Conj: Suppose that the modified Mabuchi's K-energy is bounded from below. Then

$$\sup_{g' \in 2\pi c_1(M)} \lambda(g') = (2\pi)^{-n} [nV - N_X(c_1(M))].$$

✓ It is true, was proved by Dervan-Szekelyhidi.

2023.2.24.

To prove Prop. 6.11, we need the following Key Lemma.

Lem 6.13: Let f_t be a minimizer of the $W(g_t\cdot)$-functional associated evolved Kähler metric g_t of (6.11), h_t: Ricci potential of g_t, i.e, $Ric(g_t) - g_t = \sqrt{-1}\partial\bar\partial h_t$ w/ $\int_M e^{h_t}\omega_{g_t}^n = V$

Then $\exists\ t_i \in [i, i+1]$ s.t.

(a) $\lim_{t_i\to\infty} \|\Delta(f_{t_i} + h_{t_i})\|_{L^2(M, g_{t_i})} = 0$;

(b) $\lim_{t_i\to\infty} \|\nabla(f_{t_i} + h_{t_i})\|_{L^2(M, g_{t_i})} = 0$

(c) $\lim_{t_i\to\infty} \|f_{t_i} + h_{t_i}\|_{C^0} = 0$.

Rk: The above result can be improved as $t\to\infty$.

Pf of Lem 6.13: By the first variation of Perelman's μ-entropy

$$\frac{d}{dt}\lambda(g_t) = (2\pi)^{-n}\int_M |\mathrm{Ric}(g_t) - g_t + \nabla^2 f_t|^2_{g_t} e^{-f_t}\omega^n_{g_t}$$

$$\Rightarrow \frac{d}{dt}\lambda(g_t) \geq (2\pi)^{-n}\int_M \frac{1}{2n}(R - n + \Delta f_t)^2 e^{-f_t}\omega^n_{g_t}$$

$$\left[\mathrm{Ric}(g_t) - g_t = \sqrt{-1}\partial\bar\partial h_t \Rightarrow R - n = \Delta h_t\right]$$

$$= \frac{(2\pi)^{-n}}{2n}\int_M \underbrace{|\Delta(h_t + f_t)|}^2 e^{-f_t}\omega^n_{g_t} \geq 0$$

$$= \frac{(2\pi)^{-n}}{2n}\|\Delta(h_t + f_t)\|^2_{L^2(e^{-f_t}\omega^n_{g_t})}$$

Since $\lambda(g_t) \leq W(g_t, 0) = (2\pi)^{-n} nV$ are uniformly bounded, $\exists\ t_i \in [i, i+1]$ s.t.

$$\lim_{i\to\infty}\int_M |\Delta(h_{t_i} + f_{t_i})|^2 e^{-f_{t_i}}\omega^n_{g_{t_i}} = 0$$

Tian-Zhu (Convergence of Kähler-Ricci flow on Fano manifolds, Crelle, 2013)

- f_t is uniformly bounded. (Since f_t satisfy the following equ

(6.8) $\Delta f_t + f_t + \frac{1}{2}(R - |\nabla f_t|^2_{g_t}) = (2\pi)^n V^{-1}\lambda(g_t)$

$\Longrightarrow$ (a)

For (b). Using (a), we get

$$\lim_{t_i\to\infty} \|\nabla(f_{t_i}+h_{t_i})\|^2_{L^2(M,g_{t_i})} \leq \lim_{t_i\to\infty} \int_M |f_{t_i}+h_{t_i}||\Delta(f_{t_i}+h_{t_i})|\omega^n_{g_{t_i}}$$

$\because$ f_t, h_t are bounded.

$$\leq C\cdot \lim_{t_i\to\infty} \|\Delta(f_i+h_t)\|_{L^2(g_{t_i})}$$

$$\longrightarrow 0.$$

For (c). Let $q_t := f_t + h_t$. We can calculate

$$-\Delta q_t = -\Delta f_t - \Delta h_t$$

(6.27) $$= f_t + \tfrac{1}{2}(R - |\nabla f_t|^2) - (2\pi)^n V^{-1}\lambda(g_t) - \Delta h_t$$

$$\leq C$$

Define $\tilde{q}_t := q_t - c_t$, where $c_t = \frac{1}{V}\int_M q_t e^{h_t}\omega^n_{g_t}$.

