Persistence approximation property for maximal Roe algebras and applications

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- Let $X$ be a proper metric space (i.e. closed balls are compact) and let $\pi : C_0(X) \to \mathcal{L}(\mathcal{H})$ be a representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$.
- Example: $\mathcal{H} = L^2(X, \mu)$ for $\mu$ Borelian measure on $X$ and $\pi$ the pointwise multiplication.

**Definition**

- If $T$ is an operator of $\mathcal{L}(\mathcal{H})$, then $\text{Supp}T$ is the complementary of the open subset of $X \times X$
  $$\{(x, y) \in X \times X \text{ such that there exists } f \text{ and } g \text{ such that } f(x) \neq 0, g(y) \neq 0 \text{ and } \pi(f) \cdot T \cdot \pi(g) = 0\}$$
- $T$ has propagation less than $r$ if $d(x, y) \leq r$ for all $(x, y) \in \text{Supp}T$. 
Let $D$ be an elliptic differential operator on a compact manifold $M$.

Let $Q$ be a parametrix for $D$.

Then $S_0 := \text{Id} - QD$ and $S_1 := \text{Id} - DQ$ are smooth kernel operators on $M \times M$:

$$P_D = \begin{pmatrix} S_0^2 & S_0(\text{Id} + S_0)Q \\ S_1D & \text{Id} - S_1^2 \end{pmatrix} \quad (1.1)$$

is an idempotent and we can choose $Q$ such that $P_D$ has arbitrary small propagation.
\[ \text{Ind} (D) = [P_D] - \begin{pmatrix} 0 & 0 \\ 0 & \text{Id} \end{pmatrix} \in K_0(K(L^2(M))) \cong \mathbb{Z} \quad (1.2) \]

- \( D \) is a Fredholm operator and

- How can we keep track of the propagation and have homotopy invariance?
**Quasi-projection**

**Definition**

If $X$ is a proper metric space and $\pi : C_0(X) \to L(H)$ is a representation of $C_0(X)$ on a Hilbert space $H$, let $0 < \epsilon < \frac{1}{4}$ (control) and $r > 0$ (propagation). Then $q \in \mathcal{L}(\mathcal{H})$ is an $\epsilon$-$r$-projection if $q = q^*$, $\|q - q^2\| < \epsilon$ and $q$ has propagation less than $r$.

- If $q$ is an $\epsilon$-$r$-projection, then its spectrum has a gap around $\frac{1}{2}$.
- Hence there exists $k_0 : \sigma(q) \to \{0, 1\}$ continuous and such that $k_0(t) = 0$ if $t < \frac{1}{2}$ and $k_0(t) = 1$ if $t > \frac{1}{2}$.
- By continuous functional calculus, $k_0(q)$ is a projection such that $\|k_0(q) - q\| \leq 2\epsilon$. 
Let $D$ be a differential elliptic operator on a manifold, let $Q$ be a parametrix. Set $S_0 := Id - QD$ and $S_1 := Id - DQ$ and

$$P_D = \begin{pmatrix}
S_0^2 & S_0(Id + S_0)Q \\
S_1D & Id - S_1^2
\end{pmatrix} \quad (1.3)$$

Then

$$((2P_D^* - 1)(2P_D - 1) + 1)^{\frac{1}{2}} P_D ((2P_D^* - 1)(2P_D - 1) + 1)^{-\frac{1}{2}}$$

is a projection conjugated to the idempotent $P_D$.
Choosing \( Q = Q_{\epsilon,r} \) with propagation small enough and approximating
\[
((2P_D^* - 1)(2P_D - 1) + 1)^{\frac{1}{2}} P_D ((2P_D^* - 1)(2P_D - 1) + 1)^{-\frac{1}{2}}
\]
by a power series, we can for all \( 0 < \epsilon < \frac{1}{4} \) and \( r > 0 \), construct a \( \epsilon-r \)-projection \( q_{\epsilon,r}^D \) such that

\[
\text{Ind } (D) = [k_0(q_{\epsilon,r}^D)] - \left[ \begin{pmatrix} 0 & 0 \\ 0 & \text{Id} \end{pmatrix} \right] \in K_0(K(L^2(M))) \cong \mathbb{Z}
\]

(1.4)
Filtered $C^*$-algebras

**Definition**

A filtered $C^*$-algebra $A$ is a $C^*$-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces indexed by positive numbers such that:

- $A_r \subseteq A_{r'}$ if $r \leq r'$;
- $A_r$ is stable by involution, i.e. for any $x \in A_r$, then $x^* \in A_r$;
- $A_r \cdot A_{r'} \subseteq A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in $A$. 
If $A$ is unital, we impose that $1 \in A_r$, for any $r > 0$. If $A$ is non unital filtered $C^*$-algebra, then its unitization $\tilde{A}$ is filtered by $(A_r + \mathbb{C})_{r>0}$. We can define the homomorphism

$$\rho_A : \tilde{A} \to \mathbb{C}; \ a + z \to z$$

for $a \in A$ and $z \in \mathbb{C}$

**Definition**

Let $A$ and $B$ be two $C^*$-algebras filtered by $(A_r)_{r>0}$ and $(B_r)_{r>0}$. A $*$-homomorphism $\phi : A \to B$ is said to be filtered if $\phi(A_r) \subseteq B_r$ for all $r > 0$. 
Examples

