2023.3.10

Prop 6.16: Let (M,J) be a Fano mfd w/ KRS (g_{KS}, X). Let $\phi = \phi(\cdot, t) = \phi_t$ be a K_X-inv. soln. of KRF

(KRF) $$\frac{\partial \phi}{\partial t} = \log \frac{\omega_\phi^n}{\omega_{KS}^n} + \phi + X(\phi), \quad t \in (0,1)$$

Suppose that
$$|\phi| \leq A \quad \text{and} \quad |\dot{\phi}| = \left|\frac{\partial \phi}{\partial t}\right| \leq A$$

Then $\forall\, k \in \mathbb{N} \cup \{0\}$, $\exists$ uniform const. $C_k = C_k(\omega_{KS}, A, k)$ s.t.
$$\|\phi_t\|_{C^k(M)} < C_k, \quad \forall t \in [\tfrac{1}{4}, 1)$$

Pf of Prop. 6.16.

Lem. 6.17: $$n + \Delta\phi \leq e^{\frac{B}{t}}, \quad \forall t \in (0,1)$$

Pf of Lem. 6.17: Let $\Delta' = \Delta'_t$ be the Laplacian operator associated to ω_{ϕ_t} $(\sim g')$. Set
$$F = \frac{\partial \phi}{\partial t} - \underline{X(\phi)} - \phi.$$

is uniformly bounded by [Zhu]

By the condition, we know F is uniformly bounded and

$$(6.4.10) \qquad \log \frac{\omega_{\phi_t}^n}{\omega_{ks}^n} = F$$

Differentiating (6.4.10) for $\frac{\partial}{\partial z^k}$, we have

$$(6.4.11) \quad \sum g'^{i\bar{j}} \left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \phi}{\partial z^i \partial \bar{z}^j \partial z^k} \right) - \sum g^{i\bar{j}} \frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial F}{\partial z^k}.$$

$g = g_{ks}$

$\Delta \sim g$

$\Delta' \sim g' = g_{\omega_{\phi_t}}$

Differentiating (6.4.11) for $\frac{\partial}{\partial \bar{z}^l}$ again, we obtain

$$(6.4.12) \quad -g'^{t\bar{j}} g'^{i\bar{n}} \left(\frac{\partial g_{t\bar{n}}}{\partial \bar{z}^l} + \frac{\partial^3 \phi}{\partial z^t \partial \bar{z}^n \partial \bar{z}^l} \right) \left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \phi}{\partial z^i \partial \bar{z}^j \partial z^k} \right)$$

$$+ g'^{i\bar{j}} \left(\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \frac{\partial^4 \phi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l} \right)$$

$$+ g^{t\bar{j}} g^{i\bar{n}} \frac{\partial g_{t\bar{n}}}{\partial \bar{z}^l} - g^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} = \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l}.$$

$$\left[\frac{\partial g'^{i\bar{j}}}{\partial \bar{z}^l} = -g'^{t\bar{j}} g'^{i\bar{n}} \frac{\partial g'_{t\bar{n}}}{\partial \bar{z}^l} \right]$$

We can compute

$$\Delta'(\Delta\phi) = g'^{k\bar l}\partial_k\partial_{\bar l}\left(g^{i\bar j}\partial_i\partial_{\bar j}\phi\right)$$

$$(6.4.13)\quad = g'^{k\bar l}g^{i\bar j}\partial_k\partial_{\bar l}\partial_i\partial_{\bar j}\phi + g'^{k\bar l}\frac{\partial^2 g^{i\bar j}}{\partial z^k\partial\bar z^l}\partial_i\partial_{\bar j}\phi + g'^{k\bar l}\frac{\partial g^{i\bar j}}{\partial z^k}\partial_i\partial_{\bar j}\partial_{\bar l}\phi + g'^{k\bar l}\frac{\partial g^{i\bar j}}{\partial \bar z^l}\partial_i\partial_{\bar j}\partial_k\phi$$

We first choose a local coord. system at a given pt $p\in M$, so that

$$g_{i\bar j}(p) = \delta_{i\bar j},\quad \frac{\partial g_{i\bar j}}{\partial z^k}(p) = \frac{\partial g_{i\bar j}}{\partial \bar z^l}(p) = 0.$$

Then $(6.4.12 + 6.4.13)$

$$(6.4.14)\quad \Delta'(\Delta\phi) = \Delta F + g'^{k\bar j}g'^{i\bar n}\phi_{k\bar n\bar l}\,\phi_{i\bar j l} + g'^{i\bar j}R_{i\bar j l\bar l} - R_{i\bar i l\bar l} + g'^{k\bar l}R_{i\bar j k\bar l}\phi_{i\bar j}$$

