

# **Fuzzy sphere and the Ginsparg-Wilson index**

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## Plan of Talk

- Dirac operators in Lattice Gauge Theory
- The Ginsparg-Wilson relation
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- The  $K$ -theory of  $C_{GW}^*$
- The Ginsparg-Wilson Index
- Fuzzy sphere and a fuzzy Dirac operator
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# Dirac operators in Lattice Gauge Theory

A naive Dirac operator in lattice gauge theory is given by

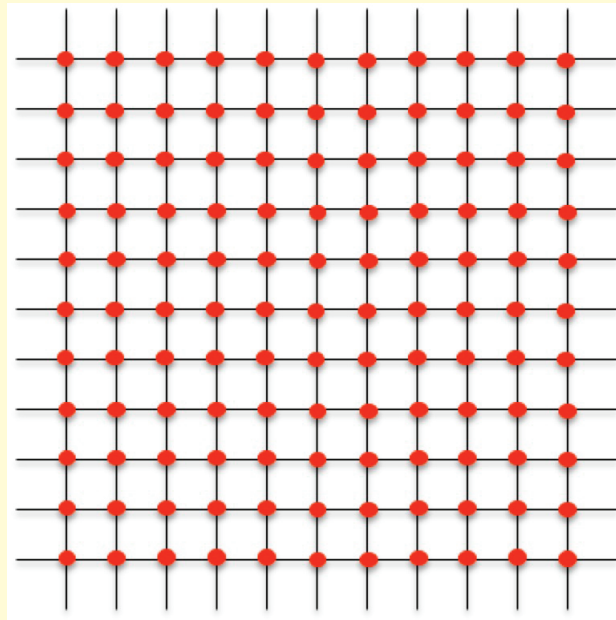
$$D = \gamma_\mu (\nabla_\mu^* + \nabla_\mu) / 2,$$

where  $\nabla_\mu$  and  $\nabla_\mu^*$  denote backward and forward finite differences:

$$(\nabla_\mu \psi)(n) = [U_{n,\mu} \psi(n + \hat{\mu}) - \psi(n)] / a$$

$$(\nabla_\mu^* \psi)(n) = [\psi(n) - U_{n,\mu} \psi(n - \hat{\mu})] / a$$

with  $a$  the lattice spacing (Here  $D$  is skew-adjoint).



But it encountered a problem of **fermion doubling**, extra freedom of fermions that appears in lattice gauge theory. In order to overcome this point Wilson proposed the **Wilson fermion operator** :

$$D_W = D + W,$$

where  $W = -\frac{a}{2}(\nabla_\mu^* \nabla_\mu) \geq 0$ . But  $D_W$  has another difficulty, namely it breaks the chirality;  $\Gamma D_W + D_W \Gamma \neq 0$ . Then the third Dirac operator, the **overlap operator**, was found by Neuberger, which is given by

$$D = \frac{1}{a} (1 + U)$$

with  $U = \frac{aD_W - 1}{|aD_W - 1|}$ . It still breaks the chirality but satisfies the Ginsparg-Wilson relation, which is a perturbation of the chirality.

## The Ginsparg-Wilson relation

$\Gamma$ : an involution, namely  $\Gamma^* = \Gamma$ ,  $\Gamma^2 = 1$

An operator  $D$  satisfies the **Ginsparg-Wilson relation** if

$$D\Gamma + \Gamma D = aD\Gamma D$$

with  $a \in \mathbb{R}$ . At  $a = 0$ , it means that  $D$  anti-commutes with  $\Gamma$ .

**Proposition 1** *Let  $U$  be a unitary operator with  $\Gamma U \Gamma = U^*$ . Set  $D = \frac{1}{a}(1 \pm U)$ . Then  $D$  satisfies the Ginsparg-Wilson relation.*

In fact,

$$D\Gamma + \Gamma D = (1 \pm U)\Gamma/a + \Gamma(1 \pm U)/a = (2\Gamma \pm U\Gamma \pm \Gamma U)/a$$

is equal to

$$aD\Gamma D = (1 \pm U)\Gamma(1 \pm U)/a = (\Gamma \pm U\Gamma \pm \Gamma U + U\Gamma U)/a$$

since  $U\Gamma U = \Gamma$ .

**Remark.**  $\text{spec} \left( \frac{1}{a}(1 \pm U) \right)$  tends to the imaginary axis  $i\mathbb{R}$  as  $a \rightarrow 0$ .

## A universal Ginsparg-Wilson algebra

Given an involution  $\Gamma$  and a unitary  $U$  with  $\Gamma U \Gamma = U^*$ .

