2023.3.17

§7. Introduction to the Hamilton–Tian Conjecture

For the complex Monge–Amperé eqn.: $\omega \in 2\pi C_1(M)$, $Ric(\omega) - \omega = \sqrt{-1}\partial\bar\partial h_\omega$, $\int_M e^{h_\omega}\omega^n = \int_M \omega^n$

$$(7.1)_t \qquad (\omega + \partial\bar\partial\varphi)^n = e^{h_\omega - t\varphi}\omega^n, \quad \omega + \sqrt{-1}\partial\bar\partial\varphi > 0$$

The soln. of $(1)_t$ for $t=1$, is a KE metric on M.

And $Ric(\omega_\varphi) \geq t\omega_\varphi$.

[Tian 97']: Kähler–Einstein metrics with positive scalar curvature, Invent. Math.

Conj. 9.1: By taking subseq. if necessary, one should have that (M, ω_t) converges to a space $(M_\infty, \omega_\infty)$, which is smooth outside a subset of real Hausdorff codimension at least 4, in the Cheeger–Gromov–Hausdorff topology. Furthermore, $(M_\infty, \omega_\infty)$ can be expanded to be an obstruction triple (M_∞, v, ξ) (possibly singular) satisfying:

$$Ric(\omega_\infty) - \omega_\infty = -L_v(\omega_\infty), \text{ on the regular part of } M_\infty,$$

where L_v denotes the Lie derivative in the direction of v. In particular, $(M_\infty, \omega_\infty)$ is a Ricci soliton if ω_∞ is not Kähler–Einstein.

[Rk: Here $\omega_t = \omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi$, φ is a soln. of $(1)_t$.]

Similar things can be said for the Ricci flow

$$\text{(KRF)}: \quad \frac{\partial\omega(t)}{\partial t} = -Ric(\omega(t)) + \omega(t), \quad \omega(0) = \omega.$$

$$\Longleftrightarrow \quad \frac{\partial\varphi}{\partial t} = \log\frac{(\omega + \sqrt{-1}\partial\bar\partial\varphi)^n}{\omega^n} + \varphi - h_\omega, \quad \varphi|_{t=0} = 0.$$

Right now, we call the above conjecture for the Kähler–Ricci flow as Hamilton–Tian conjecture.

Some explaination.

Here by the Cheeger-Gromov-Hausdorff topology, it means that

1) (M, ω_t) converges to $(M_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology;

2) For any $\{x_t\} \subset M$, $x_\infty \in M_\infty$ there are C^1 (C^r) diffeomorphisms

$\Phi_t : B_r(x_\infty) \to B_r(x_t)$ $\Phi_t(x_\infty) = x_t$ for some $r > 0$, s.t. $\Phi_t^* \omega_t$ converges to ω_∞ in the C^0-topology on $B_r(x_\infty)$, i.e.,

$$\lim_{t \to t_0} \| \Phi_t^* \omega_t - \omega_\infty \|_{C^0, \omega_\infty} = 0.$$

Definition of Gromov-Hausdoff convergence.

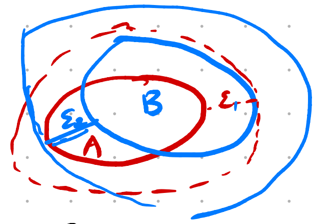
Defn (Hausdorff distance)

Assume (X, d) is a metric space, and $A, B \subset X$, then we define

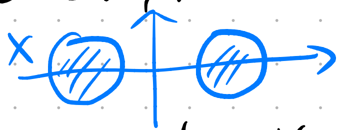
$$d(A,B) := \inf \{ d(a,b) \mid a \in A, b \in B \}$$

$$U(A, \varepsilon) := \{ x \in X \mid d(x, A) < \varepsilon \}$$

$$d_H(A,B) := \inf \{ \varepsilon \mid A \subset U(B, \varepsilon), B \subset U(A, \varepsilon) \}.$$



$d_H(A,B)$ is called the Hausdorff distance between A and B.



Defn (Gromov-Hausdorff distance)

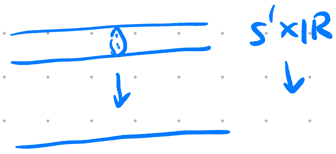
Assume (X, d_X) and (Y, d_Y) are metric spaces, then an admissible metric on disjoint union $X \sqcup Y$ is a metric d that extends the given metrics on X and Y, i.e. $d|_X = d_X$, $d|_Y = d_Y$. Then we define the Gromov-Hausdorff distance between X and Y as

$$d_{GH}(X,Y) := \inf \{ d_H(X,Y) \text{ with metric } d \mid d \text{ is an admissible metric on } X \sqcup Y \}$$

If X, Y are cpt.

