FOCK SPACE: A BRIDGE BETWEEN FREDHOLM INDEX AND THE QUANTUM HALL EFFECT

Jingbo Xia

Department of Mathematics, SUNY Buffalo

Joint work with Guo Chuan Thiang

The Fock space in this talk is the one-variable Fock space.

The Gaussian measure $d\mu$ on $\mathbf{C}$ is defined by the formula

$$d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dA(z).$$

The Fock space $\mathcal{F}^2$ is the closure of $\mathbf{C}[z]$ in $L^2(\mathbf{C}, d\mu)$. Let

$$P : L^2(\mathbf{C}, d\mu) \to \mathcal{F}^2$$

be the orthogonal projection.

Given an $f \in L^\infty(\mathbf{C})$, we have the Toeplitz operator

$$T_f h = P(fh), \quad h \in \mathcal{F}^2.$$

Connection with physics:

Consider an electron confined to a plane with a perpendicular magnetic field **B** of uniform strength. Under the right choice of orientation, identify the plane with **C**, the complex plane. We have the free Hamiltonian

$$H_b = \left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2$$

representing this system, where $b = e|\mathbf{B}|/\hbar c > 0$.

This H_b is in symmetric gauge. One can change gauge by unitary transformation.

When the Fermi energy E is in a spectral gap of H_b, we have the Fermi projection $P_{\leq E} = \chi_{(-\infty,E]}(H_b)$.

Let f_1 and f_2 be switch functions (to be explicitly given below) in the x and y directions respectively. It is known that the expression

$$\sigma_{\mathrm{Hall}}(P_{\leq E}) = -i\mathrm{tr}(P_{\leq E}[[M_{f_1}, P_{\leq E}], [M_{f_2}, P_{\leq E}]])$$

is the Kubo formula for the Hall conductance of $P_{\leq E}$, provided that the trace on the right-hand side makes sense.

With $\delta_i(A) = [M_{f_i}, A]$, we can rewrite the operator inside $\mathrm{tr}(\cdots)$ in the form

$$P_{\leq E}[\delta_1(P_{\leq E}), \delta_2(P_{\leq E})],$$

which is the Chern character of the projection $P_{\leq E}$.

The Hamiltonian H_b can be explicitly diagonalized.

Straightforward calculation shows that

$$\left(\frac{1}{i}\frac{\partial}{\partial x}+\frac{b}{2}y\right)^2+\left(\frac{1}{i}\frac{\partial}{\partial y}-\frac{b}{2}x\right)^2-b=4(-\partial+(b/4)\bar{z})(\bar{\partial}+(b/4)z).$$

Denote

$$\tilde{A}=2\bar{\partial}+(b/2)z \quad \text{and} \quad \tilde{C}=-2\partial+(b/2)\bar{z}.$$

Then

$$H_b-b=\tilde{C}\tilde{A}.$$

$\tilde{A}$, $\tilde{C}$ are called annihilation and creation operators.

By a standard exercise using the canonical commutation relation (CCR), $\tilde{A}$, $\tilde{C}$ give us an explicit diagonalization of H_b.

First of all, $[\tilde{A}, \tilde{C}] = 2b$. Define

$$\mathcal{S}_0 = \mathrm{span}\{z^k e^{-(b/4)|z|^2} : k = 0, 1, 2, \dots\}.$$

Then $\tilde{A}\mathcal{S}_0 = \{0\}$. For each $j \in \mathbf{N}$, define

$$\mathcal{S}_j = \tilde{C}^j \mathcal{S}_0.$$

The relation $[\tilde{A}, \tilde{C}] = 2b$ implies $[\tilde{A}, \tilde{C}^j] = 2bj\tilde{C}^{j-1}$ for $j \geq 1$. Let $\varphi \in \mathcal{S}_j$ for some $j \geq 0$. Then there is a $\psi \in \mathcal{S}_0$ such that $\varphi = \tilde{C}^j \psi$. Therefore

$$(H_b - b)\varphi = \tilde{C}\tilde{A}\varphi = \tilde{C}\tilde{A}\tilde{C}^j\psi = \tilde{C}[\tilde{A}, \tilde{C}^j]\psi = 2bj\tilde{C}^j\psi = 2bj\varphi.$$

That is,

$$(2.1) \qquad \mathcal{S}_j \subset \ker(H_b - (2j+1)b) \quad \text{for every} \quad j \geq 0.$$

Since H_b is self-adjoint, this means that $\mathcal{S}_i \perp \mathcal{S}_j$ for $i \neq j$.