Set  $\Gamma' = \Gamma U$ , which gives another involution:

$$(\Gamma')^* = U^* \Gamma = \Gamma U \Gamma U = 1, \quad (\Gamma')^2 = \Gamma U \Gamma U = U^* U = 1$$

Conversely, given two involutions  $\Gamma$  and  $\Gamma'$ , we set  $U = \Gamma \Gamma'$ . Then  $U$  is a unitary with  $\Gamma U \Gamma = U^*$ . This argument implies:

**Proposition 2** *Let  $S^1$  be the unit circle in  $\mathbb{C}$  and  $\epsilon \in \mathbb{Z}_2$  an involution that acts on  $S^1$  by the complex conjugation:  $\epsilon(z) = \bar{z}$  ( $z \in S^1$ ). The resulting crossed product  $C(S^1) \rtimes \mathbb{Z}_2$  is isomorphic to the universal  $C^*$ -algebra generated by two involutions.*

**Definition 1** *The crossed product  $C(S^1) \rtimes \mathbb{Z}_2$  called a **universal Ginsparg-Wilson algebra** and denoted by  $C_{GW}^*$ .*

## The $K$ -theory of $C_{GW}^*$

Let  $x \in \mathbb{R}$  be the **Cayley coordinate** for the unit circle  $S^1 \subset \mathbb{C}$ :

$$e^{i\theta} = \frac{x - i}{x + i} \in S^1.$$

Thus  $\mathbb{R}$  is identified with  $S^1 \setminus \{1\}$  and there is a short exact sequence:

$$0 \longrightarrow C_o(\mathbb{R}) \rtimes \mathbb{Z}_2 \longrightarrow C_{GW}^* \xrightarrow{\pi} C^*\mathbb{Z}_2 \longrightarrow 0$$

where  $\pi$  is the evaluation at  $z = 1 \in S^1$ . Then one has the 6-term exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

Thus the  $K$ -group of  $C_{GW}^*$  are:

$$K_0(C_{GW}^*) \cong \mathbb{Z}^3, \quad K_1(C_{GW}^*) \cong 0.$$



**Proposition 3** Define projections in  $C_{GW}^*$  such as

$$p = (1 + \epsilon)/2, \quad p' = (1 + \epsilon e^{i\theta})/2.$$

$K_0(C_{GW}^*)$  admits a basis given by  $[p] - [p']$ ,  $[p] - [1 - p']$ ,  $[1]$ , where

$$\begin{aligned} [p] - [p'] &\in K_0(C_o(\mathbb{R}) \rtimes \mathbb{Z}_2) \cong \mathbb{Z}, \\ \pi_*([p] - [1 - p']), \pi_*[1] &\in K_0(C^*\mathbb{Z}_2) \cong \mathbb{Z}^2 \end{aligned}$$

generate those groups. Let  $e_x = (x + \epsilon)^{-1}p(x + \epsilon)$  be the **graph projection**. Then the (index) class

$$[e_x] - [1 - p] \in K_0(C_o(\mathbb{R}) \rtimes \mathbb{Z}_2)$$

coincides with  $[p] - [p']$ .

### Definition 2 Set

$$\text{Ind}(U, -1) := [p] - [p'], \quad \text{Ind}(U, +1) := [p] - [1 - p'].$$

We call  $\text{Ind}(U, -1)$  a (universal) **index class** and  $\text{Ind}(U, +1)$  a (universal) **doubler class**. *A life is not simple but the K-theory is.*

# The Ginsparg-Wilson Index theorem

Given an involution  $\Gamma$  and a unitary  $U$  with  $\Gamma U \Gamma = U^*$  on a Hilbert space  $\mathcal{H}$ , there exists a  $*$ -homomorphism

$$\rho : C_{GW}^* \rightarrow \mathcal{L}(\mathcal{H})$$

to the bounded operators on  $\mathcal{H}$  in such a way that  $\rho(e^{i\theta}) = U$ ,  $\rho(\epsilon) = \Gamma$ . Associated to  $\Gamma$  and  $U$ , the **Ginsparg-Wilson index** can be defined:  $\rho_*(\text{Ind}(U, -1))$  is called the **index class** and  $\rho_*(\text{Ind}(U, +1))$  the **doubler class**, denoted often suppressing  $\rho_*$ .