- \( K(L^2(X, \mu)) \) for \( X \) a metric space and \( \mu \) probability measure on \( X \). More generally \( A \otimes K(L^2(X, \mu)) \) for \( A \) is a \( C^* \)-algebra.
- Roe algebras:
  - \( \Sigma \) proper discrete metric space, \( \mathcal{H} \) separable Hilbert space.
  - \( C[\Sigma]_r \) : space of locally compact operators on \( l^2(\Sigma) \otimes \mathcal{H} \) (i.e. \( T \) satisfies \( fT \) and \( Tf \) are compact for all \( f \in C_c(\Sigma) \)) and with propagation less than \( r \).
  - The Roe algebra of \( \Sigma \) is \( C^*(\Sigma) = \bigcup_{r>0} C[\Sigma]_r \subset L(l^2(\Sigma) \otimes \mathcal{H}) \) (filtered by \( (C[\Sigma]_r)_{r>0} \)).
- Also the maximal Roe algebras is filtered \( C^* \)-algebras.
- \( C^* \)-algebras of groups and cross-products.
\( \epsilon \)-\( r \)-projections and \( \epsilon \)-\( r \)-unitaries

**Definition**

Let \( A \) be a unital filtered C\(^*\) algebra. For any \( r > 0 \) and \( \epsilon \in (0, \frac{1}{4}) \), we call:

- An element \( p \) in \( A \) an \( \epsilon \)-\( r \)-projection if \( p \) belongs to \( A_r, p = p^* \) and \( \|p^2 - p\| < \epsilon \). The set of \( \epsilon \)-\( r \)-projections will be denoted by \( P^{\epsilon, r}(A) \).

- An element \( u \) in \( A \) is an \( \epsilon \)-\( r \)-unitary if \( u \) belongs to \( A_r, \|u^*u - 1\| < \epsilon \) and \( \|uu^* - 1\| < \epsilon \). The set of \( \epsilon \)-\( r \)-unitaries in \( A \) will be denoted by \( U^{\epsilon, r}(A) \).

We can construct a projection by continuous functional calculus on \( \sigma(p) \) denoted by \( k_0(p) \) and a unitary \( k_1(u) = u(u^*u)^{-\frac{1}{2}} \).
Definition

For a unital filtered C* algebra $A$, we can define the following equivalent relation on $P_{\infty}^{\epsilon,r}(A) \times \mathbb{N}$ and $U_{\infty}^{\epsilon,r}(A)$:

- if $p$ and $q$ are elements of $P_{\infty}^{\epsilon,r}(A)$, $l$ and $l'$ are positive integers, $(p, l) \sim (q, l')$ if there exists a positive integer $k$ and an element $h$ of $P_{\infty}^{\epsilon,r}(A[0, 1])$ such that $h(0) = \text{diag}(p, I_{k+l'})$ and $h(1) = \text{diag}(q, I_{k+l})$

- if $u$ and $v$ are elements of $U_{\infty}^{\epsilon,r}(A), u \sim v$ if there exists an element $h$ of $U_{\infty}^{3\epsilon,2r}(A[0, 1])$ such that $h(0) = u$ and $h(1) = v$.

If $p$ is an element of $P_{\infty}^{\epsilon,r}(A)$ and $l$ is an integer, we denote by $[p, l]_{\epsilon,r}$ the equivalent class of $(p, l)$ modulo $\sim$. And if $u$ is an element of $U_{\infty}^{\epsilon,r}(A)$ we denote by $[u]_{\epsilon,r}$ its equivalent class modulo $\sim$. 
Quantitative K-theory

Definition

Let \( r > 0 \) and \( \epsilon \in (0, \frac{1}{4}) \). We define:

(i) \( K_{0}^{\epsilon,r}(A) = P_{\infty}^{\epsilon,r}(A) \times \mathbb{N}/ \sim \) unital and 
\( K_{0}^{\epsilon,r}(A) = P_{\infty}^{\epsilon,r}(\tilde{A}) \times \mathbb{N}/ \sim \) such that \( \dim k_{0}(\rho_{A}(p)) = l \) for 
\( A \) non unital.

(ii) \( K_{1}^{\epsilon,r}(A) = U_{\infty}^{\epsilon,r}(\tilde{A})/ \sim \) (with \( A = \tilde{A} \) if \( A \) is already unital).

Then \( K_{0}^{\epsilon,r}(A) \) turns to be an abelian group where

\[
[p, l]_{\epsilon,r} + [p', l']_{\epsilon,r} = [\text{diag}(p, p'), l + l']_{\epsilon,r}
\]

\( K_{1}^{\epsilon,r}(A) \) is also an abelian group with

\[
[u]_{\epsilon,r} + [u']_{\epsilon,r} = [\text{diag}(u, u')]_{\epsilon,r}
\]
Lemma

If $A$ is a filtered $C^*$- algebra, then $K_{*}^{\epsilon, r}(A) = K_{0}^{\epsilon, r}(A) \oplus K_{1}^{\epsilon, r}(A)$ is a $\mathbb{Z}_2$- graded abelian group.