We secondly choose another coord. system so that

$$g_{ij}(p) = \delta_{ij} \quad\text{and}\quad \phi_{i\bar j}(p) = \delta_{ij}\,\phi_{i\bar i}(p),\ \text{then}$$

$$g'^{i\bar j}(p) = \delta_{ij}(1+\phi_{i\bar i})^{-1} \qquad (1+\phi_{i\bar i} > 0)$$

and

$$(6.4.15)\quad g'^{i\bar j}R_{i\bar j l\bar l} - R_{i\bar i l\bar l} + g'^{k\bar l}R_{i\bar j k\bar l}\phi_{i\bar j} = \frac{1}{2}R_{i\bar i l\bar l}\frac{(\phi_{i\bar i}-\phi_{l\bar l})^2}{(1+\phi_{i\bar i})(1+\phi_{l\bar l})}$$

$(6.4.14) + (6.4.15) \Rightarrow$

$$(6.4.16)\quad \Delta'(\Delta\phi) \geqslant \Delta F + g'^{k\bar j}g'^{i\bar n}\phi_{k\bar n l}\,\phi_{i\bar j l} + \inf_{i\neq l} R_{i\bar i l\bar l}\left[\sum_{i,l}\frac{1+\phi_{i\bar i}}{1+\phi_{l\bar l}} - n^2\right]$$

Since $\Delta'\phi = \sum_i \frac{\phi_{i\bar i}}{1+\phi_{i\bar i}} = n - \sum_{i=1}^n \frac{1}{1+\phi_{i\bar i}}$,

Schwart ineq.

using Schwarz's ineq., we get

$$(6.4.17)\quad \Delta'(e^{-c\phi}(n+\Delta\phi)) \geq -ce^{-c\phi}\cdot g'^{i\bar j}\phi_i(\Delta\phi)_{\bar j}$$
$$-ce^{-c\phi}(\Delta'\phi)(n+\Delta\phi) + e^{-c\phi}\Delta'(\Delta\phi)$$

$(6.4.16) \Rightarrow$

$$(n+\Delta\phi)^{-1}(g'^{i\bar j}(\Delta\phi)_i(\Delta\phi)_{\bar j})$$
$$\geq -(n+\Delta\phi)^{-1}\sum_i(1+\phi_{i\bar i})^{-1}\left|\sum\phi_{k\bar k i}\right|^2$$
$$(6.4.18)\quad +\Delta F + \sum_{k,i,j}(1+\phi_{k\bar k})^{-1}(1+\phi_{i\bar i})^{-1}\phi_{ki\bar j}\,\phi_{\bar i k\bar j}$$
$$+\sum_{i,l}\inf_{i\neq l} R_{i\bar i l\bar l}\left[\sum\frac{1+\phi_{l\bar l}}{1+\phi_{i\bar i}} - n^2\right]$$

We can estimate

$$(6.4.19)\quad (n+\Delta\phi)^{-1}\sum_i(1+\phi_{i\bar i})^{-1}\left|\sum\phi_{k\bar k i}\right|^2 \leq \sum_{k,i,j}(1+\phi_{k\bar k})^{-1}(1+\phi_{i\bar i})^{-1}\phi_{k\bar i j}\,\phi_{i\bar k\bar j}$$

$(6.4.17)-(6.4.19)$, we get

$$\Delta'(e^{-c\phi}(n+\Delta\phi)) \geq e^{-c\phi}\Big[(\Delta F - n^2\inf_{i\neq l} R_{i\bar i l\bar l} - cn(n+\Delta\phi)$$
$$+(c+\inf_{i\neq l} R_{i\bar i l\bar l})(n+\Delta\phi)\Big(\sum_{i=1}^n\frac{1}{1+\phi_{i\bar i}}\Big)\Big]$$

We can choose c large, s.t.

$$\inf R_{i\bar i l\bar l} \geq -\frac{c}{2}.$$

$$\Rightarrow \quad \Delta'\left(e^{-c\phi}(n+\Delta\phi)\right) \geq e^{-c\phi}\Big[\Delta F - C_1 - cn(n+\Delta\phi) + \frac{c}{2}(n+\Delta\phi)\Big(\sum_{i=1}^{n}\frac{1}{1+\phi_{i\bar{i}}}\Big)\Big] \tag{6.4.20}$$