By the weighted Poincaré inequality (Tian-Zhu, JAMS)

we have
$$\int_M \tilde{q}_t^2 e^{h_t}\omega^n_{g_t} \leq \int_M |\nabla q_t|^2 e^{h_t}\omega^n_{g_t}$$

$+$ h_t is bounded

$$\Longrightarrow \lim_{i\to\infty} \int_M \tilde{q}^2_{t_i}\omega^n_{g_{t_i}} = 0$$

By (6.27), we have

$$\Delta \tilde{q}_t \geq -C.$$

By using the standard Moser's iteration, we have

$$(6.28) \quad \|\tilde{q}^+_{t_i}\|_{C^0} \leq C \|\tilde{q}^+_{t_i}\|_{L^2(g_{t_i})}$$

$$\leq C \|\tilde{q}_{t_i}\|_{L^2(g_{t_i})} \longrightarrow 0 \text{ as } i \to \infty$$

and $(6.29) \quad \lim_{i\to\infty} \int_M \tilde{q}^-_{t_i} \omega^n_{g_{t_i}} = 0$

$$\left[\left(\int_M \tilde{q}^-_{t_i} \omega^n_{g_{t_i}}\right)^2 \leq \left(\int_M 1 \, \omega^n_{g_{t_i}}\right) \cdot \int_M (\tilde{q}^-_{t_i})^2 \omega^n_{g_{t_i}} \right.$$

$$\left. \leq V \cdot \int_M \tilde{q}^2_t \, \omega^n_{g_{t_i}} \longrightarrow 0 \right]$$

where $q^+_t = \max\{q_t, 0\}$, $q^-_t = \min\{q_t, 0\}$.

$(6.28) + (6.29) \Rightarrow$

$$(6.30) \quad \lim_{i\to\infty} \int_M \tilde{q}_{t_i} \omega^n_{g_{t_i}} = 0$$

$\forall \varepsilon > 0$. Let $E_i(\varepsilon) := \{x \in M \mid |\tilde{q}_{t_i}| \geq \varepsilon\}$, then

$$\lim_{i\to\infty} \mathrm{Vol}_{g_{t_i}}(E_i(\varepsilon)) = 0$$

$$\left[\text{Since } \underset{\sim\sim\sim\sim\sim\sim\sim\sim}{\varepsilon^2 \cdot \mathrm{Vol}_{g_{t_i}}(E_i(\varepsilon))} \le \int_{E_i(\varepsilon)} |\tilde{q}_{t_i}|^2 \omega_{g_{t_i}}^n \le \int_M |\tilde{q}_{t_i}|^2 \omega_{g_{t_i}}^n \longrightarrow 0\right]$$

By the normalization condition,

$$V = \int_M e^{-f_{t_i}} \omega_{g_{t_i}}^n = \int_M e^{-(f_{t_i}+h_{t_i})} e^{h_{t_i}} \omega_{g_{t_i}}^n$$

$$= \int_M e^{-q_{t_i}} e^{h_{t_i}} \omega_{g_{t_i}}^n$$

$$= \int_M e^{-(q_{t_i}-c(t_i))} e^{-c(t_i)} e^{h_{t_i}} \omega_{g_{t_i}}^n$$

$$= e^{-c(t_i)} \int_M e^{-\tilde{q}_{t_i}} e^{h_{t_i}} \omega_{g_{t_i}}^n$$

$$= e^{-c(t_i)} \left[\int_{E_i(\varepsilon)} e^{-\tilde{q}_{t_i}} e^{h_{t_i}} \omega_{g_{t_i}}^n + \int_{M \setminus E_i(\varepsilon)} e^{-\tilde{q}_{t_i}} e^{h_{t_i}} \omega_{g_{t_i}}^n\right]$$

$$= e^{-c(t_i)} \left[\int_{E_i(\varepsilon)} e^{-\tilde{q}_{t_i}} e^{h_{t_i}} \omega_{g_{t_i}}^n + \int_{M \setminus E_i(\varepsilon)} (e^{-\tilde{q}_t} - 1) e^{h_{t_i}} \omega_{g_{t_i}}^n + \int_{M \setminus E_i} e^{h_{t_i}} \omega_{g_{t_i}}^n\right]$$