Prop 7.1: $d_{GH}(X,Y)=0$ iff X and Y are isometric.

Let $(\mathcal{M}, d_{GH})$ denote the collection of compact metric spaces. It is a pseudometric space.



Defn (Gromov–Hausdorff convergence)

For a family of metric spaces (X_i, d_i), and a metric space (X, d), we say (X_i, d_i) converge to (X, d) in the Gromov–Hausdorff sense, if

$$\lim_{i\to+\infty} d_{GH}(X_i, d)=0.$$

For noncompact metric space.

Defn (Pointed Gromov–Hausdorff distance)

$$d_{GH}((X,x),(Y,y)) := \inf\{d_H(X,Y)+d(x,y)\},\ x\in X,\ y\in Y.$$

$$\mathcal{M}_* = \{(X, x, d)\}.$$

Defn. (Pointed Gromov–Hausdorff convergence)

We say that $(X_i, x_i, d_i)\longrightarrow(X, x, d)$ in the pointed Gromov–Hausdorff topology if for all $R>0$,

$$(\bar{B}(x_i,R), x_i, d_i)\longrightarrow(\bar{B}(x,R), x, d)$$

converge with respect to the pointed Gromov–Hausdorff metric.

Defn (Cheeger–Gromov convergence)

For closed complete Riemannian mfds (M_i, g_i) and a closed Riem. mfd (M_∞, g_∞). We say (M_i, g_i) converge to (M_∞, g_∞) in the sense of Cheeger–Gromov, if $\exists$ diffeomorphisms $\phi_i: M_\infty\longrightarrow M_i$ s.t.

$$\lim_{i\to\infty} \| \phi_i^* g_i - g_\infty \|_{C^k(M_\infty, g_\infty)} = 0 .$$

For noncompact mfds, we define pointed Cheeger-Gromov convergence.

For complete Riemannian mfds $\{(M_i, g_i, p_i)\}$, $p_i \in M_i$ and a complet Riem. mfd $(M_\infty, g_\infty, p_\infty)$, $p_\infty \in M$, we say (M_i, g_i, p_i) converge to $(M_\infty, g_\infty, p_\infty)$ in the sense of Cheeger-Gromov, if $\exists$ a seq. of increasing open subsets $p_\infty \in U_k \subset M_\infty$, s.t. $+ \overline{U_k}$ is cpt.

① $U_k \subset U_{k+1}$ and ② $\bigcup_{k=1}^{\infty} U_k = M_\infty$.

and $\exists$ diffeomorphisms $\phi_k : U_k \to \phi_k(U_k) \subset M_k$

with $\phi_k(p_\infty) = p_k$, and for any compact subset A of M_∞ with

$$\lim_{k\to+\infty} \| \phi_k^* g_k - g_\infty \|_{C^k(A, g_\infty)} = 0 .$$

[Tian-Zhang, 2016] Regularity of Kähler-Ricci flow on Fano manifolds, Acta Math.

For $\dim M \leq 3$.

[Chen-Wang, 2017] Space of Ricci flows (II)

Part A: moduli of singular Calabi-Yau spaces, Forum Math. Sigma

Part B: weak compactness of the flows, J. Differential Geometry (2020)

For all dimension.

[Bamler, 2018] Convergence of Ricci flows with bounded scalar curvature, Ann. Math.

Proved a Hamilton-Tian Conjecture in Riemannian case.

[Wang-Zhu, 2021] Tian's partial C^0-estimate implies Hamilton-Tian's conjecture, Adv. Math. (2021).

Hamilton-Tian conjecture $\Longrightarrow$ Tian's partial C^0-estimate

[Blum-Liu-Xu-Zhuang, 2021, Arxiv]

The existence of the Kähler-Ricci soliton degeneration.

They proved an algebraic version of the Hamilton-Tian conjecture for all log Fano pairs.

- Tian-Zhang's proof.

Due to Perelman's estimate, $\mathrm{diam}(M, g(t)) \le C$, $|R(g(t))| \le C$

and volume non-collapsing along Ricci flow, i.e., $\exists\ \kappa = \kappa(g_0, n)$ s.t. $\forall x \in M$

(7.2) $$\mathrm{Vol}_{g(t)}(B_{g(t)}(x,r)) \ge \kappa r^{2n}, \quad \forall t \ge 0 \text{ and } r \le 1.$$

and $\mathrm{Vol}_{g(t)}(M) = \text{const}.$

Then $\exists\ t_i \to \infty$ s.t. $(M, g(t_i)) \xrightarrow{d_{GH}} (M_\infty, d)$, (M_∞, d) is a length space.

