Random Dynamic Systems with Switching and Applications

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This talk reports some of our recent findings involving joint work with

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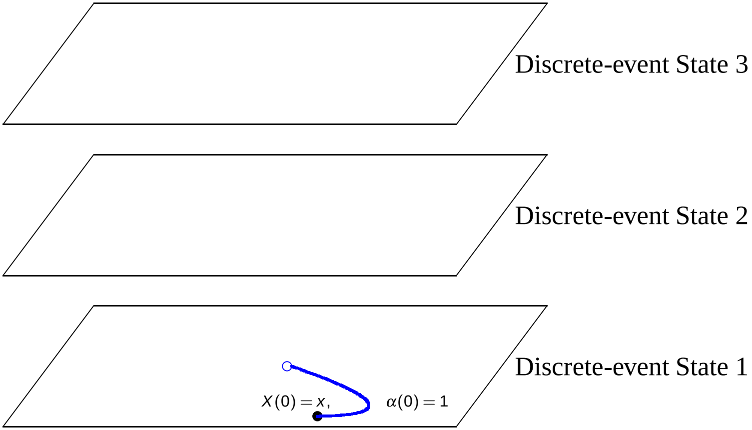
Outline

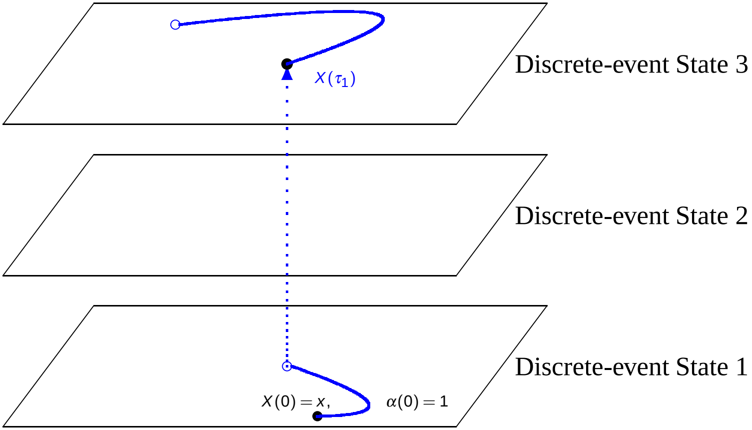
1. Switching Random Dynamic Systems
2. Switching Diffusion Examples
3. Properties
   - Recurrence
   - Ergodicity
   - Stability
4. Explosion Suppression & Stabilization
5. Numerical Approximations, Controlled Switching Diffusions, Games
6. Concluding Remarks

Switching Random Dynamic Systems

Switched Dynamic System: An Illustration



Switched Dynamic System: An Illustration



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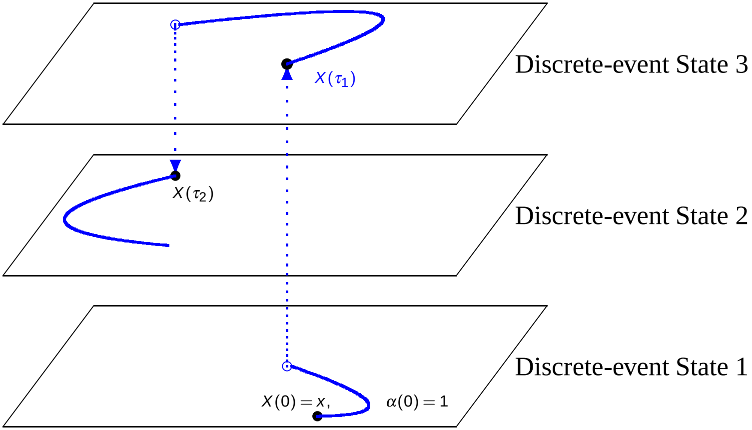

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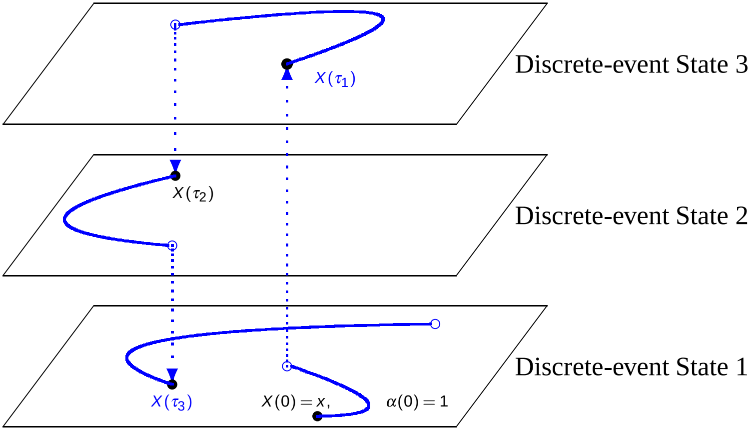

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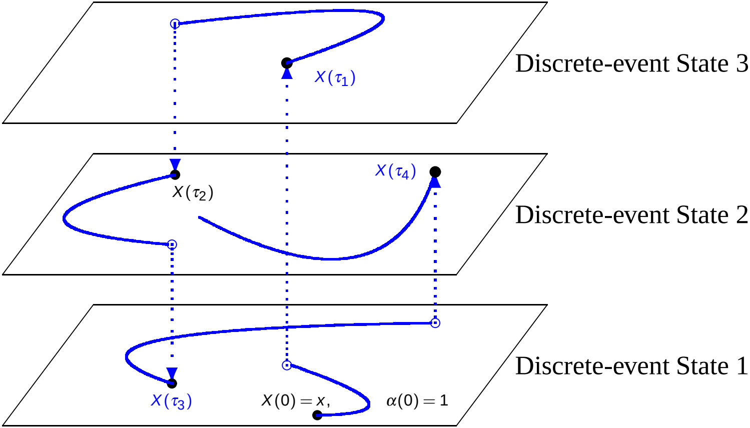


Figure: A “Sample Path” of A Switching Dynamic System $(X(t), \alpha(t))$.

Main Features

- continuous dynamics & discrete events coexist
- switching is used to model random environment or other random factors that cannot be formulated by the usual differential equations
- problems naturally arise in applications such as distributed, cooperative, and non-cooperative games, wireless communication, target tracking, reconfigurable sensor deployment, autonomous decision making, learning, etc.
- traditional ODE or SDE models are no longer adequate
- non-Gaussian distribution

Switching Diffusions

$\mathscr{M} = \{1, \ldots, m\}$
$\alpha(\cdot)$: taking values in $\mathscr{M}$.
$w(t)$: d-dimensional standard Brownian motion
$b(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r$
$\sigma(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r \times \mathbb{R}^d$

$$\begin{aligned} &dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t), \\ &X(0) = x,\ \alpha(0) = \alpha, \end{aligned} \tag{1}$$

$$\mathbf{P}\{\alpha(t+\Delta) = j | \alpha(t) = i, (X(s), \alpha(s)), s \le t\} = q_{ij}(X(t))\Delta + o(\Delta),\ i \ne j. \tag{2}$$

Formulation (cont.)

