Volume comparison with respect to scalar curvature

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Outline

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Classic volume comparison theorem

Classic volume comparison theorem
Suppose (M^n, g) is a closed Riemannian manifold with

$$Ric_g \geq (n-1)g,$$

then

$$V_M(g) \leq |\mathbb{S}^n|.$$

Key of proofs

- Assumptions about Ricci curvature:

$$Ric_g \geq (n-1)g;$$

- Ricci curvature controls Jacobi field.

A natural question

Question: Can we replace the assumptions on Ricci curvature with scalar curvature?

Answer: It depends.

Outline

Volume functional with constraint on scalar curvature

For a given manifold M^n, consider the volume functional

$$V_M(g) = \int_M dv_g,$$

with constraint

$$g \in \mathcal{C} := \{g : R_g = constant\}$$

and

$$g|_{T\partial M} = \gamma.$$

Equation for critical points

Miao and Tam calculated the Euler-Lagrange's equation of the functional:

$$\gamma_{\bar{g}}^{*} f := \nabla_{\bar{g}}^{2} f - \bar{g} \Delta_{\bar{g}} f - f Ric_{\bar{g}} = \kappa \bar{g},$$

where $f \not\equiv 0$ is a smooth function on M and $\kappa \in \mathbb{R}$.

Metrics satisfies such a equation, is called **V-static metrics.**

Examples of V-static metrics

- For $\kappa = 0$, V-static metrics are simply *vacuum static metrics*: typically spatial slices of Schwarzchild, de Sitter/Anti-de Sitter, Narai space-times etc.

- For $f = 1$ constantly, V-static metrics are simply Einstein metrics with scalar curvature $-n\kappa$: typically $\mathbb{S}^n$, $\mathbb{H}^n$ and Ricci flat manifolds.

A negative answer to volume comparison

Theorem (Corvino-Eichmair-Miao, 2013)

Let $(M, \bar{g})$ be a Riemannian manifold and $\Omega \subset M$ be a pre-compact domain with smooth boundary. Suppose $(\Omega, \bar{g})$ is not V-static, i.e the V-static equation only admits trivial solutions in $C^\infty(\Omega) \times \mathbb{R}$. Then there exists an $\delta_0 > 0$ such that for any $(\rho, V) \in C^\infty(M) \times \mathbb{R}$ with $supp(R_{\bar{g}} - \rho) \subset \Omega$ and

$$||R_{\bar{g}} - \rho||_{C^1(\Omega, \bar{g})} + |Vol_\Omega(\bar{g}) - V| < \delta_0,$$

there exists a metric g on M such that $supp(g - \bar{g}) \subset \Omega$, $R_g = \rho$ and $Vol_\Omega(g) = V$.

A natural question

Question: What about V-static metrics?

Answer: Yes for geodesic balls; it depends for closed manifolds.

Outline

A constant associated to a compact manifold

Theorem A (Y., 2016)

For $n \geq 3$, suppose $(M^n, \bar{g}, f, \kappa)$ is a V-static space. For any $p \in M$ with $f(p) > 0$, there exist constants $r_0 > 0$ and $\varepsilon_0 > 0$ such that for any geodesic ball $B_r(p) \subset M$ with radius $0 < r < r_0$ and metric g on $B_r(p)$ satisfies

- $R_g \geq R_{\bar{g}}$ *in* $B_r(p)$
- $H_g \geq H_{\bar{g}}$ *on* $\partial B_r(p)$
- g *and* $\bar{g}$ *induce the same metric on* $\partial B_r(p)$
- $||g - \bar{g}||_{C^2(B_r(p),\bar{g})} < \varepsilon_0$,

the following volume comparison hold:

- *if* $\kappa < 0$, *then* $Vol_\Omega(g) \leq Vol_\Omega(\bar{g})$;
- *if* $\kappa > 0$, *then* $Vol_\Omega(g) \geq Vol_\Omega(\bar{g})$;

with equality holds in either case if and only if the metric g is isometric to $\bar{g}$.

Sketch of the proof

- Considering the functional
  $\mathcal{F}[g] = \int_\Omega R(g) f dvol_{\bar{g}} + 2\int_\Sigma H(g) f d\sigma_{\bar{g}} - 2\kappa V_M(g)$;
- Fixing the gauge: $\exists \varphi : \Omega \to \Omega$ with $\varphi|_{\partial\Omega} = id$ and

$$\delta_{\bar{g}} h = \delta_{\bar{g}} \left(\varphi^*(g) - \bar{g}\right) = 0;$$

- Considering the expansion

$$\mathcal{F}[g]-\mathcal{F}[\bar{g}]-\mathcal{F}'[\bar{g}](h)-\frac{1}{2}\mathcal{F}''[\bar{g}](h,h) = \int_\Omega \left(R_g - R_{\bar{g}}\right) f d\mathrm{vol}_{\bar{g}}+I_\Omega+B_\Omega;$$