$\because$ $F = \frac{\partial\phi}{\partial t} - X(\phi) - \phi$

$$\Delta F = \Delta(\dot{\phi}) - \Delta(X(\phi)) - \Delta\phi$$

$$\geq \frac{\partial}{\partial t}(n+\Delta\phi) - (|\nabla X|_{g_{KS}}+1)(n+\Delta\phi) - n|\nabla X|_{g_{KS}}$$

$$\Delta(X(\phi)) = \partial_k\partial_{\bar{k}}(X^i\phi_i)$$

$$= \partial_k(\cancel{\partial_{\bar{k}}X^i}^{=0}\cdot\phi_i + X^i\phi_{i\bar{k}})$$

$$= \underline{\partial_k X^i}\ \underline{\phi_{i\bar{k}}} + X^i\underline{\phi_{i\bar{k}k}}$$

$$\geq X(\Delta\phi) - |\nabla X|_{\omega_{KS}}(n+\Delta\phi) - n|\nabla X|_{\omega_{KS}}$$

$$-\frac{c}{t}(n+\Delta\phi)\sup X(\phi)$$

$$-e^{\frac{c}{t}(\phi+A+1)}\left(X\left(e^{-\frac{c}{t}(\phi+A+1)}(n+\Delta\phi)\right)\right)$$

$$\geq \frac{\partial}{\partial t}(n+\Delta\phi) - \frac{C_2 c}{t}(n+\Delta\phi) - C_3 - e^{\frac{c}{t}(\phi+A+1)}\left(X\left(e^{-\frac{c}{t}(\phi+A+1)}(n+\Delta\phi)\right)\right)$$

We need the following ineq.

(1). $\forall a_i > 0$ s.t. $\prod_{i=1}^{n} a_i = 1$, then $\left(\sum_{i=1}^{n} a_i\right)^{n-1} \geq \sum_{i=1}^{n}\frac{1}{a_i}$
$(n\geq 2)$.

(1)': $\forall a_i > 0$, we have

$$\sum_{i=1}^{n} a_i \leq \left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^{n-1}\prod_{i=1}^{n} a_i$$

Using (1)', we have

$$\sum_{i=1}^{n} \frac{1}{1+\phi_{i\bar{i}}} \geq \prod_{i=1}^{n} (1+\phi_{i\bar{i}})^{-\frac{1}{n+1}} (n+\Delta\phi)^{\frac{1}{n-1}}$$

$$(6.4.22) \qquad = e^{-\frac{E}{n-1}} (n+\Delta\phi)^{\frac{1}{n-1}}$$

$$\geq \delta (n+\Delta\phi)^{\frac{1}{n-1}}$$

$$\Longrightarrow \quad (\Delta' + X)\left(e^{-\frac{C}{t}(\phi+A+1)} (n+\Delta\phi)\right)$$

$$(6.4.23) \qquad \geq e^{-\frac{C}{t}(\phi+A+1)} \left\{ \frac{\partial}{\partial t}(n+\Delta\phi) - \frac{C_3 C}{t}(n+\Delta\phi) - C_4 + \frac{C}{C_5 t}(n+\Delta\phi)^{\frac{n}{n-1}} \right\}$$

and

$$(6.4.24) \qquad \frac{\partial}{\partial t}\left(e^{-\frac{C}{t}(\phi+A+1)} (n+\Delta\phi)\right) = e^{-\frac{C}{t}(\phi+A+1)} \frac{\partial}{\partial t}(n+\Delta\phi) - C\left(\frac{1}{t}\dot{\phi} - \frac{1}{t^2}(\phi+A+1)\right) e^{-\frac{C}{t}(\phi+A+1)} (n+\Delta\phi)$$

$\because$ $\dot{\phi}$ and ϕ are uniformly bounded, we get

$$(6.4.25) \qquad (\Delta' + X - \partial_t)\left(e^{-\frac{C}{t}(\phi+A+1)} (n+\Delta\phi)\right) \geq e^{-\frac{C}{t}(\phi+A+1)} \left(\frac{C}{C_6 t} \underline{(n+\Delta\phi)^{\frac{n}{n-1}}} - \frac{C_7 C}{t^2}(n+\Delta\phi) - C_8 \right)$$

Set $H := e^{-\frac{C}{t}(\phi+A+1)} (n+\Delta\phi) = 0$ at $t=0$, we can apply the maximum principle to H in (6.4.25), then $\exists t_0>0$ and $x_0\in M$, s.t.