Q: Regularity of M_∞?

The main theorem of Tian-Zhang's paper.

Thm 1 (Tian-Zhang, Acta Math., 2016)

For uniform constants $p > n$ and $\Lambda < \infty$, if

(7.3) $$\int_M |\mathrm{Ric}(g(t))|^p \, dV_{g(t)} \le \Lambda.$$

Then M_∞ is smooth outside a closed subset S of (real) codim ≥ 4 and d is induced by a smooth Kähler-Ricci soliton g_∞ on $M_\infty \setminus S$.

Moreover, $g(t_i) \longrightarrow g_\infty$ in C^∞ (Cheeger-Gromov) topology outside S.

The proof of Thm 1 is based on ① Perelman's pseudolocality theorem:

$\exists\ \varepsilon_p, \delta_p > 0$ and $r_p > 0$, which depend on p and Λ in Thm 1, s.t.

$\forall (x_0, t_0) \in M \times [-1, \infty)$ for $g_i(t) := g(t_i + t)$, $t \ge -1$

if $$\mathrm{vol}_{g_i(t_0)}(B_{g_i(t_0)}(x_0, r)) \ge (1-\varepsilon_p)\,\mathrm{vol}(B_r), \quad \text{for some } r \le r_p.$$

$\mathrm{vol}(B_r)$: volume of a Euclidean ball of radius r in $\mathbb{R}^{2n}$. Then

(7.4) $$|\mathrm{Rm}_{g_i}(x,t)| \le \frac{1}{t-t_0}, \quad \forall x \in B_{g_i(t)}(x_0, \varepsilon_p r),\ t_0 < t \le t_0 + \varepsilon_p^2 r^2$$

and $$\mathrm{vol}_{g_i(t)}(B_{g_i(t)}(x_0, \delta_p\sqrt{t-t_0})) \ge (1-\eta)\,\mathrm{vol}(B_{\delta_p\sqrt{t-t_0}}), \quad t_0 < t \le t_0 + \varepsilon_p^2 r^2.$$

(Choose $\eta \le \varepsilon_p$ in the application).

② A generalization of the regularity theory of Cheeger-Colding, Cheeger-Colding-Tian for manifolds with bounded Ricci curvature.

- Under condition (7.3).

For any $p > \frac{1}{2}m$ ($m = \dim_{\mathbb{R}} M$), $\exists\ C(m,p)$ s.t.

(7.5) $$\frac{d}{dr}\left\{\left(\frac{\mathrm{Vol}(B_\Gamma(x,r))}{r^m}\right)^{1/2p}\right\} \le C(m,p)\left(\frac{1}{r^m}\int_{B_\Gamma(x,r)} |Ric_-|^p dv\right)^{\frac{1}{2p}}, \quad \forall r>0$$

here $B_\Gamma(x,r) := \{\exp_x(t\theta) \mid 0 \le t < r \text{ and } \theta \in \Gamma\}$, $\Gamma \subset S_x$: unit sphere bundle. $Ric_- := \max_{|v|=1}\{0, -Ric(v,v)\}$

$\Rightarrow \forall r_2 > r_1 > 0$

(7.6) $$\left(\frac{\mathrm{Vol}(B_\Gamma(x,r_2))}{r_2^m}\right)^{\frac{1}{2p}} - \left(\frac{\mathrm{Vol}(B_\Gamma(x,r_1))}{r_1^m}\right)^{\frac{1}{2p}} \le C(m,p)\left(r_2^{2p-m}\int_{B_\Gamma(x,r_2)} |Ric_-|^p dv\right)^{1/2p}$$

$\Rightarrow$ If $\int_M |Ric_-|^p dv \le \Lambda$, then

(7.7) $$\mathrm{Vol}(B_\Gamma(x,r)) \le |\Gamma| r^m + C(m,p)\Lambda r^{2p}, \quad \forall r>0.$$

By Gromov's first convergence theorem, $\exists$ a complete length metric space (Y,d) such that

$$(M_i, g_i) \xrightarrow{d_{GH}} (Y,d).$$

The following is the structure theorem of the limit space under (7.3).