For each $j \geq 0$, let $\mathcal{E}_j$ be the closure of $\mathcal{S}_j$ in $L^2(\mathbf{C}, dA)$.

Now define $U : L^2(\mathbf{C}, dA) \to L^2(\mathbf{C}, d\mu)$ by the formula

$$(U\psi)(z) = (2/b)^{1/2}\psi((2/b)^{1/2}z)e^{|z|^2/2}, \quad \psi \in L^2(\mathbf{C}, dA).$$

Then U is a unitary operator. For each $j \geq 0$, define

$$(2.2) \qquad \mathcal{F}_j = U\mathcal{E}_j.$$

Obviously, we have $\mathcal{F}_0 = \mathcal{F}^2$, which is the classic Fock space.

The spaces $\mathcal{F}_j$, $j \geq 1$, are the higher Fock spaces.

Now define

$$A = \bar{\partial} \quad \text{and} \quad C = -\partial + \bar{z},$$

which safisfy the commutation relation $[A, C] = 1$.

It is easy to verify that $\langle Cu, v\rangle = \langle u, Av\rangle$ for all $u, v \in \mathbf{C}[z, \bar{z}]$, where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbf{C}, d\mu)$. That is, over $\mathbf{C}[z, \bar{z}]$, C and A are each-other's adjoint.

It is also easy to verify that

$$U\tilde{A}U^* = \sqrt{2b}A \quad \text{and} \quad U\tilde{C}U^* = \sqrt{2b}C.$$

Consequently,

$$UH_bU^* - b = 2bCA.$$

It is easy to see that $U\mathcal{S}_0 = \mathbf{C}[z]$ and that $U\mathcal{S}_j = C^j\mathbf{C}[z]$ for every $j \geq 1$. Thus by an induction on the power of $\bar{z}$, we have

$$U\bigcup_{j=0}^{\infty}\mathcal{S}_j = \mathbf{C}[z,\bar{z}]. \tag{2.3}$$

This shows that $\cup_{j=0}^{\infty}\mathcal{S}_j$ is dense in $L^2(\mathbf{C}, dA)$.

For each $j \geq 0$, let $E_j : L^2(\mathbf{C}, dA) \to \mathcal{E}_j$ be the orthogonal projection. Combining (2.1) with (2.3), we see that

$$H_b = \sum_{j=0}^{\infty}(2j+1)bE_j.$$

This is an explicit diagonalization of the magnetic Hamiltonian H_b.

Summarizing the above, the spectrum of H_b consists of evenly-spaced eigenvalues $(2j+1)b$ called Landau levels, $j \geq 0$.

Since $\dim(\mathcal{E}_j) = \infty$, in physics jargon, each Landau level is said to be infinitely degenerate.

The j-th Landau level eigenspace $\mathcal{E}_j$ is identified with the j-th higher Fock space $\mathcal{F}_j$ under the unitary U.

