2023.3.24

Recall the definition of Gromov-Hausdorff distance, refer to Book "A course in metric geometry" D. Burago - A. Burago - S. Ivanov

Defn (Gromov-Hausdorff distance)

Let X and Y be cpt metric spaces.

$$d_{GH}(X,Y) := \inf_{(Z,X',Y')} \left\{ d_H(X',Y') \;\middle|\; \begin{array}{l} \text{metric space } Z \ (Z,d) \text{ and its subspace } X' \text{ and } Y' \\ \text{s.t. } X' \overset{\text{iso.}}{\cong} X,\ Y' \simeq Y \end{array} \right\}$$

§8 Song-Tian's analytic minimal model program with KRF

References:

[Song-Tian 1] The Kähler-Ricci flow on surfaces of positive Kodaira dimension, Invent. Math. 170 (2007), 609-653.

[Song-Tian 2] Canonical measures and Kähler-Ricci flow, JAMS, 25 (2012), 303-353.

[Song-Tian 3] The Kähler-Ricci flow through singularities, Invent. Math. 207 (2017), 519-595.

[Song-Weinkove] Introduction to Kähler-Ricci flow.

- A brief introduction of Mori's minimal model program (MMP) in birational geometry.

Let X, Y be projective varieties. A <u>rational map</u> from X to Y is given by a holomorphic map $f: X \setminus V \to Y$, where V is a subvariety of X.

We say that a rational map f from X to Y is <u>birational</u> if there exists a rational map from Y to X such that $f \circ g = \mathrm{id}_Y$ as a rational map Y to Y.

If a birational map from X to Y exists then we say X and Y are <u>birationally equivalent</u> (or <u>birational</u> or <u>in the same birational class</u>.)

The MMP is concerned with finding a "good" representative of a variety within its birational class. A "good" variety X is one satisfying either: $c_1(K_X) = -c_1(X)$

(i) K_X is nef; $(\forall \text{ curve } C \subset X,\ K_X \cdot C \geq 0)$

or

(ii) $\exists$ a holo. map $\pi : X \to Y$ to a lower dimensional variety Y such that the generic fiber $X_y = \pi^{-1}(y)$ is a manifold with $K_{X_y} < 0$.

In case of (i), we say that X is a <u>minimal model</u>. and in case of (ii), we say that X is a <u>Mori fiber space</u> (or <u>Fano fiber space</u>).

The basic idea of the MMP is to find a finite sequence of birational maps $f_1, \cdots, f_k$ and varieties $X_1, \cdots, X_k$,

$$X = X_0 \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} X_2 \dashrightarrow \cdots \overset{f_k}{\dashrightarrow} X_k \tag{8.1}$$

so that X_k is our "good" variety: either of type (i) or type (ii).

We want to find maps f_i which "remove" curves C with

For smooth variety.

$K_X \cdot C < 0$.

- If $\dim(X) = 1$, case (i) $\Rightarrow$ $c_1(X) < 0$ or $c_1(X) = 0$

  $\hphantom{xxxxxxxxxxxxxxx}$ $\Uparrow$ $\hphantom{xxxxxxx}$ $\Uparrow$

  $\exists g$, s.t. $K(g) = -1$ $\quad$ $\exists g$ s.t. $K(g) = 0$

  case (ii) $\Rightarrow$ $X = \mathbb{P}^1$. $\Rightarrow \exists g$, s.t. $K(g) = 1$.

- If $\dim(X) = 2$, by the Enriques-Kodaira classification, one can obtain the "good" variety X via a finite sequence of blow downs.

• If $\dim(X) \geq 3$, it is not possible to find such a seq. of birational maps if we wish to stay within the category of smooth varieties.

Let X be a smooth projective variety. A <u>1-cycle</u> C on X is a formal finite sum $C = \sum_i a_i C_i$ for $a_i \in \mathbb{Z}$, C_i: irreducible curves. We say that 1-cycles C and C' are <u>numerically equivalent</u> if $D \cdot C = D \cdot C'$, $\forall$ divisors D, denoted by $C \sim C'$.

$N_1(X)_{\mathbb{Z}}$: the space of 1-cycles modulo numerical equivalence.

$N_1(X)_{\mathbb{Q}} := N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\quad N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$.

D and D' are <u>numerically equivalent</u> if $D \cdot C = D' \cdot C$, $\forall$ all curve C.

$N^1(X)_{\mathbb{Z}}$: the set of divisors modulo numerical equivalence.