Sketch of the proof (continued)

- Assume $\kappa(Vol_{B_r(p)}(\varphi^* g) - Vol_{B_r(p)}(\bar{g})) \leq 0$;
- An Eigenvalue estimate:

$$I_\Omega \geq \frac{1}{8}\left(\inf_{B_r(p)} f\right) ||h||^2_{W^{1,2}(B_r(p))};$$

- Using mean curvature comparison:

$$B_\Omega \geq -C||h||_{C^1(B_r(p))}||h||^2_{W^{1,2}(B_r(p))};$$

- Obtain rigidity:

$$||h||^2_{W^{1,2}(\Omega)} \leq C||h||_{C^2(\Omega)}||h||^2_{W^{1,2}(\Omega)}$$

implies that $h \equiv 0$, since $||h||_{C^2(\Omega)}$ small. *i.e.* $\varphi^*(g) = \bar{g}$.

Remarks

- By replacing (f, κ) with $(-f, -\kappa)$, we only need to consider the case $f(p) > 0$;
- If $\kappa = 0$, V-static metrics reduce to *vacuum static metrics*. Under same assumptions on g, Qing-Y. showed rigidity holds and thus volume keeps the same.

Outline

Schoen's Conjecture A & B

Schoen's Conjecture A

For $n \geq 3$, let $(M^n, \bar{g})$ be a closed hyperbolic manifold. Then the Yamabe invariant

$$Y(M) := \sup_{[g]} Y(M, [g])$$

is achieved at the hyperbolic metric $\bar{g}$.

Schoen's Conjecture B

For $n \geq 3$, let $(M^n, \bar{g})$ be a closed hyperbolic manifold. Then for any metric g on M with

$$R_g \geq R_{\bar{g}},$$

we have the volume comparison

$$Vol_M(g) \geq Vol_M(\bar{g}).$$

Schoen's Conjecture A & B

Schoen's Conjecture A & B are equivalent!

A partial answer to Schoen's conjecture A

Corollary B (Besson-Courtois-Gallot, 1991; Y., 2016)

For $n \geq 3$, let $(M^n, \bar{g})$ be a closed hyperbolic manifold. There exists a constant $\varepsilon_0 > 0$ such that for any metric g on M with

$$R_g \geq R_{\bar{g}}$$

and

$$||g - \bar{g}||_{C^2(M,\bar{g})} < \varepsilon_0,$$

we have

$$Vol_M(g) \geq Vol_M(\bar{g})$$

with equality holds if and only if the metric g is isometric to $\bar{g}$.

Known results for Schoen's conjecture A

- $n = 3$: true. (Hamilton, Perelman, Agol-Storm-Thurston, 2007)
- $Ric_g \geq Ric_{\bar{g}}$: true. (Besson-Courtois-Gallot, 1995)

A Bray's conjecture

Bray's Conjecture

For $n \geq 3$, there is a positive constant $\varepsilon_n < 1$ such that for any closed manifold $(M^n, \bar{g})$ with scalar curvature

$$R_g \geq n(n-1)$$

and Ricci curvature

$$Ric_g \geq \varepsilon_n (n-1) g,$$

the volume comparison

$$Vol_M(g) \leq Vol_{\mathbb{S}^n}(g_{\mathbb{S}^n})$$

holds.

Known results for Bray's conjecture

- $n = 3$: true. (Bray, 1997)
- No other (even partial) results!

A partial answer to Bray's conjecture A

Corollary A (Y., 2016)

For $n \geq 3$, let $(\mathbb{S}^n, g_{\mathbb{S}^n})$ be the unit round sphere. There exists a constant $\varepsilon_0 > 0$ such that for any metric g on $\mathbb{S}^n$ with

$$R_g \geq n(n-1)$$

and

$$||g - \bar{g}||_{C^2(\mathbb{S}^n, g_{\mathbb{S}^n})} < \varepsilon_0,$$

we have

$$Vol_{\mathbb{S}^n}(g) \leq Vol_{\mathbb{S}^n}(g_{\mathbb{S}^n})$$

with equality holds if and only if the metric g is isometric to $g_{\mathbb{S}^n}$.

Volume comparison for Einstein closed manifolds

Theorem B (Y., 2016)

There is a constant $\alpha(n, \lambda)$ *such that for any closed Einstein manifold* $(M^n, \bar{g})$ *with*

$$Ric_{\bar{g}} = (n-1)\lambda \bar{g}$$

and

$$||W||_{L^\infty(M,\bar{g})} < \alpha(n, \lambda),$$

we can find a constant $\varepsilon_0 > 0$ *such that for any metric* g *on* M *satisfies*

$$R_g \geq n(n-1)\lambda$$

and

$$||g - \bar{g}||_{C^2(M,\bar{g})} < \varepsilon_0,$$

Volume comparison for Einstein closed manifolds (continued)

Theorem B (continued)

the following volume comparison hold:

- *if* $\lambda > 0$, *then*
$$Vol_M(g) \leq Vol_M(\bar{g});$$
- *if* $\lambda < 0$, *then*
$$Vol_M(g) \geq Vol_M(\bar{g});$$

with equality holds in either case if and only if the metric g is isometric to $\bar{g}$.

