2-Local derivations on von Neumann algebras and AW^* - algebras

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Outline

Outline

Outline

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Von Neumann algebras

Let H be a Hilbert space over the field $\mathbb{C}$ of complex numbers, and let $B(H)$ be the algebra of all bounded linear operators on H. Denote by $\mathbf{1}$ the identity operator on H, and let $P(H) = \{p \in B(H) : p = p^2 = p^*\}$ be the lattice of projections in $B(H)$. Consider a von Neumann algebra M on H, i.e. a *-subalgebra of $B(H)$ closed in the week operator topology and containing the operator $\mathbf{1}$. Denote by $\|\cdot\|_M$ the operator norm on M. The set $P(M) = P(H) \cap M$ is a complete orthomodular lattice with respect to the natural partial order on $M_h = \{x \in M : x = x^*\}$, generated by the cone M_+ of positive operators from M.

Von Neumann algebras

Every von Neumann algebra can be written uniquely as a sum of von Neumann algebras of types I, II_1 (finite), II_∞ (properly infinite, semifinite) and III (purely infinite).

Outline

Derivation

Given an algebra $\mathcal{A}$, a linear operator $D : \mathcal{A} \to \mathcal{A}$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (the Leibniz rule).

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Each element $a \in \mathcal{A}$ implements a derivation D_a on $\mathcal{A}$ defined as $D_a(x) = [a, x] = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be inner derivations.

Local derivation

There exist various types of linear operators which are close to derivations. Recall that a linear map Δ of $\mathcal{A}$ is called a local derivation if for each $x \in \mathcal{A}$, there exists a derivation $D : \mathcal{A} \to \mathcal{A}$, depending on x, such that $\Delta(x) = D(x)$. This notion was introduced in 1990 independently by Kadison [a] and Larson and Sourour [b].

[a] R. V. Kadison, Local derivations, J. Algebra. 130 (1990), 494–509.

[b] D. R. Larson, A. R. Sourour, Local derivations and local automorphisms of $B(X)$, Proc.
Sympos. Pure Math. 51. Providence, Rhode Island, 1990, Part 2, 187–194.

Local derivation

Kadison had proved that every norm continuous local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. The same result was obtained for the algebra of all bounded linear operators acting on a Banach space by Larson and Sourour.

2-local derivation

Recall that a map

$$\Delta : \mathcal{A} \to \mathcal{A}$$

(not linear in general) is called a 2-local derivation if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

2-local derivation

The notions of local and 2-local automorphisms of algebras are defined in the same way.

2-local derivation

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Any derivation (resp. automorphism) is a local and a 2-local derivation (resp. automorphism), but the converse is not true in general.

Example

Consider an algebra upper-triangular 2×2-matrix

$$\mathcal{A} = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

Define operator Δ on $\mathcal{A}$ by

$$\Delta(A) = \begin{cases} 0, & \text{if } a_{11} \neq a_{22}, \\ \begin{pmatrix} 0 & 2a_{12} \\ 0 & 0 \end{pmatrix}, & \text{if } a_{11} = a_{22}. \end{cases}$$

Then Δ is a 2-local derivation, which is not a derivation[a].

[a] J. H. Zhang, H. X. Li, 2-Local derivations on digraph algebras, Acta Math. Sinica, Chinese series 49 (2006), 1401–1406.

Outline

A Kowalski-Słodkowski theorem

The Gleason-Kahane-Żelazko theorem [A.M. Gleason, A characterization of maximal ideals, J. Analyse Math. 19, 171-172 (1967)], [J.P. Kahane, W. Żelazko, A characterization of maximal ideals in commutative Banach algebras, Studia Math. 29, 339-343 (1968)], a fundamental contribution in the theory of Banach algebras, asserts that every unital linear functional F on a complex unital Banach algebra A such that, $F(a)$ belongs to the spectrum, $\sigma(a)$, of a for every $a \in A$, is multiplicative. In modern terminology, this is equivalent to say that every unital linear local homomorphism from a unital complex Banach algebra A into $\mathbb{C}$ is multiplicative.