$N^1(X)_{\mathbb{Q}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{Q}$, $\quad N^1(X)_{\mathbb{R}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$

$C \in N_1(X)_{\mathbb{R}}$ is called <u>effective</u> if $C \sim \sum a_i C_i$ with $a_i \geq 0$, C_i: irr. curves

$NE(X) := \{ \text{effective elements of } N_1(X)_{\mathbb{R}} \}$, is a cone.

$\overline{NE(X)} :=$ closure of $NE(X)$ in $N_1(X)_{\mathbb{R}}$ $\quad (NE(X) \neq \overline{NE(X)})$

<u>Thm 8.1</u>: A divisor D is ample iff $D \cdot w > 0$, $\forall\, 0 \neq w \in \overline{NE(X)}$.

• MMP with scaling of BCHM.

Birkar–Cascini–Hacon–McKernan, Existence of minimal models for varieties of log general type, JAMS, 23 (2010), 405–468.

This is an algorithm for finding a specific seq. of birational maps $f_1, \cdots, f_k$.

First, choose an ample divisor H on X. Then define

$$T := \sup\{t>0 \mid H+tK_X>0\}$$

If $T=+\infty \Rightarrow K_X$ is nef, then is in case (i).

If $T<+\infty$. By the Rationaly Thm of Kawamata and Shokurov,

ample $\Rightarrow T$ is rational.

$\Downarrow$ $\Rightarrow L := H+TK_X$ defines a $\mathbb{Q}$-line bundle.

Semi-ample $\underset{\overset{?}{\Leftarrow}}{\Rightarrow}$ Nef.

Apply the Base Point Free Thm (Kawamata): If L is nef and $aL-K_X$ is nef and big for some $a>0$, then L is semi-ample.

$L=H+TK_X$ is nef,

$L-TK_X$ $\Rightarrow \exists$ sufficiently large $m\in\mathbb{N}$, L^m is globally generated

$=T(\frac{1}{T}L-K_X)$

$=H>0$ and $H^0(X,L^m)$ defines a holo. map $\pi: X\longrightarrow \mathbb{P}^N$ s.t.

$$L^m = \pi^*\mathcal{O}(1).$$

Denote $Y:=\pi(X)$, is uniquely determined for m sufficiently large.

Define a subcone $NE(\pi)$ of $\overline{NE(X)}$ by

$$NE(\pi) := \{w\in\overline{NE(X)} \mid L\cdot w=0\}$$

Thm 8.1 $\Rightarrow NE(\pi)\neq\phi$. (?)

Simplifying assumption: $NE(\pi)$ is an extremal ray of $\overline{NE(X)}$.

A ray R of $\overline{NE(X)}$ is a subcone of the form $R:=\{\lambda w \mid \lambda\in[0,\infty)\}$ for some $w\in\overline{NE(X)}$. We say that a subcone C in $\overline{NE(X)}$ is extremal if $a,b\in\overline{NE(X)}$, $a+b\in C \Rightarrow a,b\in C$.

The extremal ray $R = NE(\pi)$ has the additional property of being <u>K_X-negative</u>: if $K_X \cdot w < 0$, $\forall\, 0 \neq w \in R$.

Hence R contains "bad" curves ($K_X \cdot w < 0$) which we want to remove.

π contracts all curves whose class lies in the extremal ray $R = NE(\pi)$. The union of these curves is called <u>the locus</u> of R. In fact, the locus of $R = NE(\pi)$ is exactly the set of points where the map:

$\pi: X \to Y$ is not an isomorphism.

Case 1. The locus of R is equal to X. Then π is a fiber contraction and X is a <u>Mori fiber space</u>.

Case 2. The locus of R is an irreducible divisor D. In this case π is called a <u>divisorial contraction</u>.

Case 3. The locus of R has codimension at least 2. In this case, π is called a <u>small contraction</u>.

The process of the MMP with scaling is the as follows:

if we are in case 1, we stop, since X is already of type (ii).

In case 2, we have a map $\pi: X \to Y \subset \mathbb{P}^N$ to a subvariety Y.

Let $H_Y = \mathcal{O}(1)|_Y$. We can than repeat the process of the MMP with scaling with (Y, H_Y) instead of (X, H).

The serious difficulties occur in case 3. Here Y will have very bad singularities and it will not be possible to continue this process on Y. We need to find a new space to run MMP. This new space is a <u>flip</u>.

Let $\pi: X \to Y$ be a small contraction as in case 3.

The flip of $\pi: X \to Y$ is a variety X^+ together with a holo. birational map $\pi^+: X^+ \to Y$ satisfying the following conditions:

(a) The exceptional locus of π^+ (that is, the set of points in X^+ on which π^+ is not an isomorphism) has codim > 1.

(b) If C is a curve contracted by π^+, then $K_{X^+} \cdot C > 0$.

