Connes Character Formula for locally compact spectral triples

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In the beginning was the formula

The following assertion appears (without a proof) in the "Noncommutative Geometry" book.

Theorem (Character Formula)

If $(\mathcal{A}, H, D)$ is $p-$dimensional compact spectral triple, then

$$\fint \Omega(c) = \mathrm{Ch}(c)$$

for every Hochschild cycle $c \in \mathcal{A}^{\otimes(p+1)}$.

Here, $\mathrm{Ch}(c)$ is a Hochschild cocycle representing the Chern class and $\fint$ is the noncommutative integral. $\Omega(c)$ is the $0-$th order differential expression (to be defined later).

Connes advertised this formula as "local expression for the Hochschild class of Chern character". This formula was never proved by Connes; a number of authors attempted to prove it with various degrees of success.

General information

Let $\mathcal{L}_\infty$ be the $*-$algebra of all bounded operators on a given (separable, infinite dimensional) Hilbert space H.

An operator is called compact if it can be approximated (in norm topology, with any given precision) by a finite rank operator. Spectrum of a self-adjoint compact operator consists of non-zero eigenvalues of finite multiplicity converging to 0 (which may or may not be an eigenvalue).

If A is compact, then $|A|$ is compact. Eigenvalues of $|A|$ are called singular values of A.

For a compact operator A, we define its singular value sequence $\mu(A) = (\mu(k, A))_{k\geq 0}$ by arranging the eigenvalues of $|A|$ in the decreasing order and taking them with multiplicities.

Ideals and infinitesimals

An ideal $\mathcal{I}$ in $\mathcal{L}_\infty$ is a linear subspace (usually *not* closed in norm) such that $A \in \mathcal{I}$ and $B \in \mathcal{L}_\infty$ implies that $AB, BA \in \mathcal{I}$. Ideal is called principal if it is generated by a single element. Every non-trivial ideal in $\mathcal{L}_\infty$ consists of compact operators.

Principal ideal generated by the diagonal operator $\mathrm{diag}((\frac{1}{(k+1)^{1/p}})_{k\geq 0})$ is called $\mathcal{L}_{p,\infty}$. For every $p > 0$, it is quasi-Banach (see next page). Equivalently,

$$\mathcal{L}_{p,\infty} = \Big\{A \in \mathcal{L}_\infty : \ \mu(k, A) = O((k+1)^{-\frac{1}{p}})\Big\}.$$

In Connes ideology, these are "infinitesimals of order $\frac{1}{p}$".

Quasi-Banach ideals

Definition

An ideal $\mathcal{I}$ in $\mathcal{L}_\infty$ is called quasi-Banach when equipped with a complete quasi-norm $\|\cdot\|_{\mathcal{I}}$ such that

$$\|AB\|_{\mathcal{I}}, \|BA\|_{\mathcal{I}} \leq \|A\|_{\mathcal{I}}\|B\|_\infty.$$

For example, a natural quasi-norm on the ideal $\mathcal{L}_{p,\infty}$ is given by the formula

$$\|A\|_{p,\infty} = \sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,A).$$

When equipped with this quasi-norm, $\mathcal{L}_{p,\infty}$ becomes a quasi-Banach ideal. In fact, for $p>1$ its natural quasi-norm is equivalent to a norm.

Traces on ideals

Definition

Let $\mathcal{I}$ be an ideal in $\mathcal{L}_\infty$. Linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ is called trace if

$$\varphi(AB) = \varphi(BA), \quad A \in \mathcal{I}, \quad B \in \mathcal{L}_\infty.$$

Equivalently, $\varphi(U^{-1}AU) = \varphi(A)$ for all $A \in \mathcal{I}$ and for all unitary $U \in \mathcal{L}_\infty$. For $p > 1$, ideal $\mathcal{L}_{p,\infty}$ does not carry any trace. For $p = 1$, there is a plethora of traces. The most famous one is due to Dixmier.

Definition

Let ω be a free ultrafilter on $\mathbb{Z}_+$. The mapping

$$\mathrm{Tr}_\omega : A \to \lim_{n \to \omega} \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu(k, A), \quad 0 \le A \in \mathcal{L}_{1,\infty}$$

is additive. Its linear extension to $\mathcal{L}_{1,\infty}$ is called Dixmier trace.