Thm 2: (i) For any $r>0$ and $x_i \in M_i$ s.t. $x_i \to x_\infty \in Y$, we have

$$\mathrm{vol}(B(x_i,r)) \longrightarrow \mathcal{H}^m(B(x_\infty,r)). \qquad \mathcal{H}^m\text{: m-dim Hausdorff measure}$$

(ii) $\forall x \in Y$, and any seq. $\{r_j\}$ with $\lim_{j\to\infty} r_j = 0$, $\exists$ subseq. $(Y, r_j^{-2}d, x)$ converging to a metric space (C_x, d_x, o) as $j \to \infty$. Any such space (C_x, d_x, o) is a metric cone with vertex o and splits off lines isometrically.

(iii) $Y = S \cup R$, where S is a closed set of codim ≥ 2 and R is convex in Y; R consists of points whose tangent cone is $\mathbb{R}^m$.

(iv) $\exists$ a $C^{1,\alpha}$-smooth structure on R and C^α-metric g_∞, for all $\alpha < 2 - m/p$, which induces d; Moreover, $g_i \to g_\infty$ in C^α-topology on R.

(v) The singular set S has codim ≥ 4 if (M_i, g_i) is Kähler.

- Regularity under KRF.

Main Goal: Under KRF, $\int_M |\mathrm{Ric}(g(t))|^4 dV_{g(t)} \leq C$, $\forall t \geq 0$, $C = C(g_0)$.

Notations: $V = \int_M \omega^n$, Ricci potential. $R_{i\bar{j}} - g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u$, $\frac{1}{V}\int_M e^u dV_{g(t)} = 1$.

Some evolutions:

$$\frac{\partial}{\partial t} u = \Delta u + u + a, \qquad a = \frac{1}{V}\int_M u(t) e^{u(t)} dV_{g(t)} \geq 0$$

$$\frac{\partial}{\partial t}|\nabla u|^2 = \Delta|\nabla u|^2 - |\nabla\bar{\nabla} u|^2 - |\nabla\nabla u|^2 + |\nabla u|^2$$

$$\frac{\partial}{\partial t}\Delta u = \Delta(\Delta u) + |\nabla\bar{\nabla} u|^2 + \Delta u$$

$$\frac{\partial}{\partial t}|\nabla\bar{\nabla} u|^2 = \Delta|\nabla\bar{\nabla} u|^2 - 2|\nabla\nabla\bar{\nabla} u|^2 + 2R_{i\bar{j}k\bar{l}}\nabla_{\bar{i}}\nabla_l u \cdot \nabla_j \nabla_{\bar{k}} u$$

$$\frac{\partial}{\partial t}|\nabla\Delta u|^2 = \Delta|\nabla\Delta u|^2 - |\nabla\nabla\Delta u|^2 - |\nabla\bar{\nabla}\Delta u|^2 + |\nabla\Delta u|^2$$
$$- \nabla_i|\nabla\bar{\nabla} u|^2 \nabla_{\bar{i}}\Delta u - \nabla_i \Delta u \cdot \nabla_{\bar{i}}|\nabla\bar{\nabla} u|^2$$

Prop. 7.1. Under KRF,
$$\int_0^\infty \int_M |\nabla\nabla u|^2 dv\, dt < +\infty.$$

In particular,
$$\int_M |\nabla\nabla u|^2 dV_{g(t)} \to 0, \text{ as } t \to \infty.$$

Prop. 7.2
$$\int_t^{t+1}\int_M |\nabla(\Delta u - |\nabla u|^2 + u)|^2 dv\, dt \to 0, \text{ as } t \to \infty$$
$$\int_M (\Delta u - |\nabla u|^2 + u - a)^2 dv \to 0, \text{ as } t \to \infty.$$

If ω is a shrinking Kähler-Ricci soliton, $g \sim \omega$

$$Ric(g) + \nabla^2 f = g$$

$$\Rightarrow f = -u \quad \Rightarrow \nabla\nabla u = 0.$$

Hence for KRF $\omega(t)$, if $\omega(t) \longrightarrow \omega_{KS}$, it must be

$$\text{“} \|\nabla\nabla u\|_{g(t)} \to 0. \text{”}$$

We consider any solution $f = f(t)$ to the backward heat eqn.

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - \Delta u$$

$$\Rightarrow \frac{d}{dt} \mathcal{W}(g, f) = \frac{1}{V} \int_M \left(|\nabla\bar{\nabla}(f+u)|^2 + |\nabla\nabla f|^2 \right) e^{-f} dV_g$$

For prove Prop. 7.1 and 7.2, we need the following Lemma.

Lem 7.3: $g(t) \sim$ KRF, $\forall f \in C^\infty(M)$, we have

$$\int_M |\nabla\nabla f|^2 dV_{g(t)} \le C(g_0) \int_M |\nabla\bar{\nabla} f|^2 dV_{g(t)}.$$

Pf: Integrating by parts and the weighted Poincaré inequality.