We emphasize that

$$\bigoplus_{j=0}^{\infty} \mathcal{F}_j = L^2(\mathbf{C}, d\mu).$$

Let $P_j : L^2(\mathbf{C}, d\mu) \to \mathcal{F}_j$ be the orthogonal projection, $j \geq 0$. Then, of course, $P_0 = P$. It follows from (2.2) that

$$P_j = UE_jU^* \quad \text{for every} \quad j \geq 0.$$

For each $j \geq 0$ and $p \in \mathbf{C}[z]$, we have

$$\|C^j p\|^2 = \langle C^j p, C^j p\rangle = \langle A^j C^j p, p\rangle = j!\langle p, p\rangle = j!\|p\|^2.$$

Hence, for a given $j \geq 0$, if we define

$$V_j u = \frac{1}{\sqrt{j!}} C^j P u \quad \text{for} \quad u \in \mathbf{C}[z, \bar{z}],$$

then V_j extends to a partial isometry on $L^2(\mathbf{C}, d\mu)$ such that

$$V_j^* V_j = P = P_0 \quad \text{and} \quad V_j V_j^* = P_j.$$

Let $f \in L^\infty(\mathbf{C})$. On each $\mathcal{F}_j$, $j \geq 0$, we define the "Toeplitz operator"

$$T_{f,j}h = P_j(fh), \quad h \in \mathcal{F}_j.$$

Then $T_{f,0}$ is the classic Toeplitz operator T_f.

Given an integer (Landau level) $\ell \geq 0$, we define

$$P^{(\ell)} = \sum_{j=0}^{\ell} P_j.$$

We define "Toeplitz operator" of another kind:

$$T_f^{(\ell)}h = P^{(\ell)}(fh), \quad h \in \bigoplus_{j=0}^{\ell} \mathcal{F}_j.$$

It is easy to verify that

$$[T_f^{(\ell)}, T_g^{(\ell)}] = P^{(\ell)}[[M_f, P^{(\ell)}], [M_g, P^{(\ell)}]].$$

When the Fermi energy E is strictly between $(2\ell + 1)b$ and $(2\ell + 3)b$, if it happens that both products

$$[M_f, P^{(\ell)}][M_g, P^{(\ell)}], \quad [M_g, P^{(\ell)}][M_f, P^{(\ell)}]$$

are in the trace class, then the Kubo formula for Hall conductance reads

$$\sigma_{\mathrm{Hall}}(P_{\leq E}) = -i\mathrm{tr}[T_f^{(\ell)}, T_g^{(\ell)}].$$

This brings us to our familiar territory.

For each $z \in \mathbf{C}$, the function

$$k_z(\zeta) = e^{-|z|^2/2} e^{\zeta \bar{z}}$$

is the normalized reproducing kernel for the Fock space $\mathcal{F}^2$. We can represent the projection $P : L^2(\mathbf{C}, d\mu) \to \mathcal{F}^2$ in the form

$$P = \frac{1}{\pi} \int_{\mathbf{C}} k_z \otimes k_z dA(z).$$

Define $\Gamma = \{m + in : m, n \in \mathbf{Z}\}$ and $Q = \{x + iy : x, y \in [0, 1)\}$. Then

$$P = \sum_{u \in \Gamma} \frac{1}{\pi} \int_{Q+u} k_z \otimes k_z dA(z) = \frac{1}{\pi} \int_Q G_z dA(z),$$

where

$$G_z = \sum_{u \in \Gamma} k_{u+z} \otimes k_{u+z}, \quad z \in Q.$$

Easy calculation shows that

$$(C^j k_z)(\zeta) = (\bar{\zeta} - \bar{z})^j k_z(\zeta) \quad \text{for all} \;\; j \geq 0 \;\; \text{and} \;\; z \in \mathbf{C}.$$

We now define

$$k_z^{(j)}(\zeta) = (C^j k_z)(\zeta) = (\bar{\zeta} - \bar{z})^j k_z(\zeta),$$

$j \geq 0$ and $z \in \mathbf{C}$. For $j \geq 0$, the projection $P_j : L^2(\mathbf{C}, d\mu) \to \mathcal{F}_j$ has the representation

$$P_j = \frac{1}{j!\pi} \int_Q G_{z,j} dA(z),$$

where

$$G_{z,j} = \sum_{u \in \Gamma} k_{u+z}^{(j)} \otimes k_{u+z}^{(j)}, \quad z \in Q.$$

Thus each $\mathcal{F}_j$ has properties similar to $\mathcal{F}_0 = \mathcal{F}^2$.

