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§6.4 A convergence theorem of KRF

Thm 6.1 (Main Thm) [Tian-Zhu, JAMS; Tian-Z.-Zhang Z.L.-Zhu, TAMS]

Let (M, J) be a cpt Kähler mfd which admits a Kähler-Ricci soliton (g_{KS}, X), i.e., $Ric(g_{KS}) - g_{KS} = L_X g_{KS}$. Then the KRF

$$(KRF)\quad \begin{cases} \frac{\partial g(t)}{\partial t} = -Ric(g(t)) + g(t) \\ g(0) = g \end{cases}$$

$\omega_{KS} = \omega_{g_{KS}}$

with $g \in \mathcal{K}_X := \{ \omega_g = \omega_{KS} + \sqrt{-1}\partial\bar\partial\varphi > 0 \mid Im(X(\varphi)) = 0 \}$

will converges to a Kähler-Ricci soliton in C^∞ in the sense of Kähler potentials. Moreover, the convergence can be made fast exponentialy.

Pf of Thm 6.1: We write the soln. of (KRF) as $(g_t; g)$ $(g_t = g(t))$. We write the initial metric g as

$$\omega_g = \omega_\varphi = \omega_{KS} + \sqrt{-1}\partial\bar\partial\varphi \in 2\pi c_1(M)$$

for a Kähler potential φ on M. We define a path of Kähler forms

$$\omega_{g^s} = \omega_{s\varphi} = \omega_{KS} + \underbrace{s\cdot\sqrt{-1}\partial\bar\partial\varphi}_{= \sqrt{-1}\partial\bar\partial(s\varphi)}$$

and set

$$I = \{ s \in [0,1] \mid (g_t^s; g^s) \text{ converges a Kähler-Ricci soliton exponentially in } C^\infty \text{ in the sense of Kähler potentials} \}$$

Goal: $1 \in I$.

Since $\omega_{g^0} = \omega_0 = \omega_{KS}$, $0 \in I \Rightarrow I \neq \phi$.

Our goal is to prove that I is both open and closed.

$\Rightarrow I = [0,1] \Rightarrow 1 \in I$, i.e., Thm 6.1.

KRS: Kähler-Ricci soliton

Lem 6.14 (Zhu, Math. Ann. 2013)

Let (M, J) be a cpt Kähler mfd which admits a KRS (g_{KS}, X). Let ψ be a Kähler potential of a K_X-inv. initial metric g of (KRF), i.e., $\omega_g = \omega_{KS} + \sqrt{-1}\partial\bar\partial\psi$. Then $\exists\, \varepsilon > 0$ (small) such that if $\|\psi\|_{C^3(M)} \leq \varepsilon$ the soln. $g(t)$ of (KRF) converges to a KRS with respect to X in C^∞ in the sense of Kähler potentials. Moreover, the convergence can be made fast exponentially.

Pf of openness of I.

Suppose that $s_0 \in I$. Hence $(g_t^{s_0}; g^{s_0}) \xrightarrow{\text{exponentially}}$ KRS. Using the unique theorem of Tian-Zhu, $\exists\, \sigma \in \mathrm{Aut}_r(M)$ s.t

$$\sigma^* \omega_{g_t^{s_0}} = \omega_{KS} + \sqrt{-1}\partial\bar\partial(\phi_t^{s_0})_\sigma$$

with

$$\|(\phi_t^{s_0})_\sigma\|_{C^k(M), g_{KS}} \leq C_k e^{-\alpha_k t}$$

Where $C_k, \alpha_k > 0$ are two uniform constants.