- Regularity of the limit.

For any seq. $t_i \to \infty$, we define $g_i = g_i(t)$ (are soln. of KRF) by

$$(M, g_i(t)) = (M, g(t_i + t)), \quad t \ge -1.$$

Let $u_i(t)$ be the associated Ricci potentials, Perelman's estimates.

$$\|u_i(t)\|_{C^0} + \|\nabla u_i(t)\|_{C^0} + \|\Delta u_i(t)\|_{C^0} \le C(g_0), \quad \forall t \ge -1.$$

and by Prop. 7.1, $\forall t \ge -1$.

$$\int_M |\nabla\nabla u_i(t)|^2 dV_{g_i(t)} \longrightarrow 0, \text{ as } i \to \infty.$$

Under the condition $(p>n)$

$$\int_M |Ric(g(t))|^p dV_{g(t)} \le \Lambda,$$

Fix time $t=0$,

$$(M, g_i(0)) \xrightarrow{d_{GH}} (M_\infty, d).$$

$M_\infty = S \cup R$,

R: smooth complex mfd w/ a C^α-complex structure J_∞ and a C^α-metric g_∞ which induces d.

S: closed singular set of codim $\geqslant 4$.

Moreover,

$$(g_i(0), u_i(0)) \xrightarrow{C^\alpha \cap L^{2,p}} (g_\infty, u_\infty) \text{ on } R.$$

Elliptic regularity of $\Delta u_i(0) = n - s(g_i(0)) \in L^p \Rightarrow u_i(0) \xrightarrow{L^{2,p}} u_\infty$.

Perelman's estimate $\Rightarrow$ u_∞ is globally Lipschitz on M_∞.

Using the bootstrap argument argument of elliptic eq$\underline{n}$, then g_∞ is smooth and satisfies

$$Ric(g_\infty) + Hess\, u_\infty = g_\infty \quad \text{on } R.$$

Moreover, J_∞ is smooth on R and g_∞ is Kähler with respect to J_∞.

Then using Perelman's pseudolocality theorem, they can proved the smooth convergence on the regular set.

This finish the proof of Thm 1.

Thm 2: Under KRF, $\exists\, C = C(g_0)$ s.t.

$$\int_M |Ric(g(t))|^4 dV_{g(t)} \le C.$$

Thm 1 + Thm 2 $\Rightarrow$ Hamilton-Tian conj. is true for $n \le 3$.

Thm 2 $\Leftrightarrow$ $\int_M |\nabla\bar\nabla u|^4 dV_{g(t)} \le C$.

Using Perelman's estimate, Chern-Weil theory, and integration by parts.

Lem 7.4: $\int_M \left(|\nabla\nabla u|^2 + |\nabla\bar\nabla u|^2 + |Rm|^2\right) dV_g \le C, \quad \forall t \ge 0.$

Lem 7.5: $\int_M |\nabla\bar\nabla u|^4 dv \le C \int_M \left(|\nabla\nabla\bar\nabla u|^2 + |\bar\nabla\nabla\nabla u|^2\right) dv$

$$\int_M |\nabla\nabla u|^4 dV \le C \int_M \left(|\nabla\nabla\nabla u|^2 + |\nabla\nabla\bar\nabla u|^2 + |\bar\nabla\nabla\nabla u|^2\right) dV.$$

Pf: Bochner formula:

$$\Delta |\nabla u|^2 = |\nabla\nabla u|^2 + |\nabla\bar\nabla u|^2 + \Delta\nabla_i u \nabla_{\bar i} u + \nabla_i u \Delta \nabla_{\bar i} u.$$

$\Rightarrow \int_M \left(|\nabla\nabla u|^2 + |\nabla\bar\nabla u|^2\right) |\nabla\bar\nabla u|^2 dV$

$\le \frac{1}{2} \int_M |\nabla\bar\nabla u|^2 \left(|\nabla\bar\nabla u|^2 + |\nabla\nabla u|^2\right) dV$

$+ 8(n^2+1) \int_M |\nabla u|^2 \left(|\nabla\nabla\bar\nabla u|^2 + |\bar\nabla\nabla\nabla u|^2\right) dv.$ #

Lem. 7.6:

$$\int_M \left(|\bar\nabla\nabla\nabla u|^2 + |\nabla\nabla\bar\nabla u|^2 + |\nabla\nabla\nabla u|^2\right) dV \le C \int_M \left(|\nabla\Delta u|^2 + |Rm|^2 + |\nabla\nabla u|^2\right) dV.$$

Now it suffices to estimate $\int_M |\nabla\Delta u|^2 dV$.