A Kowalski-Słodkowski theorem

After the Gleason-Kahane-Żelazko theorem was established, Kowalski and Słodkowski
[S.Kowalski, Z. Słodkowski, A characterization of multiplicative linear functionals in Banach algebras, Studia Math. 67, 215-223 (1980)]
showed that at the cost of requiring the local behavior at two points, the condition of linearity can be dropped, that is, suppose A is a complex Banach algebra (not necessarily commutative nor unital), then every (not necessarily linear) mapping $T : A \to \mathbb{C}$ satisfying $T(0) = 0$ and $T(x - y) \in \sigma(x - y)$, for every $x, y \in A$, is multiplicative and linear.

A Kowalski-Słodkowski theorem

According to the above notation, the result established by Kowalski and Słodkowski proves that every (not necessarily linear) 2-local homomorphism T from a (not necessarily commutative nor unital) complex Banach algebra A into the complex field $\mathbb{C}$ is linear and multiplicative. Consequently, every (not necessarily linear) 2-local homomorphism T from A into a commutative C*-algebra is linear and multiplicative.

Outline

2-local derivation on $B(H)$. Separabel case

The notion of 2-local derivations it was introduced in 1997 by P. Šemrl [a] and in this paper he described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. A similar description for the finite-dimensional case appeared later by S. O. Kim and J. S. Kim [b]

[a] Šemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc. 125 (1997) 2677–2680.

[b] S. O. Kim, J. S. Kim, Local automorphisms and derivations on M_n, Proc. Amer. Math. Soc. 132 (2004) 1389–1392.

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The methods of the proofs above mentioned results essentially based the fact that an algebra $B(H)$ is single generated.
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$$A = \begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 \\ 0 & \frac{1}{2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2^n} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

2-local derivation on $B(H)$. General case

In 2012 [a] we suggested a new technique and have generalized the above mentioned results for arbitrary Hilbert spaces. Namely we considered 2-local derivations on the algebra $B(H)$ of all bounded linear operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation.

[a] Sh. A. Ayupov, K. K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, J. Math. Anal. Appl. 395 (2012) 15–18.

2-local derivation on $B(H)$. General case

Our proof essentially use existence a faithful normal semi-finite trace on $B(H)$. Namely, the main ingredient of our paper is the following indentity

$$\mathrm{tr}(\Delta(x)y) = -\mathrm{tr}(x\Delta(y)) \tag{1}$$

for all $x \in B(H)$, and for finite-dimensional operator $y \in B(H)$, where tr is the canonical trace on $B(H)$.

Semi-finite von Neumann algebra

A similar result for 2-local derivations on finite von Neumann algebras was obtained by Sh. A. Ayupov and etal. [a]. Finally, Sh. A. Ayupov and F. N. Arzikulov[b] extended all above results and give a short proof of this result for arbitrary semi-finite von Neumann algebras.

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[a] Sh. A. Ayupov, K. K. Kudaybergenov, B. O. Nurjanov, A. K. Alauatdinov, Local and 2-local derivations on noncommutative Arens algebras, Mathematica Slovaca. 64 (2014) 1–10.

[b] Sh. A. Ayupov, F. N. Arzikulov, 2-local derivations on semi-finite von Neumann algebras, Glasgow Math. Jour. 56 (2014) 9–12.

Outline

Main result

The following theorem is the main result of this section.

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Theorem 3.1

Let M be an arbitrary von Neumann algebra. Then any 2-local derivation $\Delta : M \to M$ is a derivation.

Main result

For a self-adjoint subset $S \subseteq M$ denote by S' is the commutant of S, i.e.

$$S' = \{y \in B(H) : xy = yx, \forall\, x \in S\}.$$

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$$S' = \{y \in B(H) : xy = yx, \forall x \in S\}.$$

Let $g \in M$ be a self-adjoint element and let $\mathcal{W}^*(g) = \{g\}''$ be the abelian von Neumann algebra generated by the element g. Then there exists an element $a \in M$ such that

$$\Delta(x) = ax - xa$$

for all $x \in \mathcal{W}^*(g)$. In particular, Δ is additive on $\mathcal{W}^*(g)$.

Main result

Let $P(M)$ denote the lattice of all projections of the von Neumann algebra M. Recall that a map $m : P(M) \to \mathbb{C}$ is called a signed measure (or charge) if $m(e_1 + e_2) = m(e_1) + m(e_2)$ for arbitrary orthogonal projections e_1, e_2 in M. A signed measure m is said to be bounded if $\sup\{|m(e)| : e \in P(M)\}$ is finite.

