*-algebra generated by $T_{z^1}, \cdots, T_{z^m}$ and $\mathcal{K}(L^2_a(Q_I))$.

An index problem

We are interested in the following index problem.

Question (R. Douglas)

Suppose that the Arveson-Douglas conjecture holds true for an ideal I. Identify the K-homology class $[\mathcal{T}(Q_I)]$ defined by the following extension

$$0 \longrightarrow \mathcal{K}(L_a^2(Q_I)) \longrightarrow \mathcal{T}(Q_I) \longrightarrow C(\sigma_e(Q_I)) \longrightarrow 0,$$

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Example

In the case that $I = \{0\}$ and $m = 1$, the Toeplitz index theorem for $S^1 = \partial\mathbb{D}$ gives the answer to the above question.

Geometry of the zero set

Let Z_I be the zero set of the ideal I, i.e.

$$Z_I := \{z | f(z) = 0, \forall f \in I\}.$$

Let Ω_I be the intersection $\Omega_I := Z_I \cap \mathbb{B}^m$ with the boundary $\partial\Omega_I \subset \mathbb{S}^{2m-1} = \partial\mathbb{B}^m$.

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Denote by $\mathcal{Y}_I := \partial\Omega_I / S^1$, an algebraic subset of $\mathbb{C}P^{m-1}$.

Grothendieck Riemann Roch Theorem

Theorem (Grothendieck)

Let $i : \mathcal{Y}_I \hookrightarrow \mathbb{C}P^{m-1}$ be the natural embedding. Assume that $\mathcal{Y}_I$ is smooth. The following commutative diagram holds.

$$\begin{array}{ccc} K_0(\mathcal{Y}_I) & \xrightarrow{i_!} & K_0(\mathbb{C}P^{m-1}) \\ \Big\downarrow{\scriptstyle \mathrm{Ch}} & & \Big\downarrow{\scriptstyle \mathrm{Ch}} \\ A(\mathcal{Y}_I) & \xrightarrow{i_*} & A(\mathbb{C}P^{m-1}) \end{array} .$$

In particular, for $\mathcal{E} \in K_0(\mathcal{Y}_I)$,

$$i_*(\mathrm{Ch}(\mathcal{E}) \cup Td(\mathcal{Y}_I)) = \mathrm{Ch}(i_!(\mathcal{E})) \cup Td(\mathbb{C}P^{m-1}).$$

Analytic Grothendieck Riemann Roch Theorem

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Observation : Though the geometric fundamental class is not unique, the extension class $[\mathcal{T}(Q_I)]^{S^1} \in K_0(\mathcal{Y}_I)$ is canonically defined.

Analytic Grothendieck Riemann Roch Theorem

When $\mathcal{Y}_I$ is not smooth, the geometric fundamental class of $\mathcal{Y}_I$ is not unique. Algebraic geometers have developed intersection homology, perverse sheaves, $\cdots$, to study the generalization of Grothendieck Riemann Roch Theorem.

Observation : Though the geometric fundamental class is not unique, the extension class $[\mathcal{T}(Q_I)]^{S^1} \in K_0(\mathcal{Y}_I)$ is canonically defined.

Conjecture (Douglas-T-Yu)

The extension class $[\mathcal{T}(Q_I)]^{S^1}$ is a fundamental class of $\mathcal{Y}_I$.

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We make the following assumptions.

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1. $M \leq m - 2$;
2. The Jacobian matrix $(\partial p_i / \partial z_j)_{i,j}$ is of maximal rank on the boundary $\partial\Omega_I = Z_I \cap \partial\mathbb{B}^m$;
3. Z_I intersects $\partial\mathbb{B}^m$ transversely.

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Under the above assumptions, the ideal $\bar{I}$ and also the quotient Q_I are essentially normal.
The K-homology class $[\mathcal{T}(Q_I)]$ is equal to $[\not{D}]$, where $\not{D}$ is the spin^c-Dirac operator defined by the CR-structure on $\partial \Omega_I$.