Lem 7.7:

$$\int_M \left(|\nabla\nabla\bar\nabla u|^2 + |\bar\nabla\nabla\nabla u|^2 + |\nabla\nabla\nabla u|^2 + |\nabla\nabla u|^4 + |\nabla\bar\nabla u|^4\right) dv \le C.$$

Pf: $\frac{\partial}{\partial t}(\Delta u)^2 = \Delta(\Delta u)^2 - 2|\nabla\Delta u|^2 - 2\Delta u \cdot |\nabla\bar\nabla u|^2 + 2(\Delta u)^2$

Integrating the above identity,

$$2\int_M |\nabla\Delta u|^2 dv = \int_M \left(-2\Delta u|\nabla\bar\nabla u|^2 + 2(\Delta u)^2 + (\Delta u)^3\right)dv - \frac{d}{dt}\int_M (\Delta u)^2 dv.$$

$$\Rightarrow \int_t^{t+1}\int_M |\nabla\Delta u|^2 dv\, ds \le C(g_0), \quad \forall t \ge 0$$

On the other hand,

$$\frac{d}{dt}\int_M |\nabla\Delta u|^2 dv \le -\frac{1}{2}\int_M \left(|\nabla\nabla\Delta u|^2 + |\nabla\bar\nabla\Delta u|^2\right)dv + C(g_0)\left(1+\int_M |\nabla\Delta u|^2 dv\right)$$

$$\le C(g_0)\left(1+\int_M |\nabla\Delta u|^2 dv\right)$$

$$\Rightarrow \int_M |\nabla\Delta u|^2 dv \le C(g_0).$$

#.

$$(\text{Lem } 7.4 - \text{Lem } 7.7) \Rightarrow \int_M |Ric(g(t))|^4 dV_{g(t)} \le C(g_0).$$

$$\because \ Ric(\omega(t)) = \omega(t) + \sqrt{-1}\partial\bar\partial u(t).$$

- Partial C^0-estimate

Let (M, L, ω) be a polarized mfd such that ω is a kähler metric $\omega \in 2\pi C_1(L)$. Choose a hermitian metric h on L such that $R(h) = \omega$. Then for any positive integer ℓ, we have the L^2-metric on $H^0(M, L^\ell, \omega)$.

$$(7.8) \qquad \langle s_1, s_2\rangle := \int_M \langle s_1, s_2\rangle_{h^{\otimes\ell}}\, \omega^n,$$

for $s_1, s_2 \in H^0(M, L^{\otimes\ell})$

Defn. We denote this linear space $H^0(M, L^\ell, \omega)$ with inner product by (7.8). For any orthonormal basis $\{s^\alpha\}$ $(0 \le \alpha \le N = N(\ell))$ of $H^0(M, L^\ell, \omega)$, we define the Bergman kernel as

$$\rho_L(M,\omega)(x) := \sum_{i=1}^{N} |S^\alpha|^2_{h\otimes \ell}(x).$$

Partial C^0-estimate.

Thm 7.8. [Liu-Székelhidi, GAFA, 2022]

Given $n, D, v>0$, $\exists\ \ell \in \mathbb{N}$ and $b>0$ with the following:

Suppose that (M, L, ω) is a polarized Kähler mfd with $\omega \in 2\pi c_1(L)$ such that

$$\mathrm{Ric}(\omega) \geq \omega, \quad \mathrm{vol}(M,\omega) \geq v, \quad \mathrm{diam}(M,\omega) \leq D.$$

Then
$$\rho_\ell(M,\omega)(x) \geq b, \quad \forall x \in M.$$

Thm 7.9 [Liu-Székelhidi, GAFA, 2022]

Given $n, d, v>0$, let (M_i^n, L_i, ω_i) be a seq. of polarized Kähler mfds such that

- L_i is a Hermitian hol. line bundle with hermitian metric h_i such that $R(h_i) = \omega_i$;
- $\mathrm{Ric}(\omega_i) \geq -\omega_i$, $\mathrm{vol}(M,\omega_i) > v$, $\mathrm{diam}(M_i, \omega_i) < d$;
- $(M_i^n, \omega_i) \xrightarrow{d_{GH}} (X, d_\infty)$ (is a metric space).