For any filtered $C^*$ algebra $A$ and any positive numbers $\epsilon, \epsilon'$ and $r, r'$ with $\epsilon \leq \epsilon' < \frac{1}{4}$ and $r \leq r'$, there exists natural group homomorphisms:

- $\iota_{0}^{\epsilon, r}: K_{0}^{\epsilon, r}(A) \rightarrow K_{0}(A); [p, l]_{\epsilon, r} \mapsto [k_{0}(p)] - [I_l]$;
- $\iota_{1}^{\epsilon, r}: K_{1}^{\epsilon, r}(A) \rightarrow K_{1}(A); [u]_{\epsilon, r} \mapsto [k_{1}(u)]$;
- $\iota_{*}^{\epsilon, r} = \iota_{0}^{\epsilon, r} \oplus \iota_{1}^{\epsilon, r}$;
- $\iota_{0}^{\epsilon, \epsilon', r, r'}: K_{0}^{\epsilon, r}(A) \rightarrow K_{0}^{\epsilon', r'}(A); [p, l]_{\epsilon, r} \mapsto [p, l]_{\epsilon', r'}$;
- $\iota_{1}^{\epsilon, \epsilon', r, r'}: K_{1}^{\epsilon, r}(A) \rightarrow K_{1}^{\epsilon', r'}(A); [u]_{\epsilon, r} \mapsto [u]_{\epsilon', r'}$;
- $\iota_{*}^{\epsilon, \epsilon', r, r'} = \iota_{0}^{\epsilon, \epsilon', r, r'} \oplus \iota_{1}^{\epsilon, \epsilon', r, r'}$. 
Persistence approximation property for maximal Roe algebras and applications

Introduction

Control pair

Definition

A control pair is a pair \((\lambda, h)\), where

- \(\lambda > 1\);
- \(h : (0, \frac{1}{4\lambda}) \to (1, +\infty); \epsilon \mapsto h_\epsilon\) is a map such that exists a non-increasing map \(g : (0, \frac{1}{4\lambda}) \to (1, +\infty)\), with \(h \leq g\).

The set of control pairs is equipped with a partial order: \((\lambda, h) \leq (\lambda', h')\) if \(\lambda \leq \lambda'\) and \(h_\epsilon \leq h'_\epsilon\) for all \(\epsilon \in (0, \frac{1}{4\lambda'})\).

Definition

For any filtered \(C^*\)-algebra \(A\), define the families

\[ K_0(A) = \left( K_0^\epsilon, r(A) \right)_{0 < \epsilon < \frac{1}{4}, r > 0}, \]

\[ K_1(A) = \left( K_1^\epsilon, r(A) \right)_{0 < \epsilon < \frac{1}{4}, r > 0}, \]

\[ K_*(A) = \left( K_*^\epsilon, r(A) \right)_{0 < \epsilon < \frac{1}{4}, r > 0}. \]
Controlled morphism

Definition

Let \((\lambda, h)\) be a control pair, \(A\) and \(B\) be two filtered \(\mathcal{C}^*\)-algebras, and \(i, j\) be elements of \(\{0, 1, *\}\). A \((\lambda, h)\)-controlled morphism

\[\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)\]

is a family \(\mathcal{F} = (F_{\epsilon, r})_{0 < \epsilon < \frac{1}{4}, r > 0}\) of group homomorphisms

\[F_{\epsilon, r} : K_{i, \epsilon, r}^\epsilon(A) \to K_{j, \lambda \epsilon, h \epsilon r}^\lambda \epsilon, h \epsilon r (B)\]

such that for any positive numbers \(\epsilon, \epsilon'\) and \(r, r'\) with \(0 < \epsilon \leq \epsilon' < \frac{1}{4\lambda}, r \leq r'\) and \(h \epsilon r \leq h \epsilon' r'\), we have

\[F_{\epsilon', r'} \circ \iota_{i, \epsilon, \epsilon', r, r'}^\epsilon = \iota_{j, \lambda \epsilon, \lambda \epsilon', h \epsilon r, h \epsilon r'}^\lambda \epsilon, h \epsilon r, h \epsilon r' \circ F_{\epsilon, r}.\]
The composition of controlled morphism

**Definition**

If \((\lambda, h)\) and \((\lambda', h')\) are two control pairs, define

\[
h \ast h' : (0, \frac{1}{4\lambda'\lambda}) \to (0, +\infty); \epsilon \mapsto h_{\lambda'\epsilon}h'_{\epsilon}.
\]

Then \(\lambda\lambda', h \ast h'\) is a control pair. Let \(A, B_1\) and \(B_2\) be filtered 
\(C^*\)-algebras, \(i, j\) and \(l\) in \(\{0, 1, *\}\). Let

\[
\mathcal{F} = (F^{\epsilon, r})_{0 < \epsilon < \frac{1}{4\alpha_{\mathcal{F}}}, r > 0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B_1) \text{ be a } (\alpha_{\mathcal{F}}, k_{\mathcal{F}})-
\]

controlled morphism, let

\[
\mathcal{G} = (G^{\epsilon, r})_{0 < \epsilon < \frac{1}{4\alpha_{\mathcal{G}}}, r > 0} : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2) \text{ be a } (\alpha_{\mathcal{G}}, k_{\mathcal{G}})-
\]

controlled morphism. Then \(\mathcal{G} \circ \mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_l(B_2)\) is the