Definition 3.5. Let $0 < a < \infty$. Then Σ_a denotes the collection of measurable functions η on $\mathbf{R}$ satisfying the following three conditions:
(1) $0 \leq \eta(x) \leq 1$ for every $x \in \mathbf{R}$.
(2) $\eta(x) = 1$ if $x > a$.
(3) $\eta(x) = 0$ if $x < -a$.

We use elements of Σ_a to construct "switch functions", functions that will appear in Kubo's formula.

Take an $a > 0$ and pick $\eta, \xi \in \Sigma_a$. Also, fix a $0 < \theta < \pi$. Define

$$f_1(\zeta) = \eta(\mathrm{Re}(\zeta)) \quad \text{and} \quad f_2(\zeta) = \xi(\mathrm{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$.

Proposition 3.7. For all $j \geq 0$ and $k \geq 0$, the operator

$$[M_{f_1}, P_j][M_{f_2}, P_k]$$

is in the trace class. Consequently, for all $j \geq 0$ and $\ell \geq 0$, the commutators

$$[T_{f_1,j}, T_{f_2,j}] \quad \text{and} \quad [T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}]$$

are in the trace class.

But note that individually, the commutators

$$[M_{f_1}, P_j] \quad \text{and} \quad [M_{f_2}, P_k]$$

are not even compact.

Proposition 3.9. Let $s, t \in \mathbf{R}$ be such that $s < t < s + \pi$. Define the wedge

$$W = \{re^{ix} : s \leq x \leq t \textit{ and } r \geq 0\}$$

in $\mathbf{C}$. Suppose that $ie^{i\theta}\mathbf{R} \cap e^{is}\mathbf{R} = \{0\}$ and $ie^{i\theta}\mathbf{R} \cap e^{it}\mathbf{R} = \{0\}$. Then for all $j \geq 0$, $k \geq 0$ and $i \in \{1, 2\}$, the operator

$$[M_{\chi_W}, P_j][M_{f_i}, P_k]$$

is in the trace class.

Define the function

$$F = f_1 + if_2$$

on $\mathbf{C}$. Also, define the square

$$S = \{x + iy : x, y \in [0, 1]\}.$$

Theorem 4.1. On the space $\mathcal{F}_j$, $j \geq 0$, the essential spectrum of the Toeplitz operator $T_{F,j}$ is contained in ∂S, the boundary of the square S.

Proof. (1) First, we show that the essential spectrum of $T_{F,j}$ is contained in S. For this we consider the Calkin algebra

$$\mathcal{Q} = \mathcal{B}(\mathcal{F}_j)/\mathcal{K}.$$

For each $X \in \mathcal{B}(\mathcal{F}_j)$, denote its image in $\mathcal{Q}$ by $\hat{X}$.

Note that $T_{f_1,j}$ and $T_{f_2,j}$ are self-adjoint with spectra contained in $[0,1]$. Therefore $\hat{T}_{f_1,j}$ and $\hat{T}_{f_2,j}$ are also self-adjoint with spectra contained in $[0,1]$.

By Proposition 3.7, $[T_{f_1,j}, T_{f_2,j}]$ is in the trace class, which implies $[\hat{T}_{f_1,j}, \hat{T}_{f_2,j}] = 0$. That is, $\hat{T}_{F,j} = \hat{T}_{f_1,j} + i\hat{T}_{f_2,j}$ is a normal element in $\mathcal{Q}$. By the GNS representation of $\mathcal{Q}$, the spectrum of $\hat{T}_{F,j}$ is contained in S, which is equivalent to saying that the essential spectrum of $T_{F,j}$ is contained in S.