Then we can choose T sufficiently large such that

$$\|(\phi_T^{s_0})_\sigma\|_{C^3(M)} < \frac{\delta}{2}$$

(where δ is the small number, e.g, $\delta \le \varepsilon$ in Lem 6.14)

Since the KRF is stable in any fixed finite time, $\exists \varepsilon > 0$ s.t.

$$\|\hat{\phi}_T^s - (\phi_T^{s_0})_\sigma\|_{C^3(M)} < \frac{\delta}{2}, \ \forall s \in [s_0, s_0+\varepsilon]$$

where $\hat{\phi}_T^s$ is a Kähler potential of $\hat{g}_T^s$, soln. of KRF $(\hat{g}_t^s = \sigma^* g^s)$ at time T. We have

$$\|\hat{\phi}_T^s\|_{C^3(M)} < \delta, \quad \forall s \in [s_0, s_0+\varepsilon]$$

By Lem. 6.14, the flow $(g_t, \hat{g}_T^s) \xrightarrow[C^\infty]{\text{exponentially}}$ KRS.

$\Rightarrow [s_0, s_0+\varepsilon) \subset I$.

$\Rightarrow I$ is open.

For prove the closedness of I. We need the following Lemma.

Lem 6.15 Let M be a cpt Kähler mfd which admits a KRS (g_{KS}, X). Let φ be a K_X-inv. Kähler potential and

$$\varphi_\gamma = \gamma^* \varphi + \rho_\gamma, \quad \forall \gamma \in Aut_r(M)$$

where ρ_γ is defined by $\gamma^* \omega_{KS} = \omega_{KS} + \sqrt{-1}\partial\bar{\partial}\rho_\gamma$

with normalization condition $\int_M e^{-\varphi_\gamma}\omega_{KS}^n = \int_M \omega_{KS}^n$.

Then $\exists!\ \sigma \in Aut_r(M)$ s.t. $\varphi_\sigma \in \Lambda^\perp(\omega_{KS})$ with properly

$$J(\varphi_\sigma) = \inf_{\gamma \in Aut_r(M)} J(\varphi_\gamma)$$

where $\Lambda^\perp(\omega_{KS}) := \left\{ \psi \in C^\infty(M) \,\middle|\, \int_M \psi\cdot\varphi\, \omega_{KS}^n = 0,\ \forall \varphi \in Ker(\Delta_{g_{KS}} + X + I_d) \right\}$

i.e. $(\Delta_{g_{KS}} + X + I_d)(\varphi) = 0$

and

$$J(\varphi) := -\int_M \varphi e^{\theta_X(\omega_\varphi)}\omega_\varphi^n + \int_0^1 \int_M \varphi e^{\theta_X(\omega_{\lambda\varphi})}\omega_{\lambda\varphi}^n \wedge d\lambda \quad (\geq 0)$$

Moreover,
$$\|\sigma - Id\| \leq C(\|\varphi\|_{C^5(M)}),$$
where $\|\sigma - Id\|$ denotes the distance norm in Lie group $Aut_r(M)$.

Pf of closedness of I.

$\exists T_0 \leq 1$ with $[0, T_0) \subset I$.

Goal: To show $T_0 \in I$.

We want to prove that for any $\delta > 0$, $\exists T$ (large) st

$$\|(\phi_t^s)_{\sigma_{s,t}}\|_{C^5(M)} \leq \delta, \quad \forall t \geq T \text{ and } s < T_0$$

Where $\sigma_{s,t} \in Aut_r(M)$.

We will use an argument by contradiction. On the contrary by Lem. 6.15, one can find a seq. of evolved Kähler metric $g_{t_i}^{s_i}$ of KRF $(g_t^{s_i}; g^{s_i})$, where $s_i \to T_0$ and $t_i \to \infty$

$\sigma_{s_i,t_i} \in \mathrm{Aut}_r(M)$ $\left(\|(\phi^{s_i}_{t_i})_{\sigma_{s_i,t_i}}\|_{C^5(M)} \geq \delta_0 \right)$ s.t.

$$(\phi^{s_i}_{t_i})_{\sigma_{s_i,t_i}} \in \Lambda^{\perp}(W_{KS})$$

and (6.4.1) $\|(\phi^{s_i}_{t_i})_{\sigma_{s_i,t_i}}\|_{C^5(M)} \geq \delta_0$ (?)

for some const. $\delta_0 > 0$.

On the other hand, for $s < \tau_0$

$\Rightarrow (g^s_t, g^s) \to$ a KRS.