Recall  $\Gamma' = \Gamma U$  is an involution. Define selfadjoint operators such as

$$H_+ = (\Gamma + \Gamma')/2, \quad H_- = (\Gamma - \Gamma')/2.$$

Let  $V(U, \pm 1)$  be the eigenspace of  $U$  of eigenvalue  $\pm 1$ , respectively. One has

$$V(U, \pm 1) = \{\xi \mid \Gamma \xi = \pm \Gamma' \xi\} = \ker H_{\mp}$$

since  $1 \pm U = \Gamma(\Gamma \pm \Gamma')$ .

**Theorem 4 (The Ginsparg-Wilson index theorem)** *Suppose that the spectrum  $\pm 1 \in \text{spec}(U)$  are isolated and the corresponding projections are finite. Then one has  $\rho_* \text{Ind}(U, \pm 1) \in K_0(\mathcal{K})$  with  $\mathcal{K}$  the ideal of compact operators and*

$$\text{Tr}(\rho_* \text{Ind}(U, \pm 1)) = \text{Tr}(\Gamma|V(U, \pm 1)) = \text{Tr}(\Gamma| \ker H_{\mp})$$

*Moreover, if  $H_{\pm}$  is traceable, then*

$$\text{Tr}(\rho_* \text{Ind}(U, \pm 1)) = \text{Tr}(H_{\pm}/2).$$

The second identity is easily follows from

$$\rho_*(p - p') = (1 + \Gamma)/2 - (1 + \Gamma')/2 = H_-$$

$$\rho_*(p - (1 - p')) = (1 + \Gamma)/2 - (1 - \Gamma')/2 = H_+$$

Theorem justifies the notation  $\text{Ind}(U, \pm 1)$  since they are associated to the spectral projection of  $U$  corresponding to  $\pm 1$ . Also note that  $H_{\pm}/2$  are not projections in general, but the trace turned out to be integers.

## Examples of the Ginsparg-Wilson index

**Example 1** Let  $D$  be a Dirac operator on a closed manifold and  $\Gamma$  the grading operator such as  $\Gamma D + D\Gamma = 0$ . The Cayley transform  $U = (D - i)(D + i)^{-1}$  satisfies

$$\Gamma U \Gamma = \Gamma \frac{D - i}{D + i} \Gamma = \frac{D + i}{D - i} = U^*.$$

Then one has

$$U\xi = -\xi \iff (D - i)\xi = -(D + i)\xi \iff \xi \in \ker D$$

and thus

$$\mathrm{Tr}(\mathrm{Ind}(U, -1)) = \mathrm{Tr}(\Gamma|V(U, -1)) = \mathrm{Tr}(\Gamma|\ker D) = \mathrm{Ind}(D^+).$$

On the other hand, one has  $V(U, +1) = 0$  since  $(D - i)\xi = (D + i)\xi$  if and only if  $\xi = 0$ . Thus  $\mathrm{Tr}(\Gamma|V(U, +1)) = 0$  although the class  $\mathrm{Ind}(U, +1)$  does not belong to  $K_0(\mathcal{K})$ . Replacing  $D$  by  $f(D)$  with  $f$  a tempered function of  $x/|x|$  on  $\mathbb{R}$ , one has  $0 = \mathrm{Ind}(U, +1) \in K_0(\mathcal{K})$ .

**Example 2 (Finite dimensional case)** Set

$$\Gamma = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}, \quad \Gamma' = \begin{pmatrix} 1_k & 0 \\ 0 & 1_l \end{pmatrix}$$

with  $n + m = k + l$ . Then one has

$$P = (1 + \Gamma)/2 = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}, \quad P' = (1 + \Gamma')/2 = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}, \quad 1 - P' = \begin{pmatrix} 0 & 0 \\ 0 & 1_l \end{pmatrix}$$

and thus

$$\text{Tr}(\text{Ind}(U, -1)) = \text{Tr}(P - P') = n - k$$

$$\text{Tr}(\text{Ind}(U, +1)) = \text{Tr}(P - (1 - P')) = n - l.$$

Note that the Fredholm index of  $F : V \rightarrow W$  does not depend on the choice of  $F$  if  $V$  is *finite dimensional*. However, we can get a nontrivial index class  $\text{Ind}(U, -1)$  with the doubler class  $\text{Ind}(U, +1)$  trivial (take  $n=l$ ) even if  $\dim V < \infty$ .