Main result

We need the following version of the Gleason's Theorem[a].

[a] L.J. Bunce, J.D. Maitland Wright, The Mackey-Gleason problem, Bull. Amer. Math. Soc., 26 (1992) 288–293.

Main result

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[a] L.J. Bunce, J.D. Maitland Wright, The Mackey-Gleason problem, Bull. Amer. Math. Soc., 26 (1992) 288–293.

Gleason Theorem

Let $\mathcal{A}$ be a von Neumann algebra with no direct summand of Type I_2. Then each complex-valued finitely additive measure on $P(\mathcal{A})$ extends to a bounded linear functional on A.

Main result

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Lemma 3.2

Let M be an infinite von Neumann algebra. The restriction $\Delta|_{M_{sa}}$ of the 2-local derivation Δ onto the set M_{sa} of all self-adjoint of M is additive.

Main result

Lemma 3.3

There exists an element $a \in M$ such that $\Delta(x) = D_a(x) = ax - xa$ for all $x \in M_{sa}$.

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In order to prove this Lemma we consider the extension $\widetilde{\Delta}$ of $\Delta|_{M_{sa}}$ on M defined by:

$$\widetilde{\Delta}(x_1 + ix_2) = \Delta(x_1) + i\Delta(x_2),\ x_1, x_2 \in M_{sa}.$$

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Lemma 3.4

If $\Delta|_{M_{sa}} \equiv 0$, then $\Delta \equiv 0$.

For details of the proof we refer to
Sh.A.Ayupov, K. K.Kudaybergenov, "2-Local derivations on von Neumann algebras"POSITIVITY, 19, 2015, No.3, 445-455.

Outline

AW^*- algebras

The notion of AW^*-algebras was introduced by Kaplansky as an abstract generalization of von Neumann algebras. Namely, AW^*-algebra is a C^*-algebra such that the left annihilator of any subset is a principal left ideal generated by a projection. He showed that much of the "non spatial theory"of von Neumann algebras can be extended to AW^*-algebras.

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Every von Neumann algebra is a AW^*-algebra but the converse is not true as was shown by Dixmier with an abelian example [J. Dixmier, Sur certain espaces consideres par M.H.Stone, Summa Brasil Math. 2 (1951), 151-182].

AW^*- algebras

Kaplansky proved that an AW^*-algebra of type I is a von Neumann algebra if and only if its center is a von Neumann algebra and conjectured that this is true for general AW^*-algebras. But in 1970 Takenouchi and Dyer independently showed this to be false by providing examples of type III AW^*-algebras which are not von Neumann algebras [O. Takenouchi, A non-W*, AW*-factor, Lect. Notes Math., vol. 650 (1978), 135-139],
[J.Dyer, Concerning AW*-algebras, Notices Amer. Math. Soc., 17 (1970), 788].

AW^*- algebras

In the present section we extend our above results concerning 2-local derivations to the case of arbitrary AW^*-algebras. First we consider 2-local derivations on matrix algebras over unital semi-prime Banach algebras. Namely, we prove that if $\mathcal{A}$ is a unital semi-prime Banach algebra with the inner derivation property then any 2-local derivation on the algebra $M_{2^n}(\mathcal{A})$, $n \geq 2$, is a derivation. We apply this result to AW^*-algebras and prove that any 2-local derivation on an arbitrary AW^*-algebra is a derivation.

Outline

2-Local derivations on matrix algebras

If $\Delta : \mathcal{A} \to \mathcal{A}$ is a 2-local derivation, then from the definition it easily follows that Δ is homogenous. At the same time,

$$\Delta(x^2) = \Delta(x)x + x\Delta(x) \tag{2}$$

for each $x \in \mathcal{A}$.

2-Local derivations on matrix algebras

In the paper of M. Bresar
[M. Bresar, Jordan derivations on semi-prime rings. Proc. Amer. Math. Soc., 104 (1988), 1003-1006]
it is proved that any Jordan derivation (i.e. a linear map satisfying the above equation) on a semi-prime algebra is a derivation. Therefore, in the case semi-prime algebras in order to prove that a 2-local derivation $\Delta : \mathcal{A} \to \mathcal{A}$ is a derivation, it is sufficient to prove that $\Delta : \mathcal{A} \to \mathcal{A}$ is additive.