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2. There is an independent work by Engliš-Eschmeier, which is called "Geometric Arveson-Douglas conjecture".
3. Ideals satisfying the assumptions in the theorem are radical, i.e.
   $f|_{Z_I} = 0$ if and only if $f \in I$.

An example of a non-radical ideal

When I is not radical, the geometry of the space $\partial\Omega_I$ is not sufficient to catch the full information of the K-homology class $[\mathcal{T}(Q_I)]$. This can be seen in the following example.

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The quotient Q_I can be written as the sum of two space

$$L^2_{a,1}(\mathbb{D}) \oplus L^2_{a,2}(\mathbb{D}),$$

where $\mathbb{D}$ is the unit disk inside the complex plane $\mathbb{C}$, and $L^2_{a,1}(-)$ (and $L^2_{a,2}(-)$) is the weighted Bergman space with respect to the weight defined by the defining function $1 - |z|^2$ (and $(1 - |z|^2)^2$).

A resolution type of result

We generalize the example of $\langle z_1^2 \rangle$ to the following result.

Theorem (Douglas-Jabbari-T-Yu)

Let I be an ideal of $\mathbb{C}[z_1, \cdots, z_m]$ generated by monomials, and $\overline{I}$ be its closure in the Bergman space $L_a^2(\mathbb{B}^m)$. There are Bergman space like Hilbert A-modules $\mathcal{A}_0 = L_a^2(\mathbb{B}^m), \mathcal{A}_1, \cdots, \mathcal{A}_k$ together with bounded A-module morphisms $\Psi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$, $i = 0, \cdots, k-1$ such that the following exact sequence of Hilbert modules holds

$$0 \longrightarrow \overline{I} \hookrightarrow L_a^2(\mathbb{B}^m) \xrightarrow{\Psi_0} \mathcal{A}_1 \xrightarrow{\Psi_1} \cdots \xrightarrow{\Psi_{k-1}} \mathcal{A}_k \longrightarrow 0.$$

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Like the example of $\langle z_1^2 \rangle$, the Hilbert A-module $\mathcal{A}_i$, $i = 1, \cdots, k$, has a similar geometric structure as a direct sum of (weighted) Bergman spaces on lower dimensional balls.

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Let $\mathcal{T}(\mathcal{A}_i)$ be the unital C^*-algebra generated by Toeplitz operators on $\mathcal{A}_i$, and σ_i^e be the associated essential spectrum space, $i = 1, ..., k$. In $K_1(\sigma_1^e \cup \cdots \cup \sigma_k^e)$, the following equation holds,

$$[\mathcal{T}(Q_I)] = [\mathcal{T}(\mathcal{A}_1)] - [\mathcal{T}(\mathcal{A}_2)] + \cdots + (-1)^{k-1}[\mathcal{T}(\mathcal{A}_k)],$$

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Every algebra $\mathcal{T}(\mathcal{A}_i)$, $i = 1, \cdots, k$, can be identified as the algebra of Toeplitz operators on square integrable holomorphic sections of a hermitian vector bundle on a disjoint union of subsets of $\mathbb{B}^m$. In this way, we obtain a geometric identification of the class $[\mathcal{T}(Q_I)]$.

Brieskorn varieties

Take $m = 5$. Consider the following polynomials

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The intersection $Z_{p_k} \cap \mathbb{S}^9$ is a topological 7-sphere, but has a distinct smooth structure with $k = 1, ..., 28$.

New invariant is needed

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Question

Find the right analytic invariant to detect the smooth structure on $Z_{p_k} \cap \mathbb{S}^9$.

A Hilbert bundle (work in progress)

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Remark

We would like to view $(\mathcal{H}^k, \nabla)$ as the analytic analog of the Milnor fibration in his study of hypersurfaces with isolated singularities.

Thank you!