$Q(x) = (q_{ij}(x))$: generator associated with $\alpha(t)$ satisfying

$$q_{ij}(x) \geq 0, \text{ if } j \neq i, \text{ and } \sum_{j=1}^{m} q_{ij}(x) = 0, \quad i = 1, 2, \ldots, m$$

$\mathscr{L}$: generator of $(X(t), \alpha(t))$. For each $i \in \mathscr{M}$, and any $g(\cdot, i) \in C^2(\mathbb{R}^r)$,

$$\mathscr{L}g(x,i) = \frac{1}{2}\mathrm{tr}(a(x,i)\nabla^2 g(x,i)) + b'(x,i)\nabla g(x,i) + Q(x)g(x,\cdot)(i) \quad (3)$$

where

$\nabla g(\cdot, i)$ & $\nabla^2 g(\cdot, i)$: gradient & Hessian of $g(\cdot, i)$,

$a(x,i) = \sigma(x,i)\sigma'(x,i)$,

$$Q(x)g(x,\cdot)(i) = \sum_{j=1}^{m} q_{ij}(x) g(x,j).$$

Main Difficulty

- Consider $(X(t), \alpha(t))$ with two different initial data $(X(0), \alpha(0)) = (x, \alpha)$ & $(X(0), \alpha(0)) = (y, \alpha)$, $y \neq x$.

- Since $Q(x)$ depends on x, $\alpha^{x,\alpha}(t) \neq \alpha^{y,\alpha}(t)$ infinitely often even though $\alpha^{x,\alpha}(0) = \alpha^{y,\alpha}(0) = \alpha$.

Associated Poisson Measure

- $\Delta_{ij}(x)$: left closed, right open intervals of $\mathbb{R}$, with length $q_{ij}(x)$
- $h : \mathbb{R}^r \times \mathscr{M} \times \mathbb{R} \mapsto \mathbb{R}$:

$$h(x, i, z) = \sum_{j=1}^{m} (j - i) I_{\{z \in \Delta_{ij}(x)\}}. \tag{4}$$

$$d\alpha(t) = \int_{\mathbb{R}} h(X(t), \alpha(t-), z)\mathfrak{p}(dt, dz), \tag{5}$$

where

$\mathfrak{p}(dt, dz)$: a Poisson random measure with intensity $dt \times m(dz)$,
m: the Lebesgue measure on $\mathbb{R}$,
$\mathfrak{p}(\cdot, \cdot)$ independent of $w(\cdot)$.

Generalized Itô Lemma

If $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^r \times \mathscr{M})$, then for any $t \geq 0$:

$$\begin{aligned} V(t, X(t), \alpha(t)) =& V(0, X(0), \alpha(0)) \\ & + \int_0^t \left[\frac{\partial}{\partial s} + \mathscr{L}\right] V(s, X(s), \alpha(s)) ds + M_1(t) + M_2(t), \end{aligned} \tag{6}$$

where

$$\begin{aligned} M_1(t) &= \int_0^t \langle \nabla V(s-, X(s-), \alpha(s-)), \sigma(X(s-), \alpha(s-)) dw(s) \rangle. \\ M_2(t) &= \int_0^t \int_{\mathbb{R}} \big[V(s-, X(s-), \alpha(s-) + h(X(s-), \alpha(s-), z)) \\ & \qquad - V(s-, X(s-), \alpha(s-))\big] \mu(ds, dz), \end{aligned}$$

$\mu(ds, dz) = \mathfrak{p}(ds, dz) - ds \times m(dz)$ is a martingale measure.

An Example

Consider

$$\dot{x}(t) = A(\alpha(t))x(t) \tag{7}$$

where $\alpha(t)$ has two states $\{1, 2\}$,

$$A(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix},$$

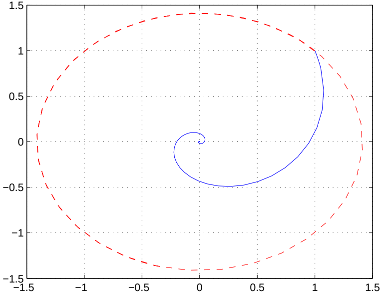
Associated with the hybrid system, there are two ODEs

$$\dot{x}(t) = A(1)x(t), \quad \text{and} \tag{8}$$

$$\dot{x}(t) = A(2)x(t) \tag{9}$$

switching back and forth according to $\alpha(t)$.

Phase Portrait of the Components



Phase portraits of the ‘component’ with a center (in dashed line) and the ‘component’ with a stable node (in solid line) with the same initial condition $x_0 = [1, 1]'$

Phase Portrait of Hybrid System

The phase portrait is given below.

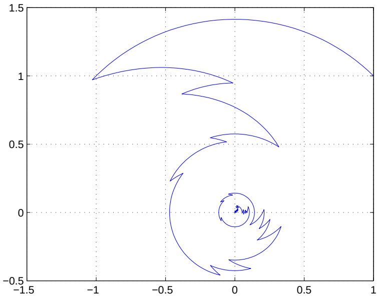


Figure: Switching linear system: Phase portrait of (7) with $x_0 = [1, 1]'$.

Switching ODEs

This example belongs to a more general class of hybrid systems:

$$\dot{x}(t) = f(x(t), \alpha(t)), \; x(0) = x, \; \alpha(0) = \alpha, \tag{10}$$

and $Q(x)$ is an x-dependent generator.

By using the Liapunov exponent, we can also obtain necessary and sufficient conditions for stability and instability.

- Some new results different from the usual Hartman-Grobman theorem

with Zhu & Song, Quarterly Appl. Math (2009)

Seemingly Not Much Different from Diffusions without Switching?

Q: When we have a coupled system with $\mathscr{M} = \{1, 2\}$ and two stable linear systems, do we always get a stable system?

Seemingly Not Much Different from Diffusions without Switching?

Q: When we have a coupled system with $\mathscr{M} = \{1,2\}$ and two stable linear systems, do we always get a stable system?

Consider $\dot{x} = A(\alpha(t))x + B(\alpha(t))u(t)$, and a state feedback $u(t) = K(\alpha(t))x(t)$. Then one gets

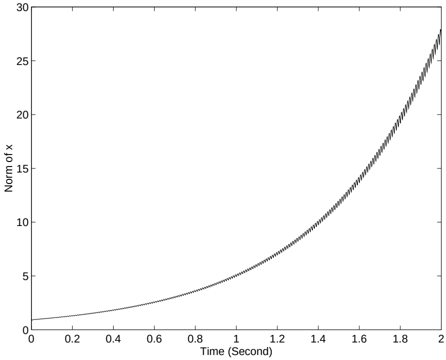
$$\dot{x} = [A(\alpha(t)) - B(\alpha(t))K(\alpha(t))]x.$$

Suppose that $\alpha(t) \in \{1,2\}$ such that

$$A(1) - B(1)K(1) = \begin{bmatrix} -100 & 20 \\ 200 & -100 \end{bmatrix}, \; A(2) - B(2)K(2) = \begin{bmatrix} -100 & 200 \\ 20 & -100 \end{bmatrix}.$$

The two feedback systems are stable individually. But if we choose $\alpha(t)$ so that it switches at $k\eta$, where $\eta = 0.01$. Then the resulting system is unstable.