\((\alpha_{\mathcal{G}}\alpha_{\mathcal{F}}, k_{\mathcal{G}} \ast k_{\mathcal{F}})\)-controlled morphism defined by the family

\[
(G^{\alpha_{\mathcal{F}} \epsilon, k_{\mathcal{F}}, \epsilon E} \circ F^{\epsilon, r})_{0 < \epsilon < \frac{1}{4\alpha_{\mathcal{F}}\alpha_{\mathcal{G}}}, r > 0}.
\]
The equivalence of controlled morphism

Definition

Let $A$ and $B$ be filtered $C^*$-algebras, and $(\lambda, h)$ is a control pair. Let $\mathcal{F} = (F^\epsilon, E)_{0 < \epsilon < \frac{1}{4\alpha_F}, r > 0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ (resp. $\mathcal{G} = (G^\epsilon, r)_{0 < \epsilon < \frac{1}{4\alpha_G}, r > 0}$) be a $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism (resp. a $(\alpha_\mathcal{G}, k_\mathcal{G})$-controlled morphism). Then we write $\mathcal{F} \sim^{(\lambda, h)} \mathcal{G}$ if

1. $(\alpha_\mathcal{F}, k_\mathcal{F}) \leq (\lambda, h)$ and $(\alpha_\mathcal{G}, k_\mathcal{G}) \leq (\lambda, h)$
2. for any $\epsilon \in (0, \frac{1}{4\lambda})$ and $r > 0$, then

$$
\iota_j^{\alpha_\mathcal{F} \epsilon, \lambda \epsilon, k_\mathcal{F}, \epsilon r, h \epsilon r} \circ F^{\epsilon, r} = \iota_j^{\alpha_\mathcal{G} \epsilon, \lambda \epsilon, k_\mathcal{G}, \epsilon r, h \epsilon r} \circ G^{\epsilon, r}.
$$
Controlled isomorphism

Definition

Let $(\lambda, h)$ be a control pair, and $\mathcal{F}: \mathcal{K}_i(A) \rightarrow \mathcal{K}_j(B)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism with $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)$.

- $\mathcal{F}$ is called left $(\lambda, h)$-invertible if there exists a controlled morphism
  
  \[ \mathcal{G}: \mathcal{K}_j(B) \rightarrow \mathcal{K}_i(A) \]

  such that $\mathcal{G} \circ \mathcal{F} \sim \text{Id}_{\mathcal{K}_i(A)}$ and $\mathcal{F} \circ \mathcal{G} \sim \text{Id}_{\mathcal{K}_j(B)}$.

- $\mathcal{F}$ is $(\lambda, h)$-isomorphism if there exists a controlled morphism
  
  \[ \mathcal{G}: \mathcal{K}_j(B) \rightarrow \mathcal{K}_i(A) \]

  which is a $(\lambda, h)$-inverse for $\mathcal{F}$. 


Let \((\lambda, h)\) be a control pair and let \(\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)\) a \((\alpha_{\mathcal{F}}, k_{\mathcal{F}})\)-controlled morphism.

**Definition**

- \(\mathcal{F}\) is called \((\lambda, h)\)-injective if \((\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)\) and for any \(0 < \epsilon < \frac{1}{4\lambda}\), any \(r > 0\) and any \(x \in K_i^{\epsilon, r}(A)\), then \(F^{\epsilon, r}(x) = 0\) in \(K_j^{\alpha_{\mathcal{F}} \epsilon, k_{\mathcal{F}} \epsilon r}(B)\) implies that \(\iota_{i}^{\epsilon, \lambda \epsilon, r, h \epsilon r}(x) = 0\) in \(K_i^{\lambda \epsilon, h \epsilon r}(A)\);

- \(\mathcal{F}\) is called \((\lambda, h)\)-surjective if for any \(0 < \epsilon < \frac{1}{4\lambda\alpha_{\mathcal{F}}}\), any \(r > 0\) and \(y \in K_j^{\epsilon, r}(B)\), there exists an element \(x \in K_i^{\lambda \epsilon, h \epsilon r}(A)\) such that \(F^{\lambda \epsilon, h \lambda \epsilon r}(x) = \iota_{j}^{\epsilon, \alpha_{\mathcal{F}} \lambda \epsilon, r, k_{\mathcal{F}} \lambda \epsilon h \epsilon r}(y)\) in \(K_j^{\alpha_{\mathcal{F}} \lambda \epsilon, r, k_{\mathcal{F}} \lambda \epsilon h \epsilon r}(B)\).
Controlled exact sequence

**Definition**

Let \((\lambda, h)\) be a control pair. Let
\[
F = (F^\epsilon, r)_{0 < \epsilon < \frac{1}{4\alpha_F}, r > 0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B_1)
\]
be a \((\alpha_F, k_F)\)-controlled morphism, and let
\[
G = (G^\epsilon, r)_{0 < \epsilon < \frac{1}{4\alpha_G}, r > 0} : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2)
\]
be a \((\alpha_G, k_G)\)-controlled morphism, where \(i, j\) and \(l\) are in \(\{0, 1, *\}\) and
\(A, B_1, B_2\) are filtered \(C^*\)-algebras. Then the composition
\[
\mathcal{K}_i(A) \xrightarrow{F} \mathcal{K}_j(B_1) \xrightarrow{G} \mathcal{K}_l(B_2)
\]
is said to be \((\lambda, h)\)-exact at \(\mathcal{K}_j(B_1)\) if \(G \circ F = 0\).
and if for any $0 < \epsilon < \frac{1}{4 \max\{\lambda \alpha_\mathcal{I}, \alpha_\mathcal{G}\}}$, any $r > 0$ and $y \in K_j^{\epsilon, r}(B_1)$ such that $G^\epsilon, r(y) = 0$ in $K_j^{\alpha_\mathcal{G} \epsilon, k_\mathcal{G}, \epsilon r}(B_2)$, there exists an element $x$ in $K_i^{\lambda \epsilon, h \epsilon r}(A)$ such that