**Example 3** *As in Example 1, let  $D$  be a selfadjoint elliptic operator on a closed manifold and take  $\Gamma$  a grading operator such as  $\Gamma D + D\Gamma = 0$ . Also take a selfadjoint operator  $A$  such that  $DA = AD$  and  $\Gamma A = A\Gamma$ . Set*

$$U = \begin{cases} \frac{D - iA}{D + iA} & \text{on } (\ker D \cap \ker A)^\perp \\ 1 & \text{on } \ker D \cap \ker A \end{cases}$$

*One has*

$$\Gamma \frac{D - iA}{D + iA} \Gamma = \frac{D + iA}{D - iA}, \quad \Gamma 1 \Gamma = 1$$

*and thus  $\Gamma U \Gamma = U^*$ . It then follows that*

$$\text{Tr}(\text{Ind}(U, -1)) = \text{Tr}(\Gamma | \ker D) = \text{Ind}(D^+)$$

$$\text{Tr}(\text{Ind}(U, +1)) = \text{Tr}(\Gamma | \ker D \cap \ker A) = \text{Tr}(\Gamma | \ker(D + iA))$$

*Since  $D + iA$  does not anti-commute with  $\Gamma$ , the second index is not an ordinary index for an odd operator. In fact, we can obtain a nontrivial doubler class for suitable  $D$  and  $A$ . *The evil twin is not bad at all.* This is generalized to the case of Dolbeault index theorem on  $SU(2)$ .*

## A Fuzzy sphere

$S^2 \subset \mathbb{R}^3$ : the standard sphere defined by  $x_1^2 + x_2^2 + x_3^2 = 1$

$\mathcal{A}(S^2)$  = the algebra of polynomials in  $x_i$  restricted to  $S^2$

$G = SU(2)$  naturally acts on  $S^2$  ( $SO(3)$ -action)

$\mathcal{A}(S^2)$  splits into the direct sum of irreducible representations of  $G$ :

$$\mathcal{A}(S^2) = \bigoplus_{l=0}^{\infty} V_l$$

where  $V_l$  is the representation space with highest weight  $l \in \mathbb{N} \cup \{0\}$ , a unique irreducible representation of  $SU(2)$  with  $\dim V = 2l + 1$  or the eigenspace of the Laplacian  $\Delta$  on  $S^2$  with eigenvalue  $l(l + 1)$ .

**Definition 3** A **fuzzy sphere** is the  $C^*$ -algebra  $\mathcal{A}_N = \text{End}(V_l)$  with  $N = 2l$ , where it decomposes into irreducible  $G$ -representations:

$$\mathcal{A}_N = \text{End}(V_l) = V_l \otimes V_l = \bigoplus_{k=0}^{2l} V_k.$$

One has  $\mathcal{A}_N \rightarrow \mathcal{A}(S^2)$  as  $N = 2l \rightarrow \infty$  “naively”.

## The fuzzy Dirac operator

$L_i \in \text{End}(V_l)$  ( $i = 1, 2, 3$ ): standard self-adjoint infinitesimal generators of  $SU(2)$ -action, namely  $L_i$  are the angular momentum operators such as  $[L_i, L_j] = \sqrt{-1}\epsilon_{ijk}L_k$ . They satisfy

$$L_1^2 + L_2^2 + L_3^2 = l(l+1)1_V$$

Set  $X_i = L_i/\sqrt{l(l+1)} \doteq L_i/l$ . One has

$$X_1^2 + X_2^2 + X_3^2 = 1, \quad [X_i, X_j] = \frac{\sqrt{-1}}{\sqrt{l(l+1)}}\epsilon_{ijk}X_k \longrightarrow 0$$

as  $l \rightarrow \infty$ . Moreover, with  $\hbar = 1/\sqrt{l(l+1)}$ , one has

$$\frac{\sqrt{-1}}{\hbar}[X_i, X_j] \longrightarrow \{x_i, x_j\} : \text{the Poisson bracket on } S^2$$

as  $\hbar \rightarrow 0$ . Thus noncommutative coordinate  $X_i$  converges to ordinary coordinate  $x_i$  as  $l \rightarrow \infty$  in a suitable sense (Rieffel, D'Andorea-Lizzi-Várilly, ... ).



$\sigma_i \in \text{End}(V_{1/2}) \cong M_2\mathbb{C}$  ( $i = 1, 2, 3$ ): the Pauli matrix, namely

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define operators on the fuzzy spinors  $\mathcal{A}_N \otimes V_{1/2} = \text{End}(V_l) \otimes V_{1/2}$  by

$$D_1 = 1/2 + \sum_i L_i^{(l)} \otimes \sigma_i, \quad D_2 = 1/2 - \sum_i L_i^{(r)} \otimes \sigma_i$$

with  $L_i^{(l)} X = L_i X$  and  $L_i^{(r)} X = X L_i$  for  $X \in \text{End}(V_l) \otimes V_{1/2}$ . Set

$$\Gamma_1 = D_1/|D_1|, \quad \Gamma_2 = D_2/|D_2|,$$

which are involutions. One has

$$\frac{\Gamma_1 + \Gamma_2}{2} = \frac{1}{2l+1} \left( 1 + \sum [L_i, X] \otimes \sigma_i \right) = \frac{\not{D}}{2l+1}.$$

Here  $\not{D}$  is the (unbounded) Dirac operator on  $S^2$ , which preserves the fuzzy spinors  $\mathcal{A}_N \otimes V_{1/2} \subset \mathcal{A}(S^2) \otimes V_{1/2}$ .