$$= e^{-c(t_i)} \Big[\int_{E_i(\varepsilon)} (e^{-\tilde{q}_{t_i}} - 1) e^{h_{t_i}} \omega_{g_{t_i}}^n + \int_{M \setminus E_i(\varepsilon)} (e^{-\tilde{q}_t} - 1) e^{h_{t_i}} \omega_{g_{t_i}}^n$$

$$+ \underbrace{\int_M e^{h_{t_i}} \omega_{g_{t_i}}^n}_{} = V \Big]$$

$\because \left|\int_{E_i(\varepsilon)} (e^{-\tilde{q}_{t_i}}-1) e^{h_{t_i}} \omega_{g_{t_i}}^n\right| \leq C \cdot \mathrm{Vol}_{g_{t_i}}(E_i(\varepsilon)) \to 0$

$$\left|\int_{M \setminus E_i(\varepsilon)} (e^{-\tilde{q}_t}-1) e^{h_{t_i}} \omega_{g_{t_i}}^n\right| \leq (e^{\varepsilon}-1) \cdot V$$

By the arbitrary of ε

$$\Rightarrow (6.32) \lim_{i\to\infty} c(t_i) = 0, \text{ i.e., } \lim_{i\to\infty} \int_M q_{t_i} e^{h_{t_i}} \omega_{g_{t_i}}^n = 0$$

Let $u_t := e^{-\frac{f_t}{2}} - e^{\frac{h_t}{2}}$.

Claim: (6.33) $\lim_{i\to\infty} \|u_{t_i}\|_{L^2(g_{t_i})} = 0$.

Pf:
$$\frac{1}{V}\int_M e^{-\frac{f_{t_i}}{2}} e^{\frac{h_{t_i}}{2}} \omega_{g_{t_i}}^n = \frac{1}{V}\int_M e^{\frac{f_{t_i}+h_{t_i}}{2}} e^{-f_{t_i}} \omega_{g_{t_i}}^n$$

$$\overset{\text{Jensen's ineq.}}{\geq} e^{\frac{1}{V}\int_M \underbrace{\frac{f_{t_i}+h_{t_i}}{2}}_{=\frac{q_{t_i}}{2}} e^{-f_{t_i}} \omega_{g_{t_i}}^n}$$

$$\xrightarrow[i\to\infty]{(6.32)} 1$$

On the other hand,

$$\frac{1}{V}\int_M e^{-\frac{f_t}{2}} e^{\frac{h_t}{2}} \omega_{g_t}^n \overset{\text{Cauchy's ineq.}}{\leq} \frac{1}{V}\left(\int_M e^{-f_t}\omega_{g_t}^n\right)^{1/2}\left(\int_M e^{h_t}\omega_{g_t}^n\right)^{1/2}$$

$$= 1.$$

$$\Rightarrow (6.34) \lim_{i\to\infty} \int_M e^{-\frac{f_{t_i}}{2}} e^{\frac{h_{t_i}}{2}} \omega_{g_{t_i}}^n = V.$$

$$\|u_{t_i}\|_{L^2(g_{t_i})} = \left(\int_M u_{t_i}^2 \omega_{g_{t_i}}^n\right)^{1/2}$$

$$= \left[2V - 2\int_M e^{-\frac{f_{t_i}}{2}} e^{\frac{h_{t_i}}{2}} \omega_{g_{t_i}}^n\right]^{1/2}$$

$$\xrightarrow[i\to\infty]{(6.34)} 0$$

End of the proof of the Claim. #

Let $v_t := e^{-\frac{f_t}{2}}$. (6.8) $\Leftrightarrow$

(6.35) $\Delta v_t - \frac{1}{2} f_t v_t - \frac{1}{4} R(g_t) v_t = -\frac{(2\pi)^n}{2V} \lambda(g_t) v_t$

Using Perelman's estimate, we also have

$$|\Delta e^{h_t}| \le C$$

+ f_t, $R(g_t)$, $\lambda(g_t)$ are uniformly bounded

$\Rightarrow$ (6.36) $|\Delta u_t| \le C$.