(2) We now show that the interior of S does not intersect the essential spectrum of $T_{F,j}$. Define the following four wedges in $\mathbf{C}$:

$$\begin{aligned}
\mathcal{A} &= \{re^{ix} : \theta/2 \leq x \leq (\pi+\theta)/2 \text{ and } r \geq 0\}, \\
\mathcal{B} &= \{re^{ix} : (\pi+\theta)/2 \leq x \leq \pi + (\theta/2) \text{ and } r \geq 0\}, \\
\mathcal{C} &= \{re^{ix} : \pi + (\theta/2) \leq x \leq (3\pi+\theta)/2 \text{ and } r \geq 0\} \quad \text{and} \\
\mathcal{D} &= \{re^{ix} : (3\pi+\theta)/2 \leq x \leq 2\pi + (\theta/2) \text{ and } r \geq 0\}.
\end{aligned}$$

Using Proposition 3.9, it is easy to verify that

$$\begin{aligned}
T_{f_2,j} T_{\chi_{\mathcal{A}},j} &= T_{\chi_{\mathcal{A}},j} + K_{\mathcal{A}}, \\
T_{f_1,j} T_{\chi_{\mathcal{B}},j} &= K_{\mathcal{B}}, \\
T_{f_2,j} T_{\chi_{\mathcal{C}},j} &= K_{\mathcal{C}} \quad \text{and} \\
T_{f_1,j} T_{\chi_{\mathcal{D}},j} &= T_{\chi_{\mathcal{D}},j} + K_{\mathcal{D}},
\end{aligned}$$

where $K_{\mathcal{A}}$, $K_{\mathcal{B}}$, $K_{\mathcal{C}}$, $K_{\mathcal{D}}$ are compact operators.

Let $\lambda \in S \backslash \partial S$. That is, $\lambda = \alpha + i\beta$, where $\alpha, \beta \in (0, 1)$. Since $T_{f_1,j}$ and $T_{f_2,j}$ are self-adjoint, the operators

$$T_{f_1,j} - \alpha + i(1-\beta), \quad T_{if_2,j} - \alpha - i\beta, \quad T_{f_1,j} - \alpha - i\beta, \quad T_{if_2,j} + (1-\alpha) - i\beta$$

are invertible on $\mathcal{F}_j$. Let A, B, C, D be their respective inverses.

Using the identities on the previous slide, it is easy to verify that

$$T_{\chi_{\mathcal{A}},j} A + T_{\chi_{\mathcal{B}},j} B + T_{\chi_{\mathcal{C}},j} C + T_{\chi_{\mathcal{D}},j} D$$

is the right Fredholm inverse of $T_{F,j} - \lambda$, and

$$A T_{\chi_{\mathcal{A}},j} + B T_{\chi_{\mathcal{B}},j} + C T_{\chi_{\mathcal{C}},j} + D T_{\chi_{\mathcal{D}},j}$$

is the left Fredholm inverse of $T_{F,j} - \lambda$.

Hence λ is not in the essential spectrum of T_F. $\square$

Similarly, we have

Theorem 4.2. Let $\ell \geq 0$. Then on the space $\mathcal{F}_0 \oplus \cdots \oplus \mathcal{F}_\ell$, the essential spectrum of the Toeplitz operator $T_F^{(\ell)}$ is contained in ∂S, the boundary of the square S.

Guo Chuan's explicit computation:

Proposition 5.1. We have

$$\mathrm{tr}[T_{f_1}, T_{f_2}] = \frac{1}{2\pi i}.$$

Proposition 5.1 only covers the setting of the classic Fock space $\mathcal{F}^2 = \mathcal{F}_0$. Presumably, we also have

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{1}{2\pi i}$$

on the higher Fock space $\mathcal{F}_j$, $j \geq 1$.

But the proof in the case $j \geq 1$ requires the Carey-Pincus theory of principal functions, which we review next.

Let A, B be bounded self-adjoint operators such that the commutator $[A, B]$ is in the trace class. Carey and Pincus showed that there is a $g_{A,B} \in L^1(\mathbf{R}^2)$, which is called the principal function for the pair A, B, such that

$$\mathrm{tr}([p(A,B), q(A,B)]) = \frac{-1}{2\pi i} \iint \{p, q\}(x,y) g_{A,B}(x,y) dxdy \tag{6.1}$$

for all $p, q \in \mathbf{C}[x, y]$, where

$$\{p, q\}(x,y) = \frac{\partial p}{\partial x}(x,y)\frac{\partial q}{\partial y}(x,y) - \frac{\partial p}{\partial y}(x,y)\frac{\partial q}{\partial x}(x,y),$$

which is called the Poisson bracket of p, q.