In the commutative limit ( $N = 2l \rightarrow \infty$ ), one has

$$H_+ = \frac{\Gamma_1 + \Gamma_2}{2} = \frac{1}{2l + 1} \left( 1 + \sum [L_i, X] \otimes \sigma_i \right) \rightarrow \frac{\not{D}}{|\not{D}|}$$

$$H_- = \frac{\Gamma_1 - \Gamma_2}{2} = \frac{1}{2l + 1} \sum \left( L_i^{(l)} + L_i^{(r)} \right) \otimes \sigma_i \rightarrow \sum x_i \otimes \sigma_i = \epsilon$$

since  $X_i \doteq L_i/l$  “converges” to  $x_i$ , where  $\epsilon$  is the grading operator on the spinor bundle on  $S^2$ . Consider the GW index:

$$\text{Ind}(U, \pm 1) = \text{Tr}(H_{\pm} | \ker H_{\mp}) = \text{Tr}(H_{\pm})$$

### Theorem 5 (The index theorem on fuzzy sphere)

$$\text{Ind}(U, -1) = \text{Tr}(H_-) = 0, \quad \text{Ind}(U, +1) = \text{Tr}(H_+) = 4l + 2$$

**Remark.** 1) In the commutative limit ( $l \rightarrow \infty$ ), one has  $\text{Tr}(H_- | \ker H_+) \rightarrow \text{Tr}(\epsilon | \ker \not{D}) = \text{Ind}(\not{D}^+) = 0$ , thus  $\text{Ind}(U, -1)$  contains no doubler fermions.

2) In a similar way we can define the Dirac operator on  $V_l \otimes V_m \otimes V_{1/2}$ , where one has  $\text{Ind}(U, -1) = 2m - 2l$ ,  $\text{Ind}(U, +1) = 2(l + m) + 2$ .

## A GW index theorem on $SU(2)$

$SU(2) \cong S^3$ : the 3-dimensional sphere

$\mathcal{E} = C^\infty(S^3) \otimes \mathbb{C}^2$ : the space of spinors with  $\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \text{End}(\mathbb{C}^2)$

$X, Y, Z$ : left-invariant differential operators on  $S^3$  corresponding to a basis of Lie algebra; Do not confuse  $\text{End}(\mathbb{C}^2)$  with  $\text{Lie}(SU(2))$ .

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \text{Lie}(SU(2)).$$

A Dolbeault operator lifted from  $S^2$  is defined by

$$D = \begin{bmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -X + iY \\ X + iY & 0 \end{bmatrix} \text{ on } \mathcal{E}.$$

Since  $[Z, X] = 2Y$ ,  $[Z, Y] = -2X$ , the Dolbeault operator  $D$  commutes with the action by  $T = \begin{bmatrix} Z & 0 \\ 0 & Z + 2i \end{bmatrix}$ .

Set  $U = \frac{D - iA_k}{D + iA_k}$  with  $A_k = -iT - k$  ( $k \in \mathbb{Z}$ ). One has  $\Gamma U \Gamma = U^*$  as is proved in Example 3.

**Theorem 6 (A GW index theorem on  $SU(2)$ )** *The doubler class  $\text{Ind}(U, +1)$  belongs to  $K_0(\mathcal{K})$  although the index class  $\text{Ind}(U, -1)$  not. One has*

$$\text{Ind}(U, +1) = \text{Tr}(\Gamma | \ker(D + iA_k)) = \text{Ind}(\bar{\partial}^+ \otimes H(k)).$$

*Here  $H(k)$  denotes the ample line bundle on  $S^2$  of degree  $k$ , namely the first Chern number  $c_1(H(k))[S^2] = k$  and*

$$\bar{\partial}^+ \otimes 1_{H(k)} : \Omega^{0,0}(S^2, H(k)) \rightarrow \Omega^{0,1}(S^2, H(k))$$

*is the Dolbeault operator coupled with  $H(k)$ .*

**Thank you for your attention**