$$F_{\lambda \epsilon, h \epsilon r}(x) = l_j^{\epsilon, \alpha_\mathcal{I} \lambda \epsilon, r, k_\mathcal{I}, \lambda \epsilon h \epsilon r}(y)$$

in $K_j^{\alpha_\mathcal{I} \lambda \epsilon, k_\mathcal{I} \lambda \epsilon h \epsilon r}(B_1)$. 
**Completely filtered extension**

**Definition**

Let $A$ be a filtered $C^*$-algebra. Let $J$ be an ideal of $A$ and set $J_r = J \cap A_r$. The extension of $C^*$-algebras

$$0 \to J \to A \to A/J \to 0$$

is called a completely filtered extension of $C^*$-algebras if the bijection continuous linear map

$$A_r/J_r \to (A_r + J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is a complete isometry i.e for any integer $n$, any $r > 0$ and $x \in M_n(A_r)$, then

$$\inf_{y \in M_n(J_r)} \|x + y\| = \inf_{y \in M_n(J)} \|x + y\|.$$
Controlled six term exact sequence

**Theorem Oyono-Oyono and G. Yu**

There exists a control pair $(\lambda, h)$ such that for any completely filtered extensions of $C^*$-algebras

$$0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0,$$

the following six-term sequence is $(\lambda, h)$-exact

$$\begin{align*}
\mathcal{K}_0(J) & \xrightarrow{j^*} \mathcal{K}_0(A) \xrightarrow{q^*} \mathcal{K}_0(A/J) \\
\mathcal{D}_{J,A} & & \mathcal{D}_{J,A} \\
\mathcal{K}_1(A/J) & \xleftarrow{q^*} \mathcal{K}_1(A) \xleftarrow{j^*} \mathcal{K}_1(J)
\end{align*}$$
Controlled Roe transformation

For any \( z \in KK_1(A, B) \), then \( z \) can be represented by a triple \((H_A, \pi, T)\) where:

- \( \pi : A \to \mathcal{L}_B(H_B) \) is a \(*\)-representation of \( A \) on \( H_B \);
- \( T \in \mathcal{L}_B(H_B) \) is a self-adjoint operator;
- \([T, \pi(a)], \pi(a)[T^2 - Id_{H_B}] \) are compact operators in \( K(H_B) \cong K(H) \otimes B \)

Let \( P = (\frac{1+T}{2}) \in \mathcal{L}_B(H_B) \) and

\[
E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in A, y \in B \otimes K(H)\}
\]

Then we have a semi-split exact extension:

\[
0 \to B \otimes K(H) \to E^{(\pi,T)} \to A \to 0
\]

where the completely positive section is

\[
s : A \to E^{\pi,T}; a \mapsto (a, P\pi(a)P).
\]
By the functor property of $C^*_\text{max}(X, \cdot)$, then we have a semi-split exact extension:

$$0 \to C^*_\text{max}(X, B) \to E_{X,\text{max}}^{\pi,T} \to C^*_\text{max}(X, A) \to 0$$

where $E_{X,\text{max}}^{\pi,T} = C^*_\text{max}(X, E_{\pi,T})$

**Proposition**

The controlled boundary map $D_{\pi,T} = D_{C^*_\text{max}(X,B),E_{X,\text{max}}^{\pi,T}}$ of the extension

$$0 \to C^*_\text{max}(X, B) \to E_{X,\text{max}}^{\pi,T} \to C^*_\text{max}(X, A) \to 0$$

only depends on the class $z$. 
Odd case

Let $A$ and $B$ be two $C^*$-algebras. Then there exists a control pair $(\alpha_X, k_X)$ such that for any $z \in KK_1(A, B)$, there exists a $(\alpha_X, k_X)$-controlled morphism

$$\hat{\sigma}_{X,max}(z) : \mathcal{K}_*(C^*_{max}(X, A)) \to \mathcal{K}_{*+1}(C^*_{max}(X, B))$$

Even case

Using Bott periodicity theorem, let $A$ and $B$ be two $C^*$-algebras. For any $z \in KK_0(A, B)$, there exists a control pair $(\alpha_X, k_X)$ and even degree $(\alpha_X, k_X)$-controlled morphism

$$\hat{\sigma}_{X,max}(z) : \mathcal{K}_*(C^*_{max}(X, A)) \to \mathcal{K}_*(C^*_{max}(X, B))$$
For any positive number $d$ and probability $\eta$ of the Rips complex $P_d(X)$ can be written as $\eta = \sum_{x \in X} \lambda_x(\eta) \delta_x$, where $\delta_x$ is the Dirac probability at $x$, and $\lambda_x : P_d(X) \to [0,1]$ is a continuous function. Let

$$h_d : \begin{cases} X \times X \to C_0(P_d(X)) \\ (x,y) \mapsto \frac{1}{2} \lambda_x \frac{1}{2} \lambda_y \end{cases}$$