The hybrid system is unstable



[L.Y. Wang, P.P. Khargonecker, and A. Beydoun, 1999, deterministic switching system]

Why is the system unstable?

$$\frac{1}{2}[A(1) - B(1)K(1) + A(2) - B(2)K(2)] = \frac{1}{2}\begin{bmatrix} -200 & 220 \\ 220 & -200 \end{bmatrix}$$

is an unstable matrix.

The averaging effect dominates the dynamics.

with Zhang, Springer book, 2nd Ed. 2013

- Consider a system

$$\dot{x}^{\varepsilon}(t) = b(x^{\varepsilon}(t), \alpha^{\varepsilon}(t)), \ \alpha^{\varepsilon}(t) \sim Q/\varepsilon \tag{11}$$

- each $\dot{x}(t) = b(x(t), i), \ i \in \mathscr{M}$ is stable.
- Q irreducible
- $x^{\varepsilon}(\cdot) \Rightarrow x(\cdot)$ such that

$$\dot{x}(t) = \overline{b}(x(t)), \ \overline{b}(x) = \sum_{i \in \mathscr{M}} \nu_i b(x, i). \tag{12}$$

- System (12) is unstable.
- Use perturbed Liapunov function to show that (11) is unstable.

Regime-switching Diffusion Examples

Average Cost Per Unit Time Problem

Consider a controlled switching diffusion $(X(t), \alpha(t))$ (drift and diffusion coefficients also depend on a control u).

Aim: find $u^*(\cdot)$ so

$$\lim_{T\to\infty} E\frac{1}{T}\int_0^T L(X(t),\alpha(t),u(t))dt$$

is minimized.

Questions: Does there exist an ergodic measure? If yes, can we replace the instantaneous measure by the ergodic one?

Two-time-scale Markov Chains

- Two-time-scale Markov chain $\alpha(t)$ with $\varepsilon > 0$ small,

$$Q(t) = Q^{\varepsilon}(t) = \frac{\widetilde{Q}(t)}{\varepsilon} + \widehat{Q}(t). \tag{13}$$

  - $\widetilde{Q}(t)$, $\widehat{Q}(t)$ are generators of Markov chains.
  - $\widetilde{Q}(t) = \mathrm{diag}(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t))$ nearly decomposable
  - $\mathscr{M} = \mathscr{M}_1 \cup \cdots \cup \mathscr{M}_l$; $\mathscr{M}_i = \{s_{i1}, \ldots, s_{im_i}\}$
- Consider the scaled sequence

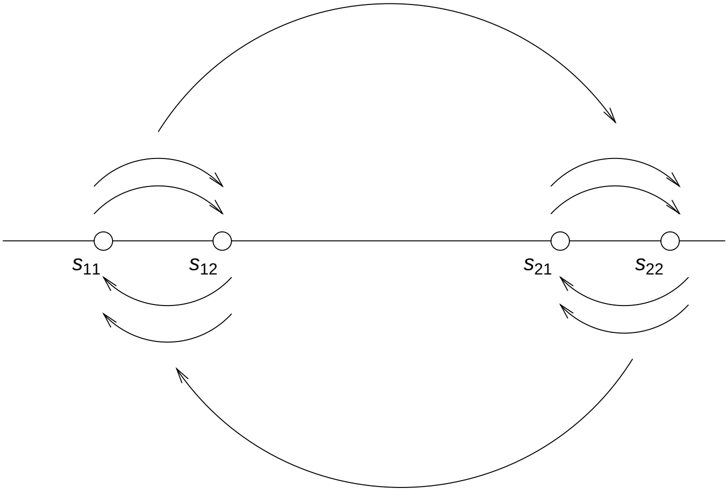
$$\frac{1}{\sqrt{\varepsilon}} \int_0^t I_{\{\alpha^{\varepsilon}(u)=s_{ij}\}} - \nu_j^i(u) I_{\{\alpha^{\varepsilon}(u)\in\mathscr{M}_i\}} du.$$

- Limit: switching diffusion.

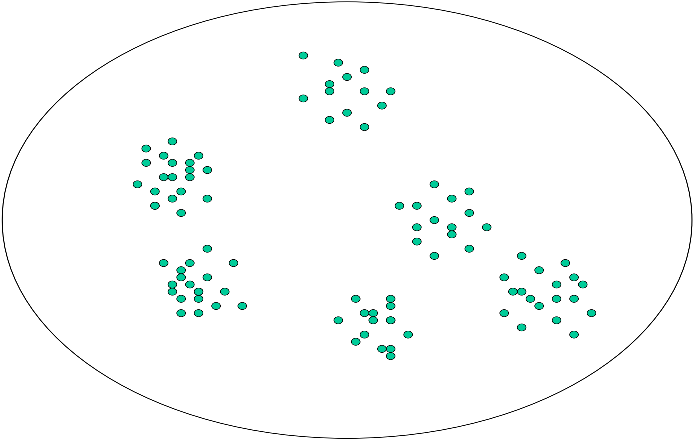
with Badowski, Zhang, Ann. Appl. Probab. (2000)

with Zhang, Ann. Appl. Probab. (2007)

Two-time Scale (a demonstration)



Aggregation for Large-scale Systems



Consensus: Flocking

Consensus: Schooling (Couzin et.al. Nature, 2005)

Consensus Problems: High Way Traffic

Consensus: Honeybee Organization (Visscher, Nature, 2003)

Consensus Issues

- multi-agent coordination
- a group objective (e.g., alignment during motion, UAVs formation)
- to maintain shared information
- some kind of agreement such as objective of operation or a condition for proceeding to further operation

- Our work: Stochastic recursive algorithm, topology switching, multi-scale systems

with L.Y. Wang and Y. Sun, SIAM MMS, Automatica (2011)

Mean-Field Model

- $\alpha(t)$: with $\mathscr{M} = \{1, 2, \ldots, m_0\}$.
- Consider an ℓ-body mean-field model For $i = 1, 2, \ldots, \ell$,

$$\begin{aligned} dX_i(t) = & \left[\gamma(\alpha(t))X_i(t) - X_i^3(t) - \beta(\alpha(t))(X_i(t) - \overline{X}(t))\right]dt \\ & + \sigma_{ii}(X(t), \alpha(t))dw_i(t), \\ \overline{X}(t) = & \ \frac{1}{\ell}\sum_{j=1}^{\ell} X_j(t), \\ X(t) = & \ (X_1(t), X_2(t), \ldots, X_\ell(t))', \end{aligned} \tag{14}$$

$\gamma(i) > 0$ and $\beta(i) > 0$ for $i \in \mathscr{M}$.