By the uniqueness of KRS, $\exists!$ $\sigma_s \in \mathrm{Aut}_r(M)$ s.t.

$$\lim_{t\to\infty} \|(\phi^s_t)_{\sigma_s}\|_{C^5(M)} = 0$$

Using Lem 6.15,

(6.4.2) $\lim_{t\to\infty} \|(\phi^s_t)_{\sigma_{s,t}}\|_{C^5(M)} = 0$, $(\phi^s_t)_{\sigma_{s,t}} \in \Lambda^{\perp}(W_{KS})$

By (6.4.1) and (6.4.2), we may further assume that

$\phi^{s_i}_{t_i}$ satisfies

(6.4.3) $\|(\phi^{s_i}_{t_i})_{\sigma_{s_i,t_i}}\|_{C^5(M)} \leq 2\delta_0$.

Claim: $\exists$ a subseq. (still denoted by $(\phi^{s_i}_{t_i})_{\sigma_{s_i,t_i}}$) of $(\phi^{s_i}_{t_i})_{\sigma_{s_i,t_i}} \longrightarrow \phi_\infty \in \Lambda^{\perp}(W_{KS})$ with

(6.4.4) $\quad \delta_0 \leq \|\phi_\infty\|_{C^5(M)} \leq 2\delta_0$

KRF $\Leftrightarrow$

(6.4.5) $\quad \dfrac{\partial \phi'}{\partial t} = \log \dfrac{\omega_{\phi'}^n}{\omega_{KS}^n} + \phi' + X(\phi'), \quad \phi'(0,\cdot) = s\varphi + c.$

where $\omega_{\phi'} = \omega_{KS} + \sqrt{-1}\,\partial\bar\partial\phi'$.

$(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}}$ satisfying the following eqn.

(6.4.6) $\quad \dfrac{\partial \phi}{\partial t} = \log \dfrac{\omega_\phi^n}{\omega_{KS}^n} + \phi + X(\phi), \quad \omega_{\phi(0,\cdot)} = (\sigma_{s_i,t_i})^* \omega_{s_i\varphi}$

Existence of KRS $(g_{KS}, X) \Rightarrow \mu(\cdot) \geqslant -C > -\infty$.

and $h_{\phi'}$ is uniformly bounded (Perelman's estimate).

one choose a suitable c such that

$$\left|\frac{\partial \phi'}{\partial t}\right| < C(\omega_{s\varphi}) \leq C.$$

$\because \quad \dfrac{\partial \phi}{\partial t} = \dfrac{\partial \phi'}{\partial t} + \theta_X(\omega_\phi) \quad$ and $\quad |\theta_X(\omega_\phi)| \leq C$

$\Rightarrow \quad \left|\dfrac{\partial \phi}{\partial t}\right| \leq C.$

In particular, (6.4.7) $\quad \left|\dfrac{\partial \phi}{\partial t}\right| < C', \quad \forall t \in (t_i - \tfrac{1}{2}, t_i + \tfrac{1}{2})$

$\overset{(6.4.8)}{\Rightarrow} \quad |\phi(t,\cdot)| < C' + 2\delta_0, \quad \forall t \in (t_i - \tfrac{1}{2}, t_i + \tfrac{1}{2})$

[since $\quad |\phi(t_i,\cdot)| = |(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}}| \leq 2\delta_0$]

Prop 6.16: Let (M, J) be a Fano mfd w/ KRS (g_{KS}, X). Let $\phi = \phi(\cdot, t) = \phi_t$ be a K_X-inv. soln. of KRF

$$\frac{\partial \phi}{\partial t} = \log \frac{\omega_\phi^n}{\omega_{KS}^n} + \phi + X(\phi), \quad t \in (0,1)$$

Suppose that

$$|\phi| \leq A \quad \text{and} \quad |\dot{\phi}| = \left|\frac{\partial \phi}{\partial t}\right| \leq A$$

Then $\forall k \in \mathbb{N} \cup \{0\}$, $\exists$ uniform const. $C_k = C_k(\omega_{KS}, A, k)$ s.t.