Then X is homeomorphic to a normal variety. More precisely, $\exists\ \ell = \ell(n,d,v) \in \mathbb{N}$ such that the orthonormal basis of $H^0(M_i, L_i^\ell, \omega_i)$ defines an embedding Φ_i. Then by taking a subseq. $\Phi_i(M_i) \longrightarrow W \subset \mathbb{CP}^N$ as cycles, and $X \overset{\text{homeo.}}{\cong} W$.

For KRF (M, ω_t), Zhang, Kewei (张科伟) proved the following partial C^0-estimate.

Thm 7.10 (Zhang, Kewei)

Assume (M, ω_t) is a soln. of KRF. Then $\exists\, \ell = \ell(M, \omega_0) \in \mathbb{N}$ and $b = b(M, \omega_0) > 0$ such that

$$\rho_\ell(M, \omega_t)(x) \geq b, \quad \forall x \in M.$$

Wang-Zhu.

The partial C^0-estimate $\Rightarrow$ Hamilton-Tian's conjecture.

Thm 7.11 (Wang-Zhu, Adv. 2020)

Assume (M, ω_t) is a soln. of KRF, $\exists\, t_i \to \infty$ and a $\mathbb{Q}$-Fano variety $\tilde{M}_\infty$ with klt singularities such that ω_{t_i} is locally C^∞-convergent to Kähler-Ricci soliton ω_∞ on $\mathrm{Reg}(\tilde{M}_\infty)$. Moreover, ω_∞ can be extended to a singular Kähler-Ricci soliton on $\tilde{M}_\infty$ with a cont. Kähler potential and the completion of $(\mathrm{Reg}(\tilde{M}_\infty), \omega_\infty)$ is isometric to the global limit $(M_\infty, \omega_\infty')$ of ω_{t_i} in sense of Gromov-Hausdorff. $(M, \omega_{t_i}) \xrightarrow{d_{GH}} (M_\infty, \omega_\infty')$

In addition, ω_∞ is a singular Kähler-Einstein metrics,

$$M_\infty \overset{\text{homeo.}}{\cong} \tilde{M}_\infty.$$

- Hamilton-Tian conj. $\Rightarrow$ Yau-Tian-Donaldson conj.

  Tian-Zhang (Zhenlei)'s proof.

Thm 7.12 (Tian-Zhang, Partial C^0-estimate).

Assume that $(M, g(t_i)) \xrightarrow{d_{GH}} (M_\infty, g_\infty)$, as in Thm 1. Then

$$(7.9) \qquad \inf_{t_i} \inf_{x \in M} \rho_{t,\ell}(x) > 0$$

for a seq. $\ell \to \infty$.

Here $\tilde{g}(t) = e^{\frac{u(t)}{n}} g(t)$, $h(t)$: induced metric of $\tilde{g}(t)$ on $K_M^{-\ell}$ s.t.

$$\mathrm{Ric}(h(t)) = \ell\, \omega(t).$$

Set $N_\ell = \dim H^0(M, K_M^{-\ell}) - 1$. Orthonormal basis $\{S_{t,\ell,k}\}_{k=0}^{N_\ell}$ of $H^0(M, K_M^{-\ell})$ relative to the L^2-norm defined by $h(t)$ (see (7.8)).

defining
$$\rho_{t,\ell}(x) := \sum_{k=0}^{N_\ell} |S_{t,\ell,k}|^2_{h(t)}(x), \quad \forall x \in M.$$

Thm 7.13 (Tian-Zhang)

Suppose that the partial C^0-estimate (7.9) holds for a seq. $t_i \to \infty$. If M is K-stable, then the K-energy is bounded from below under KRF,
$$K(\omega(0), \omega(t)) \geq -C(g_0).$$

Pf: Using S. Paul's a result:

If M is K-stable, then the K-energy is bounded below on the space of Bergman metrics which rise from Kodaira embedding via bases of $K_M^{-\ell}$.

Thm 7.13 $\Rightarrow$ M_∞ in Thm 1 must be (singular?) Kähler-Einstein. Then its automorphism group must be reductive as a corollary of the uniqueness theorem due to Berndtsson and Berman.

If M_∞ is not equal to M, then $\exists$ a $\mathbb{C}^*$-action $\{\sigma(s)\}_{s\in\mathbb{C}^*} \subset SL(N_\ell+1, \mathbb{C})$ such that $\sigma(s)\Phi_1(M) \to$ embedding of M_∞ in $\mathbb{CP}^{N_\ell}$. Contradicts the K-stability, since the Futaki inv. of M_∞ vanishes.

$\Rightarrow$ $\exists$ KE metric on $M = M_\infty$.

- Bamler's result.