- Originated from statistical mechanics, mean-field models are concerned with many-body systems with interactions. To overcome the difficulty of interactions due to the many bodies, one of the main ideas is to replace all interactions to any one body with an average or effective interaction.

with F. Xi, J. Appl. Probab. (2009)

Insurance Risk Models

The surplus at time t:

$$S(t,x,i) = x + \int_0^t c(\alpha(s))ds - \sum_{j=1}^{N(t)} X_j(\alpha(T_j)),$$

- (x,i): initial (surplus, regime);
- $c(i)$: premium rate;
- $X_j(i)$: claim size;
- T_j: claim time;
- $N(t)$: Poisson process.
- $\alpha(t)$ is used to model:
  - El Nino/La Nina phenomena in property ins.
  - economic condition in unemployment policy
  - certain epidemics in health insurance

Stock Price Models

- Stock market models
  - $S(t)$: stock price
  - $w(\cdot)$: stand Brownian motion
  - μ: return (appreciation) rate
  - σ: volatility
- traditional GBM model is given by

$$dS(t) = \mu S(t)dt + \sigma S(t)dw.$$

- Regime-switching market models

$$dS(t) = \mu(\alpha(t))S(t)dt + \sigma(\alpha(t))S(t)dw.$$

  - both the return rate & volatility depend on $\alpha(t)$
  - $\alpha(\cdot)$ and $w(\cdot)$ are independent
  - $\alpha(t)$: market mode, investor's mode, & other economic factors (e.g., bull, bear)

with X.Y. Zhou, *SIAM J. Control Optim.* (2003), IEEE T-AC, (2004)
with Bensoussan and Yan (2012), SIAM J. Fin.

Properties

Regularity & Recurrence

Definition

Regularity. A Markov process $Y^{x,\alpha}(t) = (X^{x,\alpha}(t), \alpha^{x,\alpha}(t))$ is said to be *regular*, if for any $0 < T < \infty$,

$$\mathbf{P}\{\sup_{0 \le t \le T} |X^{x,\alpha}(t)| = \infty\} = 0. \tag{15}$$

Remark

Let $\beta_n := \inf\{t : |X^{x,\alpha}(t)| = n\}$. Then $\{\beta_n\}$ is monotonically increasing and hence has a (finite or infinite) limit. It follows that the process is regular iff

$$\beta_n \to \infty \text{ almost surely as } n \to \infty. \tag{16}$$

Definition

(i) *Recurrence.* For $U := D \times J$, where $J \subset \mathcal{M}$ and $D \subset \mathbb{R}^r$ is an open set with compact closure, let $\sigma_U^{x,\alpha} = \inf\{t : Y^{x,\alpha}(t) \in U\}$. A regular process $Y^{x,\alpha}(\cdot)$ is *recurrent* w.r.t. U if

$$\mathbf{P}\{\sigma_U^{x,\alpha} < \infty\} = 1 \text{ for any } (x, \alpha) \in D^c \times \mathcal{M}.$$

(ii) *Positive and Null Recurrence.* A recurrent process with finite mean recurrence time for some set $U = D \times J$ is said to be *positive recurrent* w.r.t. U; otherwise, the process is *null recurrent* w.r.t. U.

Recurrence Is Independent of Sets

(i) The process $(X(t), \alpha(t))$ is (positive) recurrent w.r.t. $D \times \mathscr{M}$ if and only if it is (positive) recurrent w.r.t. $D \times \{\ell\}$, where $D \subset \mathbb{R}^r$ is a bounded open set with compact closure and $\ell \in \mathscr{M}$.

(ii) If the process $(X(t), \alpha(t))$ is (positive) recurrent w.r.t. some $U = D \times \mathscr{M}$, where $D \subset \mathbb{R}^r$, then it is (positive) recurrent w.r.t. $\widetilde{U} = \widetilde{D} \times \mathscr{M}$, where $\widetilde{D} \subset \mathbb{R}^r$ is any nonempty open set.

Positive Recurrence (1)

Theorem

A necessary and sufficient condition for positive recurrence with respect to a domain $U = D \times \{\ell\} \subset \mathbb{R}^r \times \mathcal{M}$ *is: For each* $i \in \mathcal{M}$, *there exists a nonnegative function* $V(\cdot, i) : D^c \mapsto \mathbb{R}$ *s.t.* $V(\cdot, i)$ *is twice continuously differentiable and that*

$$\mathcal{L}V(x,i) = -1, \quad (x,i) \in D^c \times \mathcal{M}. \tag{17}$$

Let $u(x,i) = \mathbf{E}_{x,i}\sigma_D$. *It is the* *smallest positive sol'n* *to*

$$\begin{cases} \mathcal{L}u(x,i) = -1, & (x,i) \in D^c \times \mathcal{M}, \\ u(x,i) = 0, & (x,i) \in \partial D \times \mathcal{M}. \end{cases} \tag{18}$$

with Zhu SIAM J. Control Optim. (2007), (2009)

Step 1: Positive recurrence. Show the process is positive recurrent if *exists* $V(\cdot,\cdot)$ (≥ 0) satisfying the conditions of the theorem.

- Fix any $(x,i) \in D^c \times \mathscr{M}$ and set $\sigma_D^{(n)}(t) = \min\{\sigma_D, t, \beta_n\}$. Dynkin's formula implies

$$\begin{aligned}\mathbf{E}_{x,i} V(X(\sigma_D^{(n)}(t)), \alpha(\sigma_D^{(n)}(t))) - V(x,i) &= \mathbf{E}_{x,i} \int_0^{\sigma_D^{(n)}(t)} \mathscr{L} V(X(s), \alpha(s)) ds \\ &= -\mathbf{E}_{x,i} \sigma_D^{(n)}(t).\end{aligned}$$

- Since $V(\cdot)$ is nonnegative,

$$\mathbf{E}_{x,i} \sigma_D^{(n)}(t) \leq V(x,i).$$

- Letting $n \to \infty$ and $t \to \infty$, $\mathbf{E}_{x,i} \sigma_D < \infty$. This is positive recurrence.

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Step 2: Show $u(x,i) := \mathbf{E}_{x,i}\sigma_D < \infty$ is the smallest positive solution of the BVP (18).

- Set $\sigma_D^{(n)} = \min\{\sigma_D, \beta_n\}$ & $u_n(x,i) = \mathbf{E}_{x,i}\sigma_D^{(n)}$. Then $u_n(x,i)$ solves

$$\mathscr{L}u_n(x,i) = -1,\ u_n(x,i)|_{x\in\partial D} = 0\ u_n(x,i)|_{|x|=n} = 0.$$

- $v_n(x,i) := u_{n+1}(x,i) - u_n(x,i)$ is $\mathscr{L}$-harmonic in $(D^c \cap \{|x| < n\}) \times \mathscr{M}$.
- $\mathbf{E}_{x,i}\sigma_D^{(n)} \nearrow \mathbf{E}_{x,i}\sigma_D$ by regularity and DCT. Hence we can write

$$u(x,i) = u_{n_0}(x,i) + \sum_{k=n_0}^{\infty} v_k(x,i).$$

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Step 2: Show $u(x,i) := \mathbf{E}_{x,i}\sigma_D < \infty$ is the smallest positive solution of the BVP (18).