2-Local derivations on matrix algebras

We say that an algebra $\mathcal{A}$ has the inner derivation property if every derivation on $\mathcal{A}$ is inner. Recall that an algebra $\mathcal{A}$ is said to be semi-prime if $a\mathcal{A}a = 0$ implies that $a = 0$.
Let $M_n(\mathcal{A})$ be the algebra of $n \times n$-matrices over $\mathcal{A}$ and assume that $n \geq 2$.

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Let $M_n(\mathcal{A})$ be the algebra of $n \times n$-matrices over $\mathcal{A}$ and assume that $n \geq 2$.

Lemma 4.1

Let $\mathcal{A}$ be a unital Banach algebra with the inner derivation property. Then the algebra $M_n(\mathcal{A})$ also has the inner derivation property.

2-Local derivations on matrix algebras

The following theorem is the main result of this section.

2-Local derivations on matrix algebras

The following theorem is the main result of this section.

Theorem 4.2

Let $\mathcal{A}$ be a unital semi-prime Banach algebra with the inner derivation property and let $M_{2^n}(\mathcal{A})$ be the algebra of $2^n \times 2^n$-matrices over $\mathcal{A}$. Then any 2-local derivation Δ on $M_{2^n}(\mathcal{A})$ is a derivation.

2-Local derivations on matrix algebras

The proof of Theorem 4.2. is rather technical and consists of two steps. For details we refer to our paper [Sh.A.Ayupov, K.K.Kudaybergenov, 2-Local derivations on matrix algebras over semi-prime rings and on AW*-algebras. IOP Publishing. Journal of Physics: Conference Series 697 (2016), 1-11]

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In the first step we shall show additivity of Δ on the the subalgebra of diagonal matrices from $M_{2^n}(\mathcal{A})$.

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In the first step we shall show additivity of Δ on the the subalgebra of diagonal matrices from $M_{2^n}(\mathcal{A})$.

In the second step of our proof we show that if a 2-local derivation Δ on a matrix algebra equals to zero on all diagonal matrices and on the linear span of matrix units, then it is identically zero on the whole algebra.

2-Local derivations on matrix algebras

The condition on the algebra $\mathcal{A}$ to be a Banach algebra was applied in the proof only for the invertibility of elements of the forms $\mathbf{1} + x$, where $x \in \mathcal{A}$, $\|x\| < 1$. In this connection the following question naturally arises.

2-Local derivations on matrix algebras

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Problem 4.3.

Does Theorem 4.2. hold for arbitrary (not necessarily normed) algebra $\mathcal{A}$ with the inner derivation property?

Outline

An application to AW^*-algebras

In this section we apply Theorem 4.2. to the description of 2-local derivations on AW^*-algebras.

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Theorem 5.1

Let $\mathcal{A}$ be an arbitrary AW^*-algebra. Then any 2-local derivation Δ on $\mathcal{A}$ is a derivation.

An application to AW^*-algebras

Proof.Let us first note that any AW^*-algebra is semi-prime. It is also known that AW^*-algebra has the inner derivation property [D. Olesen, Derivations AW^*-algebras are inner, Pacific J. Math., 53, 555-561 (1974)].

An application to AW^*-algebras

Let z be a central projection in $\mathcal{A}$. Since $D(z) = 0$ for an arbitrary derivation D, it is clear that $\Delta(z) = 0$ for any 2-local derivation Δ on $\mathcal{A}$. Take $x \in \mathcal{A}$ and let D be a derivation on $\mathcal{A}$ such that $\Delta(zx) = D(zx), \Delta(x) = D(x)$. Then we have $\Delta(zx) = D(zx) = D(z)x + zD(x) = z\Delta(x)$. This means that every 2-local derivation Δ maps $z\mathcal{A}$ into $z\mathcal{A}$ for each central projection $z \in \mathcal{A}$. So, we may consider the restriction of Δ onto $e\mathcal{A}$. Since an arbitrary AW^*-algebra can be decomposed along a central projection into the direct sum of an abelian AW^*-algebra, and AW^*-algebras of type $\mathrm{I}_n, n \geq 2$, type I_∞,, type II and type III , we may consider these cases separately.