Key of the proof

- The functional

$$\mathcal{F}[g] = \int_\Omega R(g) f dvol_{\bar{g}} + 2(n-1)\lambda V_M(g)$$

works for the case $\lambda < 0$, but not for $\lambda > 0$;

- For $\lambda > 0$, need to consider the scaling-invariant functional

$$\mathcal{G}[g] = (V_M(g))^{\frac{2}{n}} \int_\Omega R(g) f dvol_{\bar{g}};$$

- For $\lambda = 0$, *i.e.* Ricci-flat metric, there is no volume comparison simply from rescaling of the metric;
- Morse lemma for Banach manifold.

Outline

Q-curvature in Riemannian Geometry - 2-dimensional motivation

Gauss-Bonnet Theorem :

Let (M^2, g) be a closed surface, then

$$\int_M K_g \, dv_g = 2\pi\chi(M).$$

Uniformazation Theorem:

Any metric on M^2 is locally conformally flat.

Q-curvature in Riemannian Geometry - 4-dimensional analogue

Let (M^4, g) be a closed 4-dimensional Riemannian manifold, define the Q-curvature:

$$Q_g = -\frac{1}{6}\Delta_g R_g - \frac{1}{2}|Ric_g|_g^2 + \frac{1}{6}R_g^2.$$

It satisfies the **Gauss-Bonnet-Chern Formula**

$$\int_M \left(Q_g + \frac{1}{4}|W_g|_g^2\right) dv_g = 8\pi^2\chi(M).$$

In particular, if $W_g = 0$, *i.e.* (M, g) is locally conformally flat,

$$\int_M Q_g dv_g = 8\pi^2\chi(M).$$

Thus, Q_g is a generalization of Gaussian curvature.

Q-curvature in Riemannian Geometry - n-dimensional extensions

In 1985, for $n \geq 3$, Branson extended the definition to arbitrary n-dimensional Riemannian manifold (M^n, g):

$$Q_g = A_n \Delta_g R_g + B_n |Ric_g|_g^2 + C_n R_g^2,$$

where $A_n = -\frac{1}{2(n-1)}$, $B_n = -\frac{2}{(n-2)^2}$ and $C_n = \frac{n^2(n-4)+16(n-1)}{8(n-1)^2(n-2)^2}$.

Q-curvature in Riemannian Geometry - Principles for extensions

Define the Paneitz operator

$$P_g := \Delta_g^2 - div_g \left[(a_n R_g g + b_n Ric_g) d \right] + \frac{n-4}{2} Q_g,$$

where $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$ and $b_n = -\frac{4}{n-2}$.

Then

$$Q_{\hat{g}} = e^{-4u} \left(P_g u + Q_g \right), \qquad \text{for } n = 4 \text{ and } \hat{g} = e^{2u} g$$

$$Q_{\hat{g}} = \frac{2}{n-4} u^{-\frac{n+4}{n-4}} P_g u, \qquad \text{for } n \neq 4 \text{ and } \hat{g} = u^{\frac{4}{n-4}} g$$

Volume comparison of Q-curvture

Theorem C (Huang-Lin-Y., 2018)

There is a constant β_n such that for any closed Einstein manifold $(M^n, \bar{g})$ with $Ric_{\bar{g}} = (n-1)\bar{g}$ and $||W||_{L^\infty(M,\bar{g})} < \beta_n$, we can find a constant $\varepsilon_1 > 0$ such that for any metric g on M satisfies

$$Q_g \geq Q_{\bar{g}}$$

and

$$||g - \bar{g}||_{C^4(M,\bar{g})} < \varepsilon_0,$$

then the following volume comparison hold

$$Vol_M(g) \leq Vol_M(\bar{g})$$

with equality holds in either case if and only if the metric g is isometric to $\bar{g}$.

Volume comparison of Q-curvture

Corollary C (Huang-Lin-Y., 2018)

For $n \geq 3$, let $(\mathbb{S}^n, g_{\mathbb{S}^n})$ be the unit round sphere. There exists a constant $\varepsilon_1 > 0$ such that for any metric g on $\mathbb{S}^n$ with

$$Q_g \geq \frac{n(n-2)(n+2)}{8}$$

and

$$||g - \bar{g}||_{C^4(\mathbb{S}^n, g_{\mathbb{S}^n})} < \varepsilon_1,$$

we have

$$Vol_{\mathbb{S}^n}(g) \leq Vol_{\mathbb{S}^n}(g_{\mathbb{S}^n})$$

with equality holds if and only if the metric g is isometric to $g_{\mathbb{S}^n}$.

References

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