Define $T = A + iB$. Carey and Pincus also showed that for each point (x, y) such that $x + iy$ is not in the essential spectrum of T,

$$g_{A,B}(x, y) = \text{index}(T - (x + iy)). \tag{6.2}$$

This is an important formula for our purpose.

Apply the Carey-Pincus theory to the pair $A = T_{f_1}$ and $B = T_{f_2}$. Theorem 4.1 says that the essential spectrum of $A + iB = T_F$ is contained in ∂S. It follows from this fact that

$$\text{index}(T_F - \lambda) = 0 \quad \text{for every} \quad \lambda \in \mathbf{C} \backslash S.$$

Therefore for $A = T_{f_1}$ and $B = T_{f_2}$, we have $g_{A,B} = n\chi_S$, where

$$n = \text{index}(T_F - \lambda) \quad \text{for each} \quad \lambda \in S \backslash \partial S.$$

Applying (6.1) in the case $p(x, y) = x$ and $q(x, y) = y$, we obtain

$$\text{tr}[T_{f_1}, T_{f_2}] = \frac{-1}{2\pi i} \iint n\chi_S(x, y) dxdy = \frac{-n}{2\pi i}.$$

By Proposition 5.1, $n = -1$. Hence

$$\text{tr}[T_{f_1}, T_{f_2}] = \frac{-1}{2\pi i} \text{index}(T_F - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

Recall that we have the partial isometry

$$V_j = \frac{1}{\sqrt{j!}} C^j P$$

that maps $\mathcal{F}_0 = \mathcal{F}^2$ onto $\mathcal{F}_j$, $j \geq 1$.

Lemma 7.1. Given any $j \geq 1$, there exist coefficients $c_1^{(j)}, \ldots, c_j^{(j)}$ such that if $f \in C^\infty(\mathbf{C})$ and if f and $\partial\bar{\partial}f, \ldots, \partial^j\bar{\partial}^j f$ are all bounded on $\mathbf{C}$, then

$$V_j^* T_{f,j} V_j = T_f + \sum_{\nu=1}^{j} c_\nu^{(j)} T_{\partial^\nu \bar{\partial}^\nu f}.$$

Lemma 7.3. Suppose that the functions η, ξ in the definition of f_1, f_2 satisfy the condition $\eta, \xi \in \Sigma_a \cap C^\infty(\mathbf{R})$. Then

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{1}{2\pi i} \quad \text{for every} \quad j \geq 1.$$

Proof. By Lemma 7.1,

$$V_j^*[T_{f_1,j}, T_{f_2,j}]V_j = [T_{f_1}, T_{f_2}] + Z_1 + Z_2 + Z_3,$$

where

$$Z_1 = \sum_{\nu=1}^{j} c_\nu^{(j)}[T_{f_1}, T_{\partial^\nu \bar{\partial}^\nu f_2}], \quad Z_2 = \sum_{\nu=1}^{j} c_\nu^{(j)}[T_{\partial^\nu \bar{\partial}^\nu f_1}, T_{f_2}] \quad \text{and}$$
$$Z_3 = \sum_{\nu=1}^{j} \sum_{\nu'=1}^{j} c_\nu^{(j)} c_{\nu'}^{(j)}[T_{\partial^\nu \bar{\partial}^\nu f_1}, T_{\partial^{\nu'} \bar{\partial}^{\nu'} f_2}].$$

One then verifies that Z_1, Z_2, Z_3 are in the trace class with zero trace.