- Set $\sigma_D^{(n)} = \min\{\sigma_D, \beta_n\}$ & $u_n(x,i) = \mathbf{E}_{x,i}\sigma_D^{(n)}$. Then $u_n(x,i)$ solves

$$\mathscr{L}u_n(x,i) = -1, \; u_n(x,i)|_{x\in\partial D} = 0 \; u_n(x,i)|_{|x|=n} = 0.$$

- $v_n(x,i) := u_{n+1}(x,i) - u_n(x,i)$ is $\mathscr{L}$-harmonic in $(D^c \cap \{|x| < n\}) \times \mathscr{M}$.
- $\mathbf{E}_{x,i}\sigma_D^{(n)} \nearrow \mathbf{E}_{x,i}\sigma_D$ by regularity and DCT. Hence we can write

$$u(x,i) = u_{n_0}(x,i) + \sum_{k=n_0}^{\infty} v_k(x,i).$$

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Step 3: Show: If $Y(t) = (X(t), \alpha(t))$ is positive recurrent w.r.t. $U = D \times \{\ell\}$, then $\exists\, V$ satisfying $V \geq 0$ and the conditions of the theorem.

- The positive recurrence implies $\mathbf{E}_{x,i}\sigma_D < \infty$ for all $(x,i) \in D^c \times \mathscr{M}$. Noting $\sigma_D^{(n)} \leq \sigma_D^{(n+1)}$, Harnack's theorem for $\mathscr{L}$-elliptic systems implies that the bounded monotone increasing sequence $u_n(x,i)$ converges uniformly on every compact subset of $D^c \times \mathscr{M}$. Moreover, its limit $u(x,i)$ satisfies $\mathscr{L}u(x,i) = -1$ for every $i \in \mathscr{M}$.
- The function $V(x,i) := u(x,i)$ satisfies the required condition.

Ergodicity

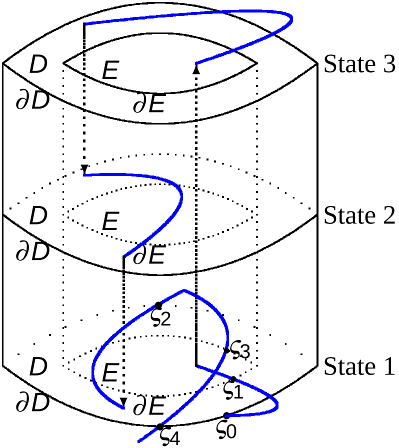

Figure 2: Cycles of $Y(t) = (X(t), \alpha(t))$; $m = 3$ & $\ell = 1$

Cycles

- Assume the process is positive recurrent w.r.t. $U = E \times \{\ell\}$; $E \subset \mathbb{R}^r$ and $\ell \in \mathscr{M}$ are fixed from now on.
- Let ∂E be sufficiently smooth. Let $D \subset \mathbb{R}^r$ be a bdd. ball with suff. smooth ∂D s.t. $E \cup \partial E \subset D$.
- Let $\varsigma_0 = 0$ and then define for $n = 0, 1, \ldots$

$$\begin{aligned}
\varsigma_{2n+1} &= \inf\{t \geq \varsigma_{2n} : (X(t), \alpha(t)) \in \partial E \times \{\ell\}\},\\
\varsigma_{2n+2} &= \inf\{t \geq \varsigma_{2n+1} : (X(t), \alpha(t)) \in \partial D \times \{\ell\}\}.
\end{aligned}$$

Then we can divide an arbitrary sample path of the process into cycles:

$$[\varsigma_0, \varsigma_2), [\varsigma_2, \varsigma_4), \ldots, [\varsigma_{2n}, \varsigma_{2n+2}) \cdots. \tag{19}$$

- Assume $Y(0) = (X(0), \alpha(0)) = (x, \ell) \in \partial D \times \{\ell\}$.
- Define $Y_n = Y(\varsigma_{2n}) = (X_n, \ell), n = 0, 1, \ldots$ It is a MC on $\partial D \times \{\ell\}$ by strong Markov property

Theorem

A positive recurrent process $(X(t), \alpha(t))$ has a unique stationary distribution $\widehat{\nu}(\cdot,\cdot) = (\widehat{\nu}(\cdot, i) : i \in \mathscr{M})$.

Strong Law of Large Numbers

Theorem

Denote by $\mu(\cdot,\cdot)$ *the stationary density associated with* $\widehat{v}(\cdot,\cdot)$ *and* $f(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}$ *is Borel measurable such that*

$$\sum_{i=1}^{m_0} \int_{\mathbb{R}^r} |f(x,i)|\mu(x,i)dx < \infty. \tag{20}$$

Then for any $(x,i) \in \mathbb{R}^r \times \mathscr{M}$

$$\mathbf{P}_{x,i}\left(\frac{1}{T}\int_0^T f(X(t),\alpha(t))dt \to \overline{f}\right) = 1, \tag{21}$$

where $\overline{f} = \sum_{i=1}^{m_0} \int_{\mathbb{R}^r} f(x,i)\mu(x,i)dx.$

Cauchy Problem

Let the assumptions of the last theorem be satisfied, and $u(t,x,i)$ be the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,x,i)}{\partial t} &= \mathscr{L}u(x,i),\ i \in \mathscr{M}, \\ u(0,x,i) &= f(x,i). \end{cases} \tag{22}$$

Then as $T \to \infty$,

$$\frac{1}{T}\int_0^T u(t,x,i)dt \to \sum_{i=1}^{m_0} \int_{\mathbb{R}^r} f(x,i)\mu(x,i)dx. \tag{23}$$

A key to establish this result is the result of law of large numbers.

Stability

Definitions

The equilibrium point $x = 0$ of system is:

(i) *stable in probability*, if for any $r > 0$, $\lim_{x\to 0} \mathbf{P}\{\sup_{t\geq 0} |X^{x,\alpha}(t)| > r\} = 0$; otherwise $x = 0$ is *unstable in probability*.

(ii) *asymptotically stable in probability*, if it is stable in prob. & $\lim_{x\to 0} \mathbf{P}\{\lim_{t\to\infty} X^{x,\alpha}(t) = 0\} = 1$.

(iii) *p-stable* (*for $p > 0$*), if $\lim_{\delta\to 0} \sup_{|x|\leq\delta,\alpha\in\mathscr{M},t\geq 0} \mathbf{E}|X^{x,\alpha}(t)|^p = 0$.

(iv) *asymptotically p-stable*, if it is p-stable & $\mathbf{E}|X^{x,\alpha}(t)|^p \to 0$ as $t\to\infty$.

(v) *exponentially p-stable*, if for some $K, k > 0$, $\mathbf{E}|X^{x,\alpha}(t)|^p \leq K|x|^p \exp\{-kt\}$, for any $\alpha\in\mathscr{M}$.