Further properties of traces

1. Every Dixmier trace is positive.
2. Every positive trace on $\mathcal{L}_{1,\infty}$ is continuous.
3. Every continuous trace on $\mathcal{L}_{1,\infty}$ is a linear combination of positive ones.
4. There are continuous traces on $\mathcal{L}_{1,\infty}$ which are not Dixmier traces.
5. There are traces on $\mathcal{L}_{1,\infty}$ which fail to be continuous.
6. There are $2^{2^{\mathbb{N}}}$ continuous traces on $\mathcal{L}_{1,\infty}$.
7. Every trace on $\mathcal{L}_{1,\infty}$ vanishes on $\mathcal{L}_1$.

Spectral triples

In Connes ideology, spectral triples are noncommutative Riemannian manifolds.

Definition

Let $\mathcal{A}$ be a $*-$algebra represented on a given Hilbert space H. Let D be an unbounded self-adjoint operator acting on H such that $\partial(a) = [D, a]$ is bounded for all $a \in \mathcal{A}$. If $a(D+i)^{-1}$ is compact for all $a \in \mathcal{A}$, then $(\mathcal{A}, H, D)$ is called spectral triple. It is called

1. compact if $1 \in \mathcal{A}$, so that $(D+i)^{-1}$ is compact operator; locally compact otherwise.
2. $p-$dimensional if $a(D+i)^{-p}$ and $\partial(a)(D+i)^{-p}$ are in $\mathcal{L}_{1,\infty}$.
3. even if equipped with unitary $\Gamma = \Gamma^* : H \to H$ such that $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$, $D\Gamma + \Gamma D = 0$. odd if such Γ is not provided.

From spectral triples to Fredholm modules

In Connes ideology, Fredholm modules are noncommutative manifolds with conformal structure.

Definition

Let $\mathcal{A}$ be a $*-$algebra represented on a given Hilbert space H. Let $F = F^* \in \mathcal{L}_\infty$ be unitary such that $[F, a]$ is compact for all $a \in \mathcal{A}$. It is called $p-$dimensional if $[F, a] \in \mathcal{L}_{1,\infty}$.

Theorem

If $(\mathcal{A}, H, D)$ is $p-$dimensional spectral triple, then $(\mathcal{A}, H, F)$ (with $F = \mathrm{sgn}(D)$) is a $p-$dimensional Fredholm module.

Noncommutative integral

Definition

Let $(\mathcal{A}, H, D)$ be a $p-$dimensional spectral triple. Let φ be a trace on $\mathcal{L}_{1,\infty}$. For $X \in \mathcal{L}_\infty$, noncommutative integral is defined by the formula

$$⨍ X = \varphi(X(1+D^2)^{-\frac{p}{2}}).$$

There is a huge choice of traces on $\mathcal{L}_{1,\infty}$ and no reasonable way to chose one. We need to specify the class of elements for which noncommutative integral is well defined.

Definition

An element $X \in \mathcal{L}_\infty$ (typically, $X \in \mathcal{A}$ or like) is measurable if

$$\varphi(X(1+D^2)^{-\frac{p}{2}})$$

is well defined and does not depend on the choice of φ.

Hochschild cycles

Elements of $\mathcal{A}^{\otimes n}$ are called $n-$chains.
The Hochschild boundary operator $b : A^{\otimes(n+1)} \to A^{\otimes n}$ is defined on elementary tensors $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ by:

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n +$$

$$+ \sum_{k=1}^{n-1} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n +$$

$$+ (-1)^n a_n a_0 \otimes a_1 \cdots \otimes a_{n-1}.$$

We say that chain c is a Hochschild cycle if $bc = 0$.

Hochschild cocycles

Multilinear functionals on $\mathcal{A}^{\otimes n}$ are called $n-$cochains.
The Hochschild coboundary operator is defined as follows: if $\theta : A^{\otimes n} \to \mathbb{C}$, then

$$(b\theta)(a_0 \otimes \cdots \otimes a_n) = \theta(a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n) +$$

$$+\sum_{k=1}^{n-1}(-1)^k \theta(a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes a_{k+2} \otimes \cdots \otimes a_n) +$$

$$+(-1)^n \theta(a_n a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}).$$

We say that cochain θ is a Hochschild cocycle if $b\theta = 0$.