Let $(e_x)_{x \in X}$ be the canonical basis of $l^2(X)$, $e$ be a rank one projection in $\mathcal{H}$, and $P_d$ be defined as the extension by linearity and continuity of

$$P_d(e_x \otimes \xi \otimes f) = \sum_{y \in X} e_y \otimes (e \xi) \otimes (h(x,y)f)$$

for every $x \in X$, $\xi \in \mathcal{H}$ and $f \in C_0(P_d(X))$. As $\sum_{x \in X} \lambda_x = 1$, $P_d$ is projection of $K(l^2(X)) \otimes C_0(P_d(X))$ of propagation less than $d$. Hence, $P_d$ define a class $[P_d, 0]_{\epsilon, r'} \in K_0^\epsilon, r'(C^*_{max}(X, C_0(P_d(X))))$ for any $\epsilon \in (0, \frac{1}{4})$ and $r' \geq d$. 
Quantitative maximal coarse Baum-Connes assembly map

**Definition**

Let $A$ be a $C^*$-algebra, $\epsilon \in (0, \frac{1}{4})$ and positive numbers $d, r$ satisfying that $k_X(\epsilon)d \leq r$. The quantitative assembly map $\hat{\mu}_{X,A,max,*} = (\mu_{\epsilon,d,r}^{X,A,max,*})_{\epsilon,r}$ is defined as the family of maps

\[ \mu_{\epsilon,d,r}^{X,A,max,*} : \begin{cases} KK_*(C_0(P_d(X)), A) \to K_*^{\epsilon,r}(C_{max}^*(X, A)) \\ z \mapsto \iota_{\epsilon,d,r}^{X,A,max,*}(\alpha_{X, \epsilon', r, r'} \circ \hat{\sigma}_{X,max}(z)[P_d, 0]_{\epsilon', r'}) \end{cases} \]

where $\epsilon'$ and $r'$ satisfy:

- $\epsilon' \in (0, \frac{1}{4})$ such that $\alpha_{X, \epsilon'} \leq \epsilon$.
- $d \leq r'$ such that $k_X(\epsilon')r' \leq r$. 
Let $KK_\ast(P_d(X), A)$ denote $KK_\ast(C_0(P_d(X)), A)$.

**Definition**

Let $A$ be a $G$-algebra, We say that:

- (Quantitative injectivity) $\mu_{X,A,max,\ast}$ is quantitative injective if for any $d > 0$, there exists $\epsilon \in (0, \frac{1}{4})$ such that for any $r > 0$ satisfying $k_X(\epsilon)d \leq r$, there exists $d' > d$ such that for any $z \in KK_\ast(P_d(X), A)$, $\mu_{X,A,max,\ast}^{\epsilon,d,r}(z) = 0$ implies that $(q_d^{d'})_\ast(z) = 0$.

- (Quantitative surjectivity) $\mu_{X,A,max,\ast}$ is quantitative surjective if there exists $\epsilon \in (0, \frac{1}{4})$ such that for any $r > 0$ such that, there exists $\epsilon' \in (\epsilon, \frac{1}{4})$ and positive numbers $d, r'$ such that $r \leq r'$ and $k_X(\epsilon')d \leq r'$, for any $y \in K^{\epsilon,r'}_\ast(C_{max}^\ast(X, A))$ there exists $z \in KK_\ast(P_d(X), A)$ such that $\mu_{X,A,max,\ast}^{\epsilon',d,r'}(z) = l^{\epsilon,\epsilon',r,r'}_\ast(y)$.
Proposition

Let $X$ be a discrete metric space with bounded geometry and $A$ be a $C^*$-algebra.

- If $\mu_{X,A,max,*}$ is quantitative injective then $\mu_{X,A,max,*}$ is one-to-one.
- If $\mu_{X,A,max,*}$ is quantitative surjective then $\mu_{X,A,max,*}$ is onto.

Definition

- $QI_{X,A,max,*}(d,d',\epsilon,r)$: for any $x \in KK_*((P_d(X),A)$, then $\mu_{X,A,max,*}^\epsilon,d,r(x) = 0$ implies $(q_d^{d'})^*(x) = 0$ in $KK_* (P_{d'}(X), A)$
- $QS_{X,A,max,*}(d,\epsilon,\epsilon',r,r')$: for any $y \in K^\epsilon,r (C_{max}^*(X,A)$, then there exist a $x \in KK_* (P_d(X), A)$ such that $\mu_{X,A,max,*}^{\epsilon',d,r'}(x) = \iota_{\epsilon,\epsilon',r,r'}^*(y)$. 
Theorem

Let $X$ be a discrete metric space with bounded geometry and $A$ is a $C^*$-algebra. The following are equivalent:

(1) $\mu_{X,l^\infty (\mathbb{N}, K(H)\otimes A), max,*}$ is one to one,

(2) For any $d > 0$, $\epsilon \in (0, \frac{1}{4})$ and $r > 0$ with $k_X(\epsilon)d \leq r$, there exists $d'$ such that $d \leq d'$ and $QI_{X,A,max,*}(d, d', \epsilon, r)$ holds.