An application to AW^*-algebras

Let $\mathcal{A}$ be an abelian AW^*-algebra. It is well-known that any derivation on a such algebra is identically zero. Therefore any 2-local derivation on an abelian AW^*-algebra is also identically zero.

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If $\mathcal{A}$ is an AW^*-algebra of type I_n, $n \geq 2$, with the center $Z(\mathcal{A})$, then it is isomorphic to the algebra $M_n(Z(\mathcal{A}))$. By the proof of Theorem 4.2. there exists a derivation D on $\mathcal{A} \equiv M_n(Z(\mathcal{A}))$ such that $\Delta \equiv D$. So, Δ is a derivation.

An application to AW^*-algebras

Let the AW^*-algebra $\mathcal{A}$ have one of the types I_∞, II or III. Then the halving Lemmata for type I_∞, type II and type III AW^*-algebras from
[I. Kaplansky, Projections in Banach algebras, Ann. Math. 53, 235-249 (1951)],
imply that the unit of the algebra $\mathcal{A}$ can be represented as a sum of mutually equivalent orthogonal projections e_1, e_2, e_3, e_4 from $\mathcal{A}$. Then the map $x \mapsto \sum\limits_{i,j=1}^{4} e_i x e_j$ defines an isomorphism between the algebra $\mathcal{A}$ and the matrix algebra $M_4(\mathcal{B})$, where $\mathcal{B} = e_{1,1}\mathcal{A}e_{1,1}$. Therefore Theorem 4.2. implies that any 2-local derivation on $\mathcal{A}$ is a derivation. The proof is complete.

Outline

2-Local automorphisms on AW^*- algebras

In 1997, Šemrl [a] also considered so-called 2-local automorphisms on algebras. Namely, he proved that such maps on the algebra $B(H)$ of all bounded linear operators on an infinite dimensional separable Hilbert space H are automatically global automorphisms.

[a] P. Šemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc. 125, 2677-2680 (1997).

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[a] P. Šemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc. 125, 2677-2680 (1997).

Recall that a map $\Delta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a 2-local automorphism if, for every $x, y \in \mathcal{A}$, there exists an automorphism $\Phi_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Phi_{x,y}(x) = \Delta(x)$ and $\Phi_{x,y}(y) = \Delta(y)$.

2-Local automorphisms on AW^*- algebras

In[a] it was established that every 2-local $*$-homomorphism from a von Neumann algebra into a C^*-algebra is a linear $*$-homomorphism.In particular this implies that every 2-local automorphism of a von Neumann algebra is an automorphism. We are going to show that this is true also for AW^*- algebras.

[a] M.J. Burgos, F.J. Fernandez Polo, J.J. Garces, A.M. Peralta, A Kowalski-Slodkowski theorem for 2-local $*$-homomorphisms on von Neumann algebras, Revista Serie A Matematicas 109, Issue 2 (2015), Page 551-568.

2-Local automorphisms on AW^*- algebras

For this sake in the present section we extend the result of the previous section, obtained for 2-local derivations on AW^*-algebras, to the case of 2-local automorphisms on AW^*-algebras.

2-Local automorphisms on AW^*- algebras

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If $\Delta : \mathcal{A} \to \mathcal{A}$ is a 2-local automorphism, then from the definition it easily follows that Δ is homogenous. At the same time,

$$\Delta(x^2) = \Phi_{x,x^2}(x^2) = \Phi_{x,x^2}(x)\Phi_{x,x^2}(x) = \Delta(x)^2$$

for each $x \in \mathcal{A}$. This means that additive (and hence, linear) 2-local automorphism is a Jordan automorphism.

2-Local automorphisms on AW^*- algebras

The following Theorem is the main result of this section.

2-Local automorphisms on AW^*- algebras

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Theorem 6.1

Let M be an arbitrary AW^*-algebra without finite type I direct summands. Then any 2-local automorphism Δ on M is an automorphism.