Lemma 7.4. Let the η, ξ in the definition of f_1, f_2 be arbitrary functions in Σ_a. Given a $j \geq 1$, let g_j be the Carey-Pincus principal function for the pair $T_{f_1,j}$, $T_{f_2,j}$. Then

$$g_j = -\chi_S.$$

Proof. Theorem 4.1 tells us that the essential spectrum of $T_{F,j}$ is contained in ∂S, whose two-dimensional Lebesgue measure is 0. Therefore

$$g_j = n_j \chi_S,$$

where

$$n_j = \text{index}(T_{F,j} - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

The above holds true for an arbitrary pair of $\eta, \xi \in \Sigma_a$.

Now take a pair of $\tilde{\eta}, \tilde{\xi} \in \Sigma_a \cap C^\infty(\mathbf{R})$. Accordingly, define

$$\tilde{f}_1(\zeta) = \tilde{\eta}(\mathrm{Re}(\zeta)) \quad \text{and} \quad \tilde{f}_2(\zeta) = \tilde{\xi}(\mathrm{Re}(e^{-i\theta}\zeta)).$$

We then define $\tilde{F} = \tilde{f}_1 + i\tilde{f}_2$. By the preceding slide, the pair $T_{\tilde{f}_1,j}$, $T_{\tilde{f}_2,j}$ has a principal function $\tilde{g}_j$ of the form $\tilde{g}_j = \tilde{n}_j \chi_S$, where

$$\tilde{n}_j = \mathrm{index}(T_{\tilde{F},j} - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

Applying Lemma 7.3, we have

$$\frac{1}{2\pi i} = \mathrm{tr}[T_{\tilde{f}_1,j}, T_{\tilde{f}_2,j}] = \frac{-\tilde{n}_j}{2\pi i} \iint \chi_S(x,y)dxdy = \frac{-\tilde{n}_j}{2\pi i}.$$

From this we conclude that $\tilde{n}_j = -1$.

Thus the lemma will follow if we can show $n_j = \tilde{n}_j$.

To prove that $n_j = \tilde{n}_j$, we define

$$\eta_t = t\eta + (1-t)\tilde{\eta} \quad \text{and} \quad \xi_t = t\xi + (1-t)\tilde{\xi},$$

$0 \leq t \leq 1$. We then define, for each $0 \leq t \leq 1$, the functions

$$f_{1,t}(\zeta) = \eta_t(\mathrm{Re}(\zeta)) \quad \text{and} \quad f_{2,t}(\zeta) = \xi_t(\mathrm{Re}(e^{-i\theta}\zeta)),$$

$\zeta \in \mathbf{C}$, and $F_t = f_{1,t} + if_{2,t}$. By Theorem 4.1, the essential spectrum of $T_{F_t,j}$ is contained in ∂S, $0 \leq t \leq 1$. Moreover, the map $t \mapsto T_{F_t,j}$ is obviously continuous with respect to the operator norm. Therefore for each $\lambda \in S \backslash \partial S$, the map

$$t \mapsto \mathrm{index}(T_{F_t,j} - \lambda)$$

remains constant on the interval $[0,1]$. Since $F_0 = \tilde{F}$ and $F_1 = F$, we have $n_j = \tilde{n}_j$ as promised. This completes the proof. □

Proposition 7.5. Suppose that the η, ξ in the definition of f_1, f_2 are arbitrary functions in Σ_a. Then for every $j \geq 1$ we have

$$\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] = \frac{1}{2\pi i}.$$

Proof. Applying the Care-Pincus trace formula and Lemma 7.4, we have

$$\begin{aligned}\mathrm{tr}[T_{f_1,j}, T_{f_2,j}] &= \frac{-1}{2\pi i}\iint g_j(x,y)dxdy \\ &= \frac{1}{2\pi i}\iint \chi_S(x,y)dxdy = \frac{1}{2\pi i}.\end{aligned}$$