Necessary Conditions

Theorem

$x = 0$ is exponentially p-stable iff b and σ have continuous bdd. derivatives w.r.t. x up to the 2nd order. Then for $i \in \mathscr{M}$, $\exists$ $V(\cdot,i) : \mathbb{R}^r \mapsto \mathbb{R}$ s.t.

$$\begin{aligned} & k_1|x|^p \le V(x,i) \le k_2|x|^p, \quad x \in N, \\ & \mathscr{L}V(x,i) \le -k_3|x|^p \quad \textit{for all} \quad x \in N - \{0\}, \\ & \left|\frac{\partial V}{\partial x_j}(x,i)\right| < k_4|x|^{p-1}, \ \left|\frac{\partial^2 V}{\partial x_j \partial x_k}(x,i)\right| < k_4|x|^{p-2}, \end{aligned} \tag{24}$$

for all $1 \le j,k \le n$, $x \in N - \{0\}$, and for some $k_i > 0$ ($i = 1,2,3,4$), where N is a neighborhood of 0.

with Khasminskii & Zhu, Stochastic Proc. Appl. (2007)
with Mao & Yuan, Automatica (2007); (Markov switching)
with Xi SIAM J. Control Optim. (2010)

Linearized Systems

- For each $i \in \mathscr{M}$, $\exists\, b(i)$, $\sigma_j(i) \in \mathbb{R}^{r\times r}$, $j = 1,\ldots,d$, and $\widehat{Q} = (\widehat{q}_{ij})$ s.t.

$$\left.\begin{array}{l} b(x,i) = b(i)x + o(|x|), \\ \sigma(x,i) = (\sigma_1(i)x,\ldots,\sigma_d(i)x) + o(|x|), \\ Q(x) = \widehat{Q} + o(1), \end{array}\right\} \text{ as } x \to 0 \qquad (25)$$

  $\widehat{Q}$ is an irreducible generator of a Markov chain $\widehat{\alpha}(t)$. Use $\pi = (\pi_1,\ldots,\pi_m) \in \mathbb{R}^{1\times m}$ to denote the stationary dist. associated with $\widehat{Q}$.

Easily Verifiable Conditions

Theorem

Suppose that $\dfrac{\sigma_j'(i)+\sigma_j(i)}{2} \geq 0.$

(a) *Then* $x = 0$ (i) *is asymptotically stable in prob. if*

$$\sum_{i=1}^{m} \pi_i \left(\Lambda_{b(i)} + \frac{1}{2} \sum_{j=1}^{d} [\Lambda_{a_j(i)} - 2(\lambda_{\sigma_j(i)})^2] \right) < 0 \tag{26}$$

and (ii) *is unstable if*

$$\sum_{i=1}^{m} \pi_i \left(\lambda_{b(i)} + \frac{1}{2} \sum_{j=1}^{d} [\lambda_{a_j(i)} - 2(\Lambda_{\sigma_j(i)})^2] \right) > 0, \tag{27}$$

where Λ_A *and* λ_A *denote the max. and min. eigenvalue of* $\dfrac{1}{2}(A+A')$, *resp,*

(b) *If* $X(t)$ *is 1-d, then* $x = 0$ *is (i) asymptotically stable in probab if* $\sum_{i=1}^{m} \pi_i \left(b_i - \frac{\sigma_i^2}{2} \right) < 0$, *& (ii) unstable in probab if* $\sum_{i=1}^{m} \pi_i \left(b_i - \frac{\sigma_i^2}{2} \right) > 0$.

Idea of Proof

We only consider (i).
$\mu = (\mu_1, \ldots, \mu_m)' \in \mathbb{R}^m$ with $\mu_i = \Lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d} \Lambda_{a_j(i)}$. Let $\beta := -\pi\mu > 0$.
Then that $\widehat{Q}c = \mu + \beta \mathbb{1}$ has a soln. $c = (c_1, \ldots, c_m)' \in \mathbb{R}^m$.

Consider $V(x,i) = (1 - \gamma c_i)|x|^\gamma$, where $0 < \gamma < 1$ is suff. small s.t. $1 - \gamma c_i > 0$, $i \in \mathscr{M}$. $V(\cdot, i)$ is continuous, nonnegative, & vanishes only at $x = 0$. $\mathscr{L}V(x,i) < 0$ for any $(x,i) \in (N - \{0\}) \times \mathscr{M}$, where $N \subset \mathbb{R}^r$ is a small neighborhood of 0. Then we can show that 0 is asymptotically stable.

Closing the “Gap"

Consider ‘linear system’ with $Q(x) \equiv Q$, and define $Y(t) = X(t)/|X(t)|$. Itô’s formula implies that

$$dY(t) = \Phi(Y(t),\alpha(t))dt + \Psi(Y(t),\alpha(t))dw(t),$$

where $w(t) = (w_1(t),\ldots,w_d(t))' \in \mathbb{R}^d$ with $w_k(t), k = 1,\ldots,d$ being indep. 1-dim. Brownian motions and Φ,Ψ are appropriate functions. We can represent $\ln|X(t)|$ in terms of $(Y(t),\alpha(t))$.

Denote the stationary density of $(Y(t),\alpha(t))$ by $\mu(y,i), i \in \mathscr{M}$

$$\rho_0 = \sum_{i=1}^{m} \int_{\mathbb{S}} \Big[y'b(i)y + \frac{1}{2}\sum_{k=1}^{d}(|\sigma_k(i)y|^2 - 2|y'\sigma_k(i)y|^2) \Big] \mu(y,i)dy.$$

Then asymptotically stable if $\rho_0 < 0$ and unstable if $\rho_0 > 0$.

Explosion Suppression & Stabilization

Regularity Criterion (cont.)

Theorem

Suppose that $b(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^r$ *and that* $\sigma(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^{r\times d}$,

$$\begin{aligned} &dX(t)=b(X(t),\alpha(t))dt+\sigma(X(t),\alpha(t))dw(t),\ (X(0),\alpha(0))=(x,\alpha),\\ &P\{\alpha(t+\delta)=j|\alpha(t)=i,X(s),\alpha(s),s\le t\}=q_{ij}(X(t))\delta+o(\delta),\ i\ne j. \end{aligned} \tag{28}$$

Suppose that for each $i\in\mathcal{M}$, *both* $b(\cdot,i)$ *and* $\sigma(\cdot,i)$ *are local linear growth and local Lipschitzian and that* $\exists$ *a nonnegative* $V(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^+$ *that is* C^2 *in* $x\in\mathbb{R}^r$ *for each* $i\in\mathcal{M}$ *s.t.* $\exists\gamma_0>0$

$$\begin{aligned} &\mathcal{L}V(x,i)\le\gamma_0V(x,i),\ \textit{for all}\ (x,i)\in\mathbb{R}^r\times\mathcal{M},\\ &V_R:=\inf_{|x|\ge R,\ i\in\mathcal{M}}V(x,i)\to\infty\ \textit{ as }\ R\to\infty. \end{aligned} \tag{29}$$

Then the process $(X(t),\alpha(t))$ *is regular.*

Explosion Suppression

$x \in \mathbb{R}^r$

$f(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r$

$\alpha(t) \in \mathscr{M} = \{1, \ldots, m\}$

$$\frac{dX(t)}{dt} = f(X(t), \alpha(t)) \tag{30}$$

$f(\cdot, i)$ continuous but the growth rate is faster than linear

We wish to stabilize (30).

Motivational Example

- Consider an even simpler problem: the logistic system

$$\dot{x}(t) = x(t)(1 + x(t)), \; x(0) = 1.$$

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  Two things are needed:
  1) extend to a global solution;
  2) stabilization.

What have been done?

- Khasminskii's book (1981): stabilize 2-d system with two white noise
- Arnold (1972): $\dot{x} = Ax$ can be stabilized by zero mean stationary process iff $\mathrm{tr}(A) < 0$
- Mao (1994) established a general stabilization results of Brownian noise under linear growth condition.
- Wu & Hu (2009) treated one-sided growth condition
- Mao, Yin, and Yuan (2007): showed that both Brownian motion and Markov Chain can be used to stabilize systems.