$$\max_{M\times[0,1]} H = H(x_0,t_0). \text{ Hence}$$

$$(6.4.26)\quad \frac{c}{C_6 t_0}(n+\Delta\phi)^{\frac{n}{n-1}}\Big|_{(x_0,t_0)} - \frac{C_7 C}{t_0^2}(n+\Delta\phi)\Big|_{(x_0,t_0)} - C_8 \leq 0$$

$$\Longrightarrow (n+\Delta\phi)\Big|_{(x_0,t_0)} \leq \frac{C_9}{t_0^{n-1}}$$

$$\Longrightarrow (6.4.27)\quad \underbrace{e^{-\frac{C}{t}(\phi+A+1)}(n+\Delta\phi)}_{=H(x,t)}$$

$$\leq \underbrace{e^{-\frac{C}{t}(\phi+A+1)}}_{\leq 1}\underbrace{(n+\Delta\phi)\Big|_{(x_0,t_0)}}_{=H(x_0,t_0)}$$

$$\leq \frac{C_9}{t_0^{n-1}} \leq C_{10}.$$

$$\Longrightarrow (n+\Delta\phi)\leq e^{\frac{B}{t}}.$$

\#

Lem 3.18: Under the condition of Prop 6.16, we have

(1) For $0<t_0<1$, then $\forall t\geq t_0$, $\omega_{KS}+\sqrt{-1}\partial\bar\partial\phi_t$ is uniformly equivalent to ω_{KS}.

(2) $\exists$ a uniform const. $\gamma = \gamma(\omega_{KS}, A)$ s.t.

$$\|\phi\|_{C^3(M)} \le e^{\frac{\gamma}{t}}, \quad \forall t \in (0,1).$$

Pf: (i) Assume the eigenvalue of ω_ϕ with ω_{KS} are $\lambda_1, \cdots, \lambda_n$, by conditions, we know

$$C^{-1} \le \prod_{i=1}^n \lambda_i \le C$$

By Lem 6.17: $\displaystyle\sum_{i=1}^n \lambda_i \le e^{\frac{B}{t}} \le C$ for $t \ge t_0$

$$\Rightarrow \forall i, \ \lambda_i \le e^{\frac{B}{t}}$$

$$\& \ \lambda_i \ge C_1^{-1} e^{-\frac{(n-1)B}{t}}.$$

(ii) We consider Calabi's function

$$S = g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} \phi_{i\bar{j}k} \phi_{\bar{r}s\bar{t}}$$

$t \ge t_0 > 0$, $\displaystyle \approx g^{i\bar{r}} g^{s\bar{j}} g^{k\bar{t}} \phi_{i\bar{j}k} \phi_{\bar{r}s\bar{t}}.$

As in the proof of (6.4.25), for large const. $\alpha > 0$, we have

$$(6.4.29) \quad (\Delta' - \partial_t + X)\left(e^{-\frac{2\alpha}{t}}(n+\Delta\phi)\right) \ge C_1 e^{-\frac{\alpha}{t}}(n+\Delta\phi) - C_2$$

Following Yau's Calabi's computation, and using (1) of Lem 6.18, for large const. $\beta > 0$, we get

$$(6.4.30) \quad (\Delta' - \partial_t + X)\left(e^{-\frac{2\beta}{t}} S\right) \ge -C_3 e^{-\frac{\beta}{t}} S - C_4$$

(also see [Phong-Sesum-Sturm, Multiplier ideal sheaves and the Kähler-Ricci flow])

Choose $\beta > \alpha$ and large const. $A > 0$,

$$(6.4.31)\quad (\Delta' - \partial_t + X)\left(e^{-\frac{2\beta}{t}} S + A e^{-\frac{2\alpha}{t}}(n + \Delta\phi)\right) \geq C_5 e^{-\frac{\alpha}{t}} S - C_6$$

Using the maximum principle, we can get

$$S \leq C_7 \cdot e^{\frac{2\beta}{t}}. \qquad \#$$

Then using the regularity theory of the parabolic eqn we obtain Prop. 6.16.

$$\#$$

- Guo (Bin) - Phong - Sturm, On the Kähler-Ricci flow on Fano manifolds.

- **Lema.** $\mu_{X,\omega_0}(\varphi) \geq C J_G(G\varphi) - D, \ \forall \varphi \in P_X(M, \omega_0)$

$$\Downarrow$$

$$\| \sigma_j^* \phi_{t_j} \|_{C^\alpha(M,\omega_0)} \leq C, \quad \sup_M \sigma_j^* \phi_{t_j} = 0.$$

Set $\psi_j = \sigma_j^* \phi_{t_j}$

$$\Longrightarrow$$

$$C^{-1} \omega_0^n \leq \tilde{\omega}_j^n \leq C \omega_0^n$$

$$\Longrightarrow \quad \tilde{\omega}_j^n = e^{-(\psi_j - \sup_M \psi_j) + f_{\tilde{\omega}_j} - f_{\omega_0}} \omega_0^n$$

$\Rightarrow \psi$ has zero Lelong number

$\Rightarrow \forall p > 0,$ $\quad \int_M e^{-p\psi} \omega_0^n \le C_p, \ \forall \psi \in \underline{\underline{S_A}}.$

$\xrightarrow{\text{Kolodziej's Thm}}$ $\quad \underline{\|\tilde{\psi}\|_{C^\alpha(M, \omega_0)} \le C.}$