□

By combining Proposition 7.5 with more trace argument, we obtain

Theorem 8.1. For each $\ell \geq 0$, the commutator $[T^{(\ell)}_{f_1}, T^{(\ell)}_{f_2}]$ is in the trace class with

$$\mathrm{tr}[T^{(\ell)}_{f_1}, T^{(\ell)}_{f_2}] = \frac{\ell + 1}{2\pi i}.$$

Let $g^{(\ell)}$ be the principal function for the pair $T_{f_1}^{(\ell)}$, $T_{f_2}^{(\ell)}$, $\ell \geq 0$. It follows from (6.2) and Theorem 4.2 that

$$g^{(\ell)} = n^{(\ell)} \chi_S,$$

where

$$n^{(\ell)} = \text{index}(T_F^{(\ell)} - \lambda) \quad \text{for every} \quad \lambda \in S \backslash \partial S.$$

Applying (6.1), we have

$$\text{tr}[T_{f_1}^{(\ell)}, T_{f_2}^{(\ell)}] = \frac{-n^{(\ell)}}{2\pi i} \iint \chi_S(x,y) dxdy = \frac{-n^{(\ell)}}{2\pi i}.$$

By Theorem 8.1, we can express the quantized Hall conductance

$$\sigma_{\mathrm{Hall}}(P_{\leq E}) = -i\mathrm{tr}(P^{(\ell)}[[M_{f_1}, P^{(\ell)}], [M_{f_2}, P^{(\ell)}]])$$

for the case $(2\ell+1)b < E < (2\ell+3)b$, in the following two ways:

$$\sigma_{\mathrm{Hall}}(P_{\leq E}) = \frac{1}{2\pi}\mathrm{index}(T^{(\ell)}_{f_1+if_2} - \lambda), \quad \lambda \in S \backslash \partial S;$$
$$\sigma_{\mathrm{Hall}}(P_{\leq E}) = -\frac{\ell+1}{2\pi}$$

$\ell \geq 0$.

Remark. It is not a priori obvious that σ_{Hall} is additive with respect to Landau level ℓ

We now consider the pair of functions

$$\varphi_1(\zeta) = \mathrm{Re}\left(\frac{\zeta}{|\zeta|}\right) \quad \text{and} \quad \varphi_2(\zeta) = \mathrm{Im}\left(\frac{\zeta}{|\zeta|}\right),$$

$\zeta \in \mathbf{C}\backslash\{0\}$. Furthermore, define

$$\Phi = \varphi_1 + i\varphi_2.$$

That is, $\Phi(\zeta) = \zeta/|\zeta|$ for $\zeta \in \mathbf{C}\backslash\{0\}$.

φ_1 and φ_2 are NOT the kind of function suitable for the Kubo formula, but they are mathematically interesting.

For the pair of function φ_1 and φ_2 we have

Theorem 9.1. (1) The Toeplitz operator T_Φ is a compact perturbation of the unilateral shift.
(2) The commutator $[T_\Phi^*, T_\Phi]$ is in the trace class. Consequently, the commutator $[T_{\varphi_1}, T_{\varphi_2}]$ is in the trace class.
(3) We have $\mathrm{tr}[T_\Phi^*, T_\Phi] = 1$. In other words, $\mathrm{tr}[T_{\varphi_1}, T_{\varphi_2}] = (2i)^{-1}$.
(4) The Toeplitz operator T_Φ is hyponormal.

By Theorem 9.1 and the Carey-Pincus theory, we have

$$\mathrm{tr}(P^{(0)}[[M_{\varphi_1}, P^{(0)}], [M_{\varphi_2}, P^{(0)}]]) = -\frac{1}{2i}\mathrm{index}(T_\Phi^{(0)} - \lambda)$$

when $|\lambda| < 1$. The obvious question is, does the analogue of this hold at Landau levels $\ell \geq 1$?

Mathematically, the following embodies all the difficulties:

Problem 9.2. For $\ell \geq 1$, does the commutator $[T_{\varphi_1}^{(\ell)}, T_{\varphi_2}^{(\ell)}]$ belong to the trace class?

For an $\ell \geq 1$, if $T_\Phi^{(\ell)}$ is hyponormal, then $[T_{\varphi_1}^{(\ell)}, T_{\varphi_2}^{(\ell)}]$ is in the trace class. But is it?

This is the kind of problem that gets operator theorists excited.

THANK YOU!