$$\|\phi_t\|_{C^k(M)} < C_k, \quad \forall t \in [\tfrac{1}{4}, 1)$$

By (6.4.7) and (6.4.8) + Prop 6.16,

$$\|\phi(t, \cdot)\|_{C^k(M)} \leq C_k, \quad \forall t \in (t_i - \tfrac{1}{4}, t_i + \tfrac{1}{2})$$

In particular, $(\phi_{t_i}^{s_i})_{\sigma_{s_i}, t_i}$ is uniformly C^k-bounded.

$$\Rightarrow \quad (\phi_{t_i}^{s_i})_{\sigma_{s_i}, t_i} \xrightarrow[\forall k \in \mathbb{N} \cup \{0\}]{C^k} \phi_\infty$$

. Next we want to show that

(6.4.9) $$\lambda(\omega_{\phi_\infty}) = \lambda(\omega_{KS}) = (2\pi)^{-n}(nV - N_X(C_1(M))).$$

M admits a KRS $g_{KS} \Rightarrow \mu(\cdot) \geq -C > -\infty$

$\xrightarrow{\text{Prop 6.11}}$ $\exists\, T > 0$ s.t. $\lambda(g_t^{\tau_0}) \geq (2\pi)^{-n}(nV - N_X(C_1(M))) - \frac{\varepsilon}{2}$,

$\lambda(g_t^{\tau_0}) \nearrow$ as t $\qquad \forall t \geq T$

Since the KRF is stable in finite time and $\lambda(g_t^s) \nearrow$ as t

$\exists\, \delta > 0$ s.t. $\forall s \geq \tau_0 - \delta$, we have

$$\lambda(g_t^s) \geq (2\pi)^{-n}(nV - N_X(C_1(M))) - \varepsilon, \quad \forall t \geq T.$$

$$\Rightarrow \lim_{s_i \to \tau_0, t_i \to \infty} \lambda(\sigma_{s_i, t_i}^* g_{t_i}^{s_i}) = \lim_{s_i \to \tau_0, t_i \to \infty} \lambda(g_{t_i}^{s_i}) = (2\pi)^{-n}(nV - N_X(C_1(M))).$$

$\Rightarrow$ (6.4.9) holds.

From Cor 6.15, ω_{ϕ_∞}: global maximizer of $\lambda(\cdot)$ in K_X.

$\because \; \delta\lambda(g) = -(2\pi)^{-n} \int_M \langle \delta g, Ric(g) - g + \nabla^2 f \rangle e^{-f} dv_g$

$\Rightarrow$ ω_{ϕ_∞} is a KRS.

By the uniqueness of KRS, $\exists\, \sigma \in Aut_r(M)$ s.t.

$$\omega_{\phi_\infty} = \sigma^* \omega_{KS} \Rightarrow \phi_\infty = \rho_\sigma$$

$$\int_M e^{-\rho_\sigma} \omega_{KS} = \int_M \omega_{KS}^n, \text{ i.e. } \int_M e^{-\phi_\infty} \omega_{KS}^n = \int_M \omega_{KS}^n$$

Since $\phi_\infty \in \Lambda^\perp(\omega_{KS})$, by Lem 6.15.

$\Rightarrow \underline{\phi_\infty = 0}$ (?)

It is a contradiction!

Hence we prove that

$$\|(\phi_t^s)_{\sigma_{s,t}}\|_{C^5(M)} \le \delta, \quad \forall t \ge T \text{ and } s < \tau_0.$$

$\Rightarrow \exists\, T_0$ and $\sigma_0 \in \mathrm{Aut}_r(M)$ s.t

$$\|(\phi_{T_0}^{\tau_0})_{\sigma_0}\|_{C^5(M)} \le \delta$$

By Lem 6.14,

$$\left(g_t ; \omega_{(\phi_{T_0}^{\tau_0})_{\sigma_0}}\right) \xrightarrow{\text{exponentially}} \text{a KRS.}$$

$\Rightarrow \tau_0 \in I.$

$\Rightarrow I$ is closed.

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