Then by the standard Moser's iteration, we get

$$\|u_{t_i}\|_{C^0} \le C \|u_{t_i}\|_{L^2(g_t)} \xrightarrow{(6.33)} 0$$

That is

(6.37) $\lim_{i\to\infty} \|u_{t_i}\|_{C^0} = 0$

Since $u_{t_i} = e^{-\frac{f_{t_i}}{2}} - e^{\frac{h_{t_i}}{2}} = e^{\frac{h_{t_i}}{2}}\left(e^{-\frac{q_{t_i}}{2}} - 1\right)$

$\Longrightarrow \lim_{i\to\infty} \|q_{t_i}\|_{C^0} = 0$, that is (c) of Lem. 6.13 #.

Now we begin to prove the Prop. 6.11.

Pf of Prop. 6.11.

real Riem. metric ds^2

Note that $\dfrac{R(g_t)}{2} = n + \frac{1}{2}\Delta h_t$. Then

$$\int_M \frac{1}{2}(R(g_t) + |\nabla f_t|^2) e^{-f_t} dv_{g_t}$$

$$= nV + \frac{1}{2}\int_M (\Delta h_t + |\nabla f_t|^2) e^{-f_t} dv_{g_t}$$

$$= nV + \frac{1}{2}\int_M \Delta(h_t + f_t) e^{-f_t} dv_{g_t}$$

By the Cauchy's ineq. and (a) of Lem. 6.13, we get

$$(6.38) \quad \lim \int_M \frac{1}{2}(R(g_{t_i}) + |\nabla f_{t_i}|^2) e^{-f_{t_i}} dv_{g_{t_i}} = nV$$

Using (a) of Prop 6.10, (c) of Lem 6.13 and

$$\sigma_t^* \theta_X(\omega_{g_t}) = \theta_X(\omega_{g_t'}), \quad \sigma_t^* h_{g_t} = h_{\phi_t'}$$

we have

$$(6.39) \quad \lim_{i\to\infty} \| f_{t_i} + \theta_X(\omega_{g_{t_i}}) \|_{C^0} = 0$$

Hence

$$(6.40) \quad \lim \int_M f_{t_i} e^{-f_{t_i}} \omega_{g_{t_i}}^n = -\lim_{i\to\infty} \int_M \theta_X(\omega_{g_{t_i}}) e^{\theta_X(\omega_{g_{t_i}})} \omega_{g_{t_i}}^n$$

$$= -N_X(G(M)).$$

(6.38) + (6.40) $\Rightarrow$ $\lim_{i\to\infty} \lambda(g_{t_i}) = nV - N_X(G(M)).$

Since $\lambda(g_t)$ is nondecreasing along the KRF,

$$\lim_{t\to+\infty}\lambda(g_t) = nV - N_X(c_1(M)).$$

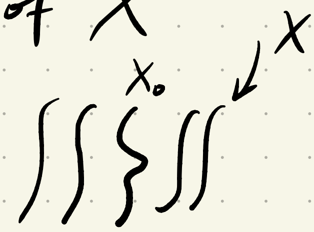
that is Prop. 6.11. #

Dervan and Szekelyhidi: "The Kähler-Ricci flow and optimal degenerations", JDG, 2020.

(1) $$\sup_{\omega\in c_1(X)} \mu(\omega) = nV - \sup_{\mathcal{X}} H(\mathcal{X})$$

here $\mathcal{X}$ is a special $(\mathbb{R}\text{-})$ degeneration of X

$\mathcal{X} \longrightarrow \mathbb{C}$, fiber $\mathcal{X}_t \overset{\text{iso}}{\cong} X$, $\forall t\neq 0$.



- $$H(\mathcal{X}) = \int_{\mathcal{X}_0} \theta_0 e^{h_0}\omega_0^n - V\log\left(\frac{1}{V}\int_{\mathcal{X}_0} e^{\theta_0}\omega_0^n\right)$$

$\omega_0\in c_1(\mathcal{X}_0)$, h_0: Ricci potential w/ ω_0,

θ_0: Hamiltonian potential for the induced h.v.f. v on $\mathcal{X}_0$

- H-functional.

$\omega\in c_1(X)$ Kähler metric, h: Ricci potential with ω

$$H(\omega) := \int_X h e^h \omega^n \quad \text{w/} \quad \int_X e^h\omega^n = 1$$

Thm: $$\inf_{\omega\in c_1(X)} H(\omega) = \sup_{\mathcal{X}} H(\mathcal{X})$$

here X is taken over all IR-degenerations..