Chern character

For $a = a_0 \otimes \cdots \otimes a_p \in \mathcal{A}^{\otimes(p+1)}$, we have

$$\prod_{k=0}^{p}[F, a_k] \in \prod_{k=0}^{p} \mathcal{L}_{p,\infty} = \mathcal{L}_{\frac{p}{p+1},\infty} \subset \mathcal{L}_1.$$

Definition

Define a $(p+1)-$cochain by setting

$$\mathrm{Ch}(a_0 \otimes \cdots \otimes a_p) = \frac{1}{2}\mathrm{Tr}(\Gamma F \prod_{k=0}^{p}[F, a_k]).$$

This is a Hochschild cocycle called Chern character.

Main hypothesis

We require that

1. spectral triple is infinitely smooth.
2. spectral triple is $p-$dimensional, that is, for all $a \in \mathcal{A}$

$$a(D+i)^{-p}, \partial(a)(D+i)^{-p} \in \mathcal{L}_{1,\infty}.$$

3. for all $a \in \mathcal{A}$

$$\left\| a(D+i\lambda)^{-p-1} \right\|_1, \left\| \partial(a)(D+i\lambda)^{-p-1} \right\|_1 = O(\lambda^{-1}).$$

Statement of the main result

Set

$$\Omega(a_0 \otimes \cdots \otimes a_p) = \Gamma a_0 \prod_{k=0}^{p} [D, a_k].$$

Theorem

Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying the hypothesis. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a (local!) Hochschild cycle, then

$$\varphi(\Omega(c)(1 + D^2)^{-\frac{p}{2}}) = \mathrm{Ch}(c)$$

for every trace φ on $\mathcal{L}_{1,\infty}$.

Intermediate result 1: heat semigroup asymptotic

Theorem

Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying the hypothesis. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a Hochschild cycle, then

$$\mathrm{Tr}(\Omega(c)(1+D^2)^{1-\frac{p}{2}}e^{-s^2D^2}) = \frac{p}{2}\mathrm{Ch}(c)s^{-2} + O(s^{-1}), \quad s \downarrow 0.$$

Proof is long and mostly combinatorial.

Intermediate result 2: analyticity of $\zeta-$function

Theorem

Let $(\mathcal{A}, H, D)$ be a spectral triple satisfying hypothesis. If $c \in \mathcal{A}^{\otimes(p+1)}$ is a Hochschild cycle, then the function

$$z \to \mathrm{Tr}(\Omega(c)(1+D^2)^{-\frac{z}{2}}), \quad \Re(z) > p, \tag{1}$$

is holomorphic and has analytic continuation to the set $\{\Re(z) > p-1\} \setminus \{p\}$. The point $z = p$ is a simple pole with residue $p\mathrm{Ch}(c)$.

Proof.

It follows from the formula

$$(1+D^2)^{-\frac{z}{2}} = \frac{1}{\Gamma(\frac{z}{2})} \int_0^\infty s^{z-1} e^{-s^2(1+D^2)} ds.$$

□

Intermediate result 3: analyticity of $\zeta-$function

The core component of the proof is:

Theorem

If $a \in \mathcal{A}$, then the function

$$(1+D^2)^{-\frac{z}{2}} a^{2z} - (a(1+D^2)^{-\frac{1}{2}} a)^z$$

is $\mathcal{L}_1-$valued analytic for $\Re(z) > p-1$.

Intermediate result 4: criterion for measurability

Theorem

If $0 \leq V \in \mathcal{L}_{1,\infty}$ *and* $A \in \mathcal{L}_\infty$ *are such that*

$$z \to \mathrm{Tr}(AV^{1+z})$$

is analytic for $\Re(z) > 1 - \epsilon$ *(except at* 0, *where it has simple pole), then*

$$\varphi(AV) = \mathrm{Res}_{z=0}\mathrm{Tr}(AV^{1+z})$$

for every trace φ *on* $\mathcal{L}_{1,\infty}$.

End of the proof

Suppose c is local, that is $(a \otimes 1^{\otimes p}) = c$. Then $a\Omega(c) = \Omega(c)$ and, moreover, $a^{2z}\Omega(c) = \Omega(c)$. We have

$$z \to \mathrm{Tr}(\Omega(c)(1 + D^2)^{-\frac{z}{2}} a^{2z})$$

is analytic for $\Re(z) > p - 1$ (with simple pole at $z = p$). Hence,

$$z \to \mathrm{Tr}(\Omega(c)(a(1 + D^2)^{-\frac{1}{2}} a)^z)$$

is analytic for $\Re(z) > p - 1$ (with simple pole at $z = p$). Computing the residue, we complete the proof.