Theorem

Let $X$ be a discrete metric space with bounded geometry and $A$ is a $C^*$-algebra. Then there exist $\lambda > 1$ such that the following are equivalent:

(1) $\mu_{X,l^\infty (\mathbb{N}, K(H)\otimes A), max,*}$ is onto;

(2) For any positive numbers $\epsilon$ with $\epsilon < \frac{1}{4\lambda}$ and $r > 0$, there exist $d > 0$ and $r' > 0$ with $k_X(\epsilon)d \leq r$ and $r \leq r'$ for which $QS_{X,A,max,*}(d, r, r', \epsilon, \lambda\epsilon)$ is satisfied.
Corollary

Let $X$ be a discrete metric space with bounded geometry and $A$ is a $C^*$-algebra. Then we have the following results:

- $\mu_{X,A,max,*}$ is one to one. Then for any $\epsilon \in (0, \frac{1}{4})$ and every $d > 0$, $r > 0$ such that $k_X(\epsilon)d \leq r$, there exists $d'$ with $d \leq d'$ such that $QI_{X,A,max,*}(d, d', \epsilon, r)$ holds.

- $\mu_{X,A,max,*}$ is onto. Then for some $\lambda \geq 1$ and any $\epsilon \in (0, \frac{1}{4\lambda})$ and every $r > 0$, there exists $d > 0$, $r' > 0$ such that $k_X(\epsilon)d \leq r$ and $r \leq r'$ such that $QS_{X,A,max,*}(d, r, r', \epsilon, \lambda \epsilon)$ holds.
Persistence approximation property was introduced by Oyono-Oyono and Guoliang Yu. It provides the geometric obstruction to Baum-Connes conjecture.

**Definition**

Let $B$ be a filtered $C^*$-algebra. we sat that $K_*(B)$ has persistence approximation property if: for any $\epsilon \in (0, \frac{1}{4})$ and $r > 0$, there exists $\epsilon' \in (\epsilon, \frac{1}{4})$ and $r' \geq r$ such that for any $x \in K_*(B)$, then

$$\iota_{\epsilon', \epsilon, r, r'}(x) \neq 0 \text{ in } K_{\epsilon', r'}(B)$$

implies that

$$\iota_{\epsilon, \epsilon', r, r'}(x) = 0 \text{ in } K_{\epsilon, r}(B).$$

$\mathcal{PA}_*(B, \epsilon, \epsilon', r, r')$: for any $x \in K_{\epsilon, r}(B)$, then

$$\iota_{\epsilon, \epsilon', r, r'}(x) = 0 \text{ in } K_{\epsilon', r'}(B).$$
Persistence approximation property for crossed with groups

**Theorem Oyono-Oyono and G.Yu**

Let $\Gamma$ be a finite generated group and $A$ be a $C^*$-algebra. Assume that:

- $\mu_{\Gamma,l^\infty}(\mathbb{N},A\otimes K(H))$ is onto and $\mu_{\Gamma,A}$ is one to one.
- $\Gamma$ admits a cocompact universal example for proper actions.

Then for some universal constant $\lambda_{PA} \geq 1$, any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$, any $r > 0$, and any $\Gamma$-$C^*$-algebra $A$ there exists $r' \geq r$ such that $\mathcal{P}A_*(A \rtimes_{red} \Gamma, \epsilon, \lambda_{PA}\epsilon, r, r')$ holds.
Persistence approximation property for maximal Roe algebras and applications

Persistence approximation property

Persistence approximation property for crossed product with groupoids

Theorem Clément Dell’Aiera

Let $G$ be an étale groupoid such that:

- $G^{(0)}$ is compact.
- $G$ admits a cocompact example for universal space for proper actions.

Then there exists a universal constant $\lambda_{PA} \geq 1$ such that for any $G$-algebra $A$, if $\mu_{G,l^\infty(N,A \otimes K(H))}$ is onto and $\mu_{G,A}$ is one to one, then for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and every $F \in \mathcal{E}$, there exists a $F$ such that $F \subseteq F'$ and $\mathcal{P} \mathcal{A}^* (\hat{A} \times_{red} G, \epsilon, \lambda_{PA} \epsilon, F, F')$ holds.
For the metric space, we need a condition to replace that the group(groupoid) admits a cocompact universal example for proper actions.

**Definition**

A discrete metric space is coarsely uniformly contractible: if for every $d > 0$, there exists $d' \geq d$ such that any compact subset of $P_d(X)$ lies in a contractible invariant compact subset of $P_{d'}(X)$.

**Example 2.5 D.Meintrup and T.Schick**

Any discrete hyperbolic metric space is coarsely uniformly contractible.
Theorem Q.Wang and Z.Wang

Let $X$ be a discrete metric space with bounded geometry and $A$ is a $C^*$-algebra. Assume that:

- $X$ is coarsely uniformly contractible.
- $\mu_{X,l^\infty(N,A\otimes K(H)),max,*}$ is onto and $\mu_{X,A,max,*}$ is one to one.