Thm 7.14: For $(M, g(t))$ is a soln. of Ricci flow (Riemannian case)

Assume $\quad R(\cdot, t) \leq \frac{C}{T-t}, \quad \forall t \in (0,T).$

Then for any $q \in M$ and any seq. of time $t_i \nearrow T$, we can choose a subseq. such that $(M, \frac{g_{t_i}}{T-t_i}, q) \longrightarrow (\mathcal{X}, q_\infty) = (X, d, R, g, q_\infty)$ (which is singular space) that has singularities of codim 4 in the sense of the following Definition 7.2 and that is Y- regular at scale 1 in the sense of Definition 7.4 for some $Y < \infty$ that only depends on g_0 and C.

Moreover, $\mathcal{X}$ is a shrinking gradient Ricci soliton in the following sense: $\exists$ bounded fcn $f_\infty \in C^\infty(R)$ that satisfies

$$Ric(g) + \nabla^2 f_\infty = \frac{1}{2} g \quad \text{on } R.$$

Cor. (Hamilton-Tian Conj.)

Let $(M, g(t))$ be a soln. of unnormalized KRF $\frac{\partial \omega}{\partial t} = -Ric(\omega(t))$ on Fano mfd M. Then $\forall$ seq. $t_i \nearrow T$, $\exists$ a subseq. s.t.

$$(M, (T-t_i)^{-1} g_{t_i}) \longrightarrow \mathcal{X} = (X, d, R, g)$$

$\mathcal{X}$ is a compact singular space which has singularities of codim 4, that is Y- regular at scale 1 for some $Y < \infty$ and that is a shrinker in the sense of Thm 7.14.

Def'n 7.1 (Singular space)

A tuple $\mathcal{X} = (X, d, R, g)$ is called an (n-dim) singular space if the following holds:

(1) (X, d) is a locally cpt, complete metric length space.

(2) $R \subset X$ is open and dense w/ the structure of a diff. mfd whose top. is equal to the top. induced by X.

(3) g is a smooth Riemannian metric on R.

(4) The length metric of $(R, g) = d|_R$, i.e.,

$$(X, d) = \overline{(R, g)}$$

(5) $\exists\ 0 < \kappa_1 < \kappa_2 < \infty$ such that $\forall\ r \in (0, 1)$,

$$\kappa_1 r^n < |\underbrace{B(x, r)}_{d} \cap \underbrace{R}_{g}| < \kappa_2 r^n.$$

If $q \in X$ is a point, then the tuple $(\mathcal{X}, q)$ or (X, d, R, g, q) is called pointed singular space. R: regular part of $\mathcal{X}$.

$X \setminus R$: singular part of $\mathcal{X}$.

We define

$$r_{Rm} = \begin{cases} 0, & \text{if } x \in X \setminus R. \\ \sup\{ r \mid |Rm|(y) < r^{-2}, \forall y \in B(x, r) \subset R \} \end{cases}$$

Def'n 7.2 (Singularities of codim p_0)

A singular space $\mathcal{X} = (X, d, R, g)$ is said to have singularities of codim p_0, if $\forall\ 0 < p < p_0$, $x \in X$ and $r_0 > 0$, $\exists\ E_{p,x,r_0} < \infty$ such that the following holds: $\forall\ 0 < r < r_0$ and $0 < s < 1$, we have

$$|\{ r_{Rm} < sr \} \cap B(x, r) \cap R| \leq E_{p,x,r_0} s^p r^n$$

"$\tilde{r}$ in Bamler's paper?"

Defn 7.3 (Mild singularities)

A singular space $\mathcal{X} = (X, d, \mathcal{R}, g)$ is said to have mild sing. if $\forall p \in X$ $\mathcal{R}$, $\exists$ a closed subset $Q_p \subset \mathcal{R}$ of measure zero such that $\forall x \in \mathcal{R} \setminus Q_p$, $\exists$ a minimizing geodesic between p and x that lies in $\mathcal{R}$.

Defn 7.4 (Y-regularity) A singular space $\mathcal{X}$ is called Y-regular at scales less than a, for some $a, Y > 0$, if $\forall p \in X$ and $0 < r < a$, the following holds:

If
$$|B(p,r) \cap \mathcal{R}| > (\omega_n - Y^{-1}) r^n,$$
then $p \in \mathcal{R}$ and $r_{Rm}(y) > Y^{-1} r$. Here ω_n denotes the volume of the standard n-dim ball in $\mathbb{R}^n$. The space $\mathcal{X}$ is said to be Y-regular at all scales if it is Y-regular at scale a for all $a > 0$.