Thus we have a diagram

birational map, or is called a flip.

$$\begin{tikzcd} X \arrow[rr, dashed, "(\pi^+)^{-1} \circ \pi"] \arrow[dr, "\pi"'] & & X^+ \arrow[dl, "\pi^+"] \\ & Y & \end{tikzcd}$$

" $K_X \cdot C < 0$, $C \subset X \rightsquigarrow C' \subset X'$ w/ $K_{X^+} \cdot C' > 0$ "

$\exists$ of flip is a diffult problem.

Denote $H^+ := (\pi^+)^* \mathcal{O}(1)|_Y$. Then we run MMP with (X^+, H^+) instead of (X, H).

An example of a flip. Let $X_{m,n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m+1})$ be the $\mathbb{P}^{m+1}$ bundle over $\mathbb{P}^n$. Let $Y_{m,n}$ be the projective cone (?) over $\mathbb{P}^m \times \mathbb{P}^n$ in $\mathbb{P}^{(m+1)(n+1)}$ by the Segre embedding

$$[z_0, \cdots, z_m] \times [w_0, \cdots, w_n] \to [z_0 w_0, \cdots, z_0 w_n, z_1 w_0, \cdots, z_1 w_n, \cdots, z_i w_j, \cdots, z_m w_n] \in \mathbb{P}^{(m+1)(n+1)-1}$$

Note that $Y_{m,n} = Y_{n,m}$. Then $\exists$ a holo. map

$$\Phi_{m,n} : X_{m,n} \longrightarrow Y_{m,n} \quad (m \geq 1)$$

contracting the zero section of $X_{m,n}$ of codim $m+1$ to the cone singularity of $Y_{m,n}$. The following diagram gives a flip from $X_{m,n}$ to $X_{n,m}$ for $1 \le m < n$.

$$\begin{tikzcd} X_{m,n} \arrow[rr, dashed, "\Phi"] \arrow[dr, "\Phi_{m,n}"'] & & X_{n,m} \arrow[dl, "\Phi_{n,m}"] \\ & Y_{m,n} & \end{tikzcd}$$

- Analytic MMP with KRF

[Song-Tian 3] (X, H, ω), X: $\mathbb{Q}$-factorial proj. variety with log terminal sing., H: big semi-ample $\mathbb{Q}$-divisor s.t. $H+\varepsilon K_X>0$ for small $\varepsilon>0$,

Let X be a smooth proj. variety with an ample divisor H.

and $\omega \in \mathcal{K}_{H,p}(X)$ for some $p>1$.

For unnormalized KRF:

$$\begin{cases} \frac{\partial}{\partial t}\omega = -Ric(\omega) \\ \omega|_{t=0} = \omega_0 \end{cases}$$

Assume $\omega_0 \in c_1([H])$. Then the maximal time existence of KRF is

$$T = \sup\{t>0 \mid H+tK_X>0\}.$$

In general, we expect that as $t \to T$, the KRF carries out a "surgery", which is equivalent to the algebraic procedure of contracting an extremal ray, as discussed above.

The following is a conjectural picture for the behavior of the KRF, as proposed by Song-Tian [1-3].

Step 1. We start with a metric $\omega_0 \in c_1([H])$. We then consider the soln. $\omega(t)$ of KRF on X starting at ω_0. The flow exists on $[0,T)$ with $T = \sup\{t>0 \mid H+tK_X>0\}$.

Conj 1 (Song-Tian3) For each $t \in (0, T_0)$, the metric completion of X_{reg} by $\omega(t)$ is homeophic to X.

Step 2. If $T=\infty$, then K_X is nef and the KRF exists for all time. The flow $\omega(t)$ should converge, after an appropriate normalization, to a canonical 'generalized Kähler-Einstein metric' on X as $t\to\infty$.

The abundance conjecture predicts that K_X is semi-ample and $\mathrm{kod}(X)\geqslant 0$.

2.1 $\mathrm{Kod}(X)=\dim X$, i.e., X is a minimal model of general type.

Conj. 2 [Song-Tian 3]

The normalized KRF $\frac{\partial\tilde{\omega}}{\partial s}=-\mathrm{Ric}(\tilde{\omega})-\tilde{\omega}$ starting with ω converges to the unique Kähler-Einstein metric ω_{KE} on X_{can} in Gromov-Hausdorff sense as $s\to\infty$.

If X is nonsingular and K_X is ample, it is Cao's classical result.

If X is nonsingular, the weak convergence in distribution and smooth convergence outside the exceptional locus is obtained by Tsuji and Tian-Zhang (Zhou).