Then there exists a universal constant $\lambda_{PA} \geq 1$ such that for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and every $r > 0$, there exists a $r' > 0$ such that $r \leq r'$ and $PA_*(C^*_{max}(X, A), \epsilon, \lambda_{PA}\epsilon, r, r')$ holds.
Theorem Q.Wang and Z.Wang

Let $X$ be a discrete metric space with bounded geometry. Assume that $X$ admits a fibred coarse embedding into Hilbert space and $X$ is coarsely uniformly contractible. then there exists a universal constant $\lambda_{PA} \geq 1$ such that: for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and any $r > 0$ there exists $r' > r$ such that $PA_*(C^*_\text{max}(X), \epsilon, \lambda_{PA}\epsilon, r, r')$ holds.
Theorem Q. Wang and Z. Wang

Let $\Gamma$ be a finite generated residually finite group with Haagerup property and admits a cocompact universal example for proper actions. Then there exists a universal constant $\lambda_{PA} \geq 1$ for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{P}A_*(C^*_{max}(X(\Gamma)), \epsilon, \lambda_{PA} \epsilon, r, r')$ holds.

Example

Both $F_2$ and $SL_2(\mathbb{Z})$ are finite generated group with Haagerup property. Since their classifying space is a tree and this tree is cocompact. So they admit a cocompact universal example for proper actions. Hence the maximal Roe algebra of their box space will have persistence approximation property.
Let $\mathcal{X} = (X_i)_{i \in \mathbb{N}}$ be a family of discrete metric space with bounded geometry and $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ be a family of C*-algebras. Denote $C^*_{max}(\mathcal{X}, \mathcal{A})$ be the closure of $\bigcup_{r > 0} (\prod_{i \in \mathbb{N}} C[X_i, A_i])_r$ of $\prod_{i \in \mathbb{N}} C^*_{max}(X_i, A_i)$. Then $C^*_{max}(\mathcal{X}, \mathcal{A})$ is filtered C*-algebra.

**Lemma**

There exist a control pair $(\lambda, h)$ and a $(\lambda, h)$-controlled isomorphism

$$K_*(C^*_{max}(\mathcal{X}, \mathcal{A})) \to \prod_i K_*(C^*_{max}(X_i, A_i))$$
Quantitative assembly map for a family of metric space

**Definition**

For any \( \epsilon \in (0, \frac{1}{4}) \) and \( d, r > 0 \) with \( k_X(\epsilon) \cdot d \leq r \). Define:

\[
\mu_{X, max,*}^{\infty, \epsilon, d, r} : \left\{ \prod_{i \in \mathbb{N}} KK_*(C_0(P_d(X_i)), \mathbb{C}) \to K_*, \epsilon, r^\infty(C_{max}^*(X)) \right. \\
\left. z \mapsto \iota^\epsilon_{X, \epsilon', \epsilon', r', r'} \circ \hat{\sigma}_{X, max}^\infty(z) [P_{d, X}, 0] \epsilon', r' \right.
\]

where \( \epsilon' \) and \( r' \) satisfy:

- \( \epsilon' \in (0, \frac{1}{4}) \) such that \( \alpha_X \cdot \epsilon' \leq \epsilon \);
- \( d \leq r' \) and \( k_X(\epsilon') \cdot r' \leq r \).
Definition

- \( QI_{X, \text{max}, *}(d, d', r, \epsilon): \) for any \( x \in \prod_{i \in \mathbb{N}} KK_*(P_d(X_i), \mathbb{C}) \), then \( \mu_{X, \text{max}, *}^{\infty, \epsilon, d, r}(x) = 0 \) implies \( (q_{d'})^*(x) = 0 \) in \( \prod_{i \in \mathbb{N}} KK_*(P_d(X_i), \mathbb{C}) \)

- \( QS_{X, \text{max}, *}(d, r, r', \epsilon, \epsilon'): \) for any \( y \in K_{*}^{\epsilon, r}(C_{\text{max}}^*(X)) \), there exists \( x \in \prod_{i \in \mathbb{N}} KK_*(P_d(X_i), \mathbb{C}) \) such that \( \mu_{X, \text{max}, *}^{\infty, \epsilon', d, r'}(x) = \iota_{*, \epsilon', r, r'}(y) \)

Let \( \Sigma = \bigcup_{i \in \mathbb{N}} X_i \), where \( (X_i)_{i \in \mathbb{N}} \) is a family of metric space satisfying: for any \( r > 0 \), there exists an integer \( N_r \) such that for any integer \( i \), any ball of radius \( r \) in \( X_i \) is no more than \( N_r \) elements.

The metric \( d \) on \( \Sigma \) is defined to be:

- on each \( X_i \), the metric is just the usual metric on \( X_i \);
- \( d(X_i, X_j) \geq i + j \) if \( i \neq j \).
Theorem Q.Wang and Z.Wang

Let $\mathcal{X} = (X_i)_{i \in \mathbb{N}}$ be a family of discrete metric space with bounded geometry. Let $\Sigma = \bigsqcup_{i \in \mathbb{N}} X_i$ defined as before. Assume that:

- For any $\epsilon \in (0, \frac{1}{4})$ and positive numbers such that $\alpha_{X}(\epsilon) \cdot d \leq r$, there exists $d'$ with $d \leq d'$, such that $QI_{X, \max, \ast}(d, d', \epsilon, r)$ is holds.

- For some $\lambda > 1$ and any $\epsilon \in (0, \frac{1}{4\lambda})$, $r > 0$, there exists $d > 0$, $r' > r$ with $\alpha_{X}(\epsilon) \cdot d \leq r'$ such that $QS_{X, \max, \ast}(d, r, r', \epsilon, \lambda \epsilon)$.

Then $\Sigma$ satisfies the maximal coarse Baum-Connes conjecture.
Thank you!