2-Local automorphisms on AW^*- algebras

The proof of this Theorem is based on representations of AW^*-algebras as matrix algebras over a unital Banach algebra with the following two properties:
(J): for any Jordan automorphism Φ on $\mathcal{A}$ there exists a decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ such that

$$x \in \mathcal{A} \mapsto p_1(\Phi(x)) \in \mathcal{A}_1$$

is a homomorphism and

$$x \in \mathcal{A} \mapsto p_2(\Phi(x)) \in \mathcal{A}_2$$

is an anti-homomorphism, where p_i is a projection from $\mathcal{A}$ onto $\mathcal{A}_i$, $i = 1, 2$, and $p_1 + p_2 = \mathbf{1}$.

2-Local automorphisms on AW^*- algebras

(M): There exist elements $x, y \in \mathcal{A}$ such that $xy = 0$ and $yx \neq 0$.

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Remark 6.2

Note that if an algebra $\mathcal{A}$ contains a subalgebra isomorphic to the matrix algebra $M_2(\mathbb{C})$, then it satisfies the condition (M). Indeed, for matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have $xy = 0$ and $yx \neq 0$.

2-Local automorphisms on AW^*- algebras

The key tool for the proof of Theorem 6.1 is the following.

2-Local automorphisms on AW^*- algebras

The key tool for the proof of Theorem 6.1 is the following.

Theorem 6.3

Let $\mathcal{A}$ be a unital Banach algebra with the properties (J) and (M) and let $M_{2^n}(\mathcal{A})$ be the algebra of all $2^n \times 2^n$-matrices over $\mathcal{A}$, where $n \geq 2$. Then any 2-local automorphism Δ on $M_{2^n}(\mathcal{A})$ is an automorphism.

2-Local automorphisms on AW^*- algebras

Now we apply Theorem 6.3 to the proof of our main result which describes 2-local automorphism on AW^*-algebras.

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Now we apply Theorem 6.3 to the proof of our main result which describes 2-local automorphism on AW^*-algebras.

First note that by Theorem 3.3[a] (see also Theorem 3.2.3[b]) any C^*-algebra, in particular, AW^*-algebra, has the property (J).

[a] E. Stormer, On the Jordan structure of C^*-algebras, Trans. Amer. Math. Soc. 120 (1965), 438-447.

[b] O. Brattelli, D. Robinson, Operator algebras and quantum statistical mechanics, 2nd Edition Springer-Verlag Berlin Heidelberg New York 2002.

2-Local automorphisms on AW^*- algebras

Let M be an arbitrary AW^*-algebra without finite type I direct summands. Then there exist mutually orthogonal central projections z_1, z_2, z_3 in M such that $M = z_1M \oplus z_2M \oplus z_3M$, where z_1M, z_2M, z_3M are algebras of types I_∞, II and III, respectively. Then the halving Lemma [p. 120, Theorem 1] [a] applied to each summand implies that the unit z_i of the algebra z_iM, $(i = 1, 2, 3)$ can be represented as a sum of mutually equivalent orthogonal projections $e_1^{(i)}, e_2^{(i)}, e_3^{(i)}, e_4^{(i)}$ from z_iM. Set $e_k = \sum\limits_{i=1}^{3} e_k^{(i)}$, $k = 1, 2, 3, 4$.

[a] S. Berberian, Bear *-rings, Springer 1972, 2nd edition 2011.

2-Local automorphisms on AW^*- algebras

Then the map $x \mapsto \sum_{i,j=1}^{4} e_i x e_j$ defines an isomorphism between the algebra M and the matrix algebra $M_4(\mathcal{A})$, where $\mathcal{A} = e_{1,1} M e_{1,1}$. Moreover, the algebra $\mathcal{A}$ has the properties (J) and (M) (see the Remark 6.2 after the definition of property (M)). Therefore Theorem 6.3 implies that any 2-local automorphism on M is an automorphism. The proof is complete.

2-Local automorphisms on AW^*- algebras

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For details of this result we refer to our joint paper "2-local automorphisms of AW^*-algebras"with K.K.Kudaybergenov and T.K.Kalandarov (to appear in "Positivity"; in the issue dedicated to the 65th anniversary of Professor Ben de Pagter.)

2-Local automorphisms on AW^*- algebras

Concerning the case of type I finite AW^*-algebras, in particular the abelian case, our technic does not work, so the problem remains open.
Of course, the case of type I_{2^n} AW^*- algebras follows from the above Theorem 6.3.

THANKS FOR YOUR ATTENTION!