If X is a singular minimal model, they expect the KRF converge to the singular Kähler-Einstein metric of Guedj-Eyssidieux-Zeriahi [Singular Kähler-Einstein metrics, JAMS, 22 (2009), 607-639.

2.2 $0<\mathrm{kod}(X)<\dim X$.

Conj 3. [Song-Tian 3] The normalized KRF $\frac{\partial\tilde{\omega}}{\partial s}=-\mathrm{Ric}(\tilde{\omega})-\tilde{\omega}$ starting with ω converges to the unique generalized Kähler-Einstein metric ω_{can} on X_{can} in Gromov-Hausdorff sense as $s\to\infty$.

If K_X is semi-ample, for large m, $H^0(X, K_X^m)$ induces a holo. map

$$\phi: X \to X_{can} \subset \mathbb{C}P^N.$$

X_{can} is called the canonical model of X and it is uniquely determined by the canonical ring of X.

If $0 < kod(X) < \dim X$, then X admits a Calabi-Yau fibration over X_{can}. Here W_{can} should be (away from a subvariety of X_{can})

$$Ric(g_{can}) = -g_{can} + g_{WP}$$

where g_{WP} is the Weil-Petersson metric induced from the Calabi-Yau fibration of X over X_{can}.

If X is nonsingular, the weak convergence in distribution is obtained by [Song-Tian 1 & 2].

2.3 $kod(X) = 0$. K_X is numerically trivial.

Conj. 4 The KRF $\frac{\partial \omega}{\partial t} = -Ric(\omega)$ converges to the unique Ricci-flat Kähler metric in $[H]$ in Gromov-Hausdorff sense as $t \to \infty$.

If X is nonsingular, it is shown by Cao that the flow converges in $C^\infty(X)$. If X has log terminal singularities, Song-Yuan, Y obtained a weak convergence.

Step 3. If $T < \infty$, then the semi-ample divisor $H + TK_X$ induces a contraction

$$\pi: X \longrightarrow Y.$$

3.1 $\dim Y = \dim X$

Conj. 5 [Song–Tian 3] As $t \to T$, $(X, \omega(t))$ converges to a metric space (Y, ω_Y) along the KRF

$$\frac{\partial \omega}{\partial t} = -Ric(\omega)$$

in Gromov–Hausdorff sense. Furthermore, $\omega(t,\cdot)$ converges in C^∞ outside a subvariety S of X, and $(Y, \omega_Y) = \overline{(X \setminus S, \omega(T))}$. In particular, Y is normal proj. variety satisfying the following diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "(\pi^+)^{-1}\circ\pi"] \arrow[dr, "\pi"'] & & X^+ \arrow[dl, "\pi^+"] \\ & Y & \end{tikzcd}$$

where $X^+: X^+ \to Y$ is a general flip of X.

We then repeat Step 1 by replacing (X, H, ω) with $(X^+, H_{X^+}, \omega_{X^+})$ even though X^+ is not necessarily $\mathbb{Q}$-factorial. They furthure conjecture that along the new KRF, $(X^+, \omega_{X^+}(t))$ converges in Gromov–Hausdorff sense to (Y, ω_Y) as $t \to 0+$.

3.2 $0 < \dim Y < \dim X$. X then admits a Fano fibration over Y.

Conj. 6. Along the KRF, $(X, \omega(t)) \xrightarrow[t\to T]{GH} (Y', \omega_Y')$, which is a metric space. Let $H_{Y'}$ be the divisor where $\omega_{Y'}$ lies. Then both $K_{Y'}$ and $H_{Y'}$ are $\mathbb{Q}$-Cartier, and $\omega_{Y'} \in \mathcal{K}_{H_{Y'}, p'}(Y')$ for some $p' > 1$.

We then repeat Step 1 by replacing (X, H, ω) by $(Y', H_{Y'}, \omega_{Y'})$.

3.3 If $\dim Y = 0$, X is Fano and $\omega \in -T[K_X]$. Then Song–Tian have the following generalized Hamilton–Tian conjecture.

Conj. 7. Then the normalized KRF

$$\frac{\partial \tilde{\omega}}{\partial s} = -Ric(\tilde{\omega}) + \frac{1}{T}\tilde{\omega}$$

starting with ω converges to a Kähler-Ricci soliton (X_∞, ω_{KR}) in Gromov-Hausdorff sense as $s \to \infty$.

Diameter estimate along KRF.

(1) Jian-Song, GAFA, X is semiample

$$\Rightarrow \mathrm{diam}(X, \omega(t)) \leq C.$$

(2) Guo-Phong-Song-Sturm, X is nef.

$$\Rightarrow \mathrm{diam}(X, \omega(t)) \leq C.$$