Motivation (diffusion case)

$$dx = \mu x dt + \sigma x dw, \; x(0) = x_0.$$

$$x(t) = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t)\right).$$

$$\text{when} \;\; \sigma^2 > 2\mu,$$

$$\limsup_t \frac{\log|x(t)|}{t} \le \left(\mu - \frac{\sigma^2}{2}\right) < 0.$$

This implies exponential stability.

How to Get a Global Solution? Stablization?

- add a diffusion perturbation

$$dX(t) = f(X(t), \alpha(t))dt + a_1(\alpha(t))|X(t)|^{\beta} X(t)dw_1(t)$$

  such that $2\beta - \beta_1 > 0$, where $w_1(\cdot)$ is scalar Brownian motion.
- add another diffusion to get stability

$$\begin{aligned} dX(t) &= f(X(t), \alpha(t))dt + a_1(\alpha(t))|X(t)|^{\beta} X(t)dw_1(t) \\ &\quad + a_2(\alpha(t))X(t)dw_2(t), \end{aligned} \tag{31}$$

  where $w_2(\cdot)$ is a scalar Brownian motion independent of $w_1(\cdot)$.
- More general,

$$dX(t) = f(X(t), \alpha(t))dt + \sigma_1(X(t), \alpha(t))dw_1 + \sigma_2(X(t), \alpha(t))dw_2. \tag{32}$$

Results

- With proper choice of the perturbations, we get a global solution
- $\limsup_{t\to\infty} P(|X(t)| \geq K_\delta) \leq \delta$
- The resulting system is stable w.p.1. In fact, $\limsup_t \log|X(t)|/t < 0$ w.p.1.

with Wu and Zhao, SIAM J. Appl. Math (2012)

Example

Begin with (30) together with initial condition $X(0)=1$. Suppose that $\alpha(t)$ is a Markov chain with two states $\mathscr{M}=\{1,2\}$ and $Q=\begin{pmatrix} -0.1 & 0.1 \\ 1 & -1 \end{pmatrix}$, $f(x,1)=x(x+1)$ and $f(x,2)=x(2x+1)$. Corresponding to the states, we have two equations

$$\begin{aligned} \frac{d}{dt}X(t) &= X(t)(X(t)+1), \\ \frac{d}{dt}X(t) &= X(t)(2X(t)+1). \end{aligned} \tag{33}$$

Neither equation has a global soln. For the 1st equation, we have $X(t)=e^t/(2-e^t)$ that will blow up at time $\ln 2$; for the second equation, $X(t)=e^t/(3-2e^t)$ that will blow up at time $\ln(3/2)$. We plot the trajectories of the switched system as well as each individual system.

To regularize the system, use a feedback control $a_1(\alpha(t))X^2(t)dw_1(t)$, where $w_1(t)$ is a 1-d Brownian motion. The resulting eq is

$$dX(t) = f(X(t), \alpha(t))dt + a_1(\alpha(t))X^2(t)dw_1(t), \tag{34}$$

$a_1(i) = 2$ for $i = 1, 2$.
Although the system has a global solution, it is not asymptotically stable. To stabilize the system, we add another feedback control $a_2(\alpha(t))X(t)dw_2(t)$, $w_2(t)$ is 1-d standard Brownian motion independent of $w_1(t)$ and $a_2(1) = 19$ and $a_2(2) = 24$.

$$dX(t) = f(X(t), \alpha(t))dt + a_1(\alpha(t))X^2(t)dw_1(t) + a_2(\alpha(t))X(t)dw_2(t). \tag{35}$$

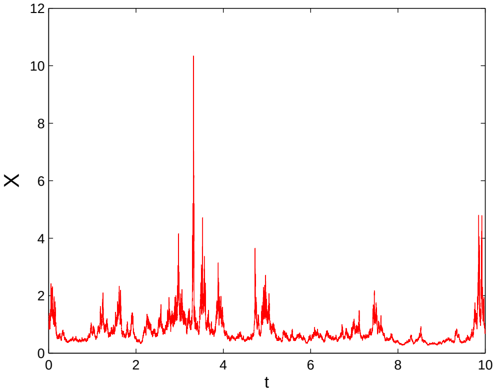


Figure: Trajectory of system (34) with stepsize $\Delta t = 10^{-4}$.

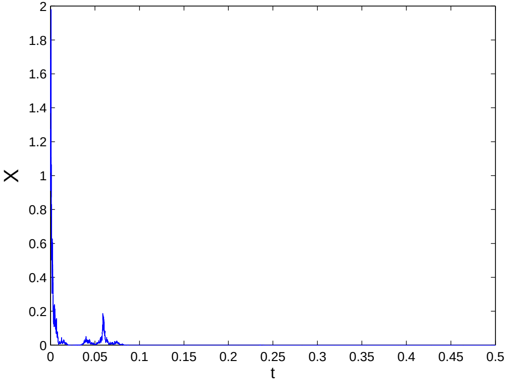
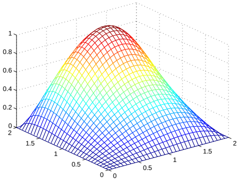


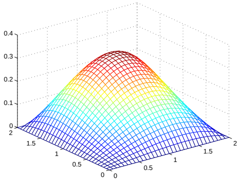
Figure: Trajectory of system (35) with stepsize $\Delta t = 10^{-6}$.

Numerical Approximations, Controlled Switching Diffusions, Games

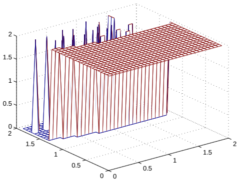
Numerical Methods for SDE, Controls, and Games



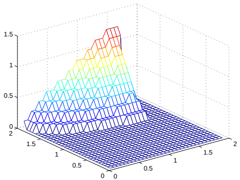
(a) $V^{h,+}(\cdot,\cdot,1)$



(b) $V^{h,+}(\cdot,\cdot,2)$



(c) $U_1(\cdot,\cdot,1)$: player1 1st



(d) $U_1(\cdot,\cdot,2)$ player1 1st

Numerics for Controlled Switching Diffusions

$$\begin{cases} X(t) = x + \int_0^t b(X(s), \alpha(s), u(s))ds + \int_0^t \sigma(X(s), \alpha(s))dw, \\ \alpha(t) \ \text{ continuous-time MC } \ \alpha(0) = i, \end{cases} \tag{36}$$

where $w(t)$ is a standard Brownian motion independent of the Markov chain $\alpha(t)$.

- Kushner & Dupuis, Springer, Markov chain approximation
- with Song & Zhang (2006), regime-switching & jump diffusion

Controlled Switching Diffusions (cont.)

Given $B > 0$, define a stopping time as

$$\tau_B^{x,i,u} = \inf\{t : X^{x,i,u}(t) \notin (-B, B)\}.$$

Objective: choose control $u.$ to minimize the expected cost function

$$\begin{cases} J_i^B(x,u) & = \mathbf{E}\displaystyle\int_0^{\tau_B^{x,i,u}} f(X(s),\alpha(s),u(s))ds, \\ & \qquad \forall x \in (-B,B),\ i \in \mathscr{M}, \\ J_i^B(x,u) & = 0,\ \forall x \notin (-B,B),\ i \in \mathscr{M}, \end{cases} \tag{37}$$

where for each $i \in \mathscr{M}$, $f(\cdot, i, \cdot)$ is an appropriate function representing the running cost function.

For each $i \in \mathscr{M}$, the value function is given by

$$V^B(x,i) = \inf_{u \in \mathscr{U}} J^B(x,i,u), \tag{38}$$

where $\mathscr{U}$ is the space of all $\mathscr{F}_t$-adapted controls taking values on a compact set U.

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Formally, the value functions satisfy Hamilton-Jacobi-Bellman (HJB) equations,

$$\begin{cases} \inf_{u \in U} \{L^u V^B(x,i) + f(x,i,u)\} = 0, & \forall x \in (-B,B), i \in \mathscr{M}, \\ V^B(x,i) = 0, & \forall x \notin (-B,B), i \in \mathscr{M}, \end{cases} \tag{39}$$

where

$$L^u \varphi(x,i) = \frac{1}{2}\sigma^2(x,i)\frac{d^2\varphi(x,i)}{dx^2} + b(x,i,u)\frac{d\varphi(x,i)}{dx} + \sum_{j \in \mathscr{M}} q_{ij}\varphi(x,j).$$

Algorithm

- $h > 0$: discretization parameter.
- $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \dots\}$. Let $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathscr{M}$
- $p^h((x,i),(y,j)|u)$: transition probabilities from $(x,i) \in S_h \times \mathscr{M}$ to $(y,j) \in S_h \times \mathscr{M}$, for $u \in U$.

Then, $\bar{V}^{B,h}(x,i)$, the discretization of $V^B(x,i)$ with step size $h>0$, is the solution of

$$\begin{cases} \inf\limits_{u\in U}\{L_h^u \bar{V}^{B,h}(x,i)+f(x,i,u)\}=0, & \forall x\in(-B,B)_h,\ i\in\mathscr{M}, \\ \bar{V}^{B,h}(x,i)=0, & \forall x\notin(-B,B)_h,\ i\in\mathscr{M}, \end{cases} \tag{40}$$

where

$$(-B,B)_h=(-B,B)\cap S_h,\ \ [-B,B]_h=(-B,B)_h\cup\{B,-B\}. \tag{41}$$

$$\begin{aligned} \bar{V}^{B,h}(x,i) &= \inf_{u\in U}\Big\{\bar{p}_i^{h,+}(x,u)\bar{V}^{B,h}(x+h,i)+\bar{p}_i^{h,-}(x,u)\bar{V}^{B,h}(x-h,i) \\ &\qquad +\sum_{j\neq i}\bar{p}_{ij}^{h}(x)\bar{V}^{B,h}(x,j)+f(x,i,u)\Delta\bar{t}_i^h(x)\Big\} \end{aligned} \tag{42}$$

Rates of Convergence

Theorem

Under suitable conditions, $\exists \gamma \in (2,3]$ *and* $\rho \in (0,1]$ *s.t. the Markov chain approximation algorithm converges at the rate* $(\gamma - 2) \wedge \rho \wedge \frac{1}{2}$*. That is,*

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \le K h^{\frac{1}{2} \wedge \rho \wedge (\gamma - 2)}, \quad \forall (i,x) \in \mathcal{M} \times G.$$

- Note that $\gamma \in (2,3]$ comes from Markov chain $\approx$, ρ is the Hölder exponent of the cost function.
- PDE approach for controlled diffusions (finite difference approx of PDEs)
  - Menaldi, SIAM J. Control Optim. (1989)
  - Krylov, Probab. Theory Related Fields, (2000)
  - Dong & N.V. Krylov, Appl. Math Optim.
- we use probabilistic approach for controlled switching diffusions
  - with Q.S. Song, SIAM J. Control Optim. (2009)

Main Ideas

- Use relaxed controls (measures)
- Construct strong approximation
- Consider boundary perturbations
  - usual notion of cost $J_i(x, \widetilde{m})$;
  - ours $J_i^{B}(x, \widetilde{m})$

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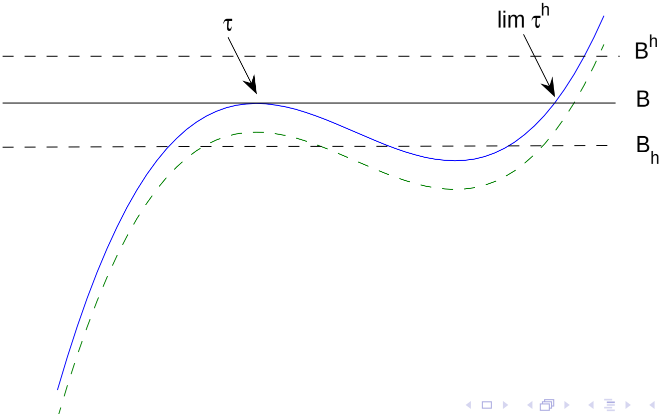
Tangency Problem

- τ and τ^h: the first hitting time of $X(t)$ and $x^h(t)$ to the boundary.
- Objective: $\approx \mathbf{E}\tau$ by $\mathbf{E}\tau^h$
- In the Figure, $\tau^h \not\to \tau$, even though $x^h(\cdot)$ converges to $X(\cdot)$.
- Q: extra conditions needed?



Concluding Remarks

In this talk, we

- presented several switching diffusion examples
- considered such properties as recurrence, ergodicity, stability etc.
- developed numerical algorithms for control and game problems
- ascertained rates of convergence and treated tangency problems

Further work:

- rates of convergence for games
- large deviations
- null-recurrent switching diffusion systems ...

- Past dependent switching and countable switching space:

$$\mathbf{P}\{\alpha(t+\Delta)=j|\alpha(t)=i,X_s,\alpha(s),s\le t\}=q_{ij}(X_t)\Delta+o(\Delta) \text{ if } i\neq j$$
$$\mathbf{P}\{\alpha(t+\Delta)=i|\alpha(t)=i,X_s,\alpha(s),s\le t\}=1-q_i(X_t)\Delta+o(\Delta),$$
$$q_i(\phi)=\sum_{j\neq i}q_{ij}(\phi) \text{ for any}(\phi,i)\in\mathscr{C}\times\mathbb{Z}_+.$$

With D. Nguyen, SIAM J. Control Optim. (2016), Potential Anal. (2017)

- Switching jump diffusion:

$$\begin{aligned}\mathscr{L}f(x) &= \sum_{k,l=1}^{r} a_{kl}(x)\frac{\partial^2 f(x)}{\partial x_k\partial x_l}+\sum_{k=1}^{r} b_k(x)\frac{\partial f(x)}{\partial x_k}\\ &\quad+\int_{\mathbb{R}^r}\big(f(x+z)-f(x)-\nabla f(x)\cdot z\mathbf{1}_{\{|z|<1\}}\big)\,\pi(x,dz).\end{aligned}$$

with Chen, Chen, Tran, Bernoulli (2018), Appl. Math Optim. (2018)

Thank you