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§6.2. Estimates for modified Ricci potentials.

NKRF:

$$(6.11)\quad \frac{\partial g(t)}{\partial t} = -\mathrm{Ric}(g(t)) + g(t), \quad g(0) = g \in 2\pi C_1(M).$$

$\Rightarrow \exists$ Kähler potential $\phi = \phi_t$ s.t.

$$\omega_\phi = \omega_{\phi_t} = \omega_g + \sqrt{-1}\partial\bar\partial\phi_t$$

is a soln. of (6.11).

Let $X \in \eta_r(M)$ be the extremal holomorplic vecter field on M as defined before. and $\sigma_t = \exp(tX)$ be a one-parameter subgp generated by X.

Let $\phi_t' = \phi_{\sigma_t}$ be the corresponding Kähler potentials of $\sigma_t^*\omega_{\phi_t}$, i.e.,

$$(6.12)\quad \omega_{\phi_{\sigma_t}} := \sigma_t^*\omega_{\phi_t} = \omega_g + \sqrt{-1}\partial\bar\partial\phi_{\sigma_t} = \omega_g + \sqrt{-1}\partial\bar\partial\phi_t'$$

Denote $\omega_{\phi_t'} := \omega_{\phi_{\sigma_t}}$, then

$$(6.13)\quad \frac{\partial\omega_{\phi_t'}}{\partial t} = -\mathrm{Ric}(\omega_{\phi_t'}) + \omega_{\phi_t'} + L_X\omega_{\phi_t'}$$

[here using $(\sigma_t^{-1})_*X = X$] modified Kähler-Ricci flow.

(6.13) $\Leftrightarrow$ The following MA-eqn for ϕ_t' (modulo a constant).

$$(6.14)\quad \frac{\partial\phi_t'}{\partial t} = \log\frac{\omega_{\phi_t'}^n}{\omega_g^n} + \phi_t' + \theta_X(\omega_{\phi_t'}) - h_g, \quad \underline{\phi'(0,\cdot) = c}.$$

#

Analogous to the case of the K-energy and the Futaki inv, we have

$$\frac{d\mu_{\omega_g}(\varphi_{f_t})}{dt} = \mathrm{Re}\left(F_X(v)\right) \qquad [PSSW]$$

where $f_t = \exp\{tv\}$ is one-parameter subgp generated by a holo. v.f. v on M, $f_t^* \omega_\varphi = \omega_g + \sqrt{-1}\partial\bar\partial \varphi_{f_t}$

[PSSW]: Phong–Song–Sturm–Weinkov, The convergence of the modified KRF and solitons, CMH.

By (6.13), we have

(6.15): $h_{\phi_t'} - \theta_X(\omega_{\phi_t'}) = -\frac{\partial}{\partial t}\phi_t' + C_t$, for some const. C_t.

$\Rightarrow$ (6.16) $$\frac{d\mu(\phi_t')}{dt} = -\frac{1}{V}\int_M \left\|\bar\partial \frac{\partial \phi_t'}{\partial t}\right\|^2_{\omega_{\phi_t'}} e^{\theta_X(\omega_{\phi_t'})}\omega_{\phi_t'}^n \leq 0$$

$$\mu(\phi_t') \searrow \Rightarrow \mu(\phi_t') \leq \mu(\phi_0'), \qquad \boxed{\mu(0)=0}$$

If we assume $\mu(\cdot)$ is bounded from below.

$$\int_0^t \left[\frac{d}{ds}\mu(\phi_s')\right] ds = -\frac{1}{V}\int_0^t \int_M \left\|\bar\partial \frac{\partial \phi_s'}{\partial s}\right\|^2 e^{\theta_X}\omega_{\phi'}^n\, ds$$

$$\Rightarrow \int_0^{+\infty}\int_M \left\|\bar\partial \frac{\partial \phi'}{\partial t}\right\|^2 e^{\theta_X}\omega_{\phi'}^n\, dt < +\infty.$$

Denote $u_{X,\phi_t'} = u_{X,g_t'} := h_{\phi_t'} - \theta_X(\omega_{\phi_t'})$ — modified Ricci potential.

Recalling Perelman's estimate:

Lem 6.8: $\exists$ consts $c>0$ and $C>0$ depending only on g. such that

(a) $\operatorname{diam}(M, \omega_{\phi'}) \leq C$; (b) $\operatorname{Vol}(B_r(p), \omega_{\phi_t'}) \geq c \cdot r^{2n}$

(c) $\|h_{\phi_t'}\|_{C^0(M)} \leq C$; (d) $\|\nabla h_{\phi_t'}\|_{\omega_{\phi_t'}} \leq C$; (e) $\|\Delta h_{\phi_t'}\|_{C^0(M)} \leq C$.

Lem 6.9: $\|\nabla u_{X,\phi_t'}\|_{\omega_{\phi_t'}} + \|\Delta u_{X,\phi'}\|_{C^0} + \|X\|_{\underbrace{C^0}_{(\omega_{\phi_t'})}} + \|\Delta\theta_X(\omega_{\phi_t'})\|_{C^0} \leq C$.

[PSSW, Prop 2]

Prop 6.10: Suppose that $\mu(\cdot)$ is bounded from below in $\mathcal{J}_X(M,\omega)$. Then we have:

(a) $\lim\limits_{t\to+\infty} \|u_{X,\phi_t'}\|_{C^0} = 0$;

(b) $\lim\limits_{t\to\infty} \|\nabla u_{X,\phi_t'}\|_{\omega_{\phi_t'}} = 0$;

(c) $\lim\limits_{t\to\infty} \|\Delta u_{X,\phi_t'}\|_{C^0} = 0$.

Pf: Set $H(t) := \int_M |\nabla u_{X,\phi_t'}|^2 e^{\theta_X(\omega_{g_t'})} \omega_{g_t'}^n$

$$\mu(\cdot) > -\infty \Rightarrow \int_0^{+\infty} H(t)\,dt < +\infty$$

$$\Rightarrow \exists\, t_i \in [i, i+1] \text{ s.t. } \lim_{i\to+\infty} H(t_i) = 0$$

<u>Claim</u>: $\frac{dH(t)}{dt} \le C\,H(t)$.

pf of claim:

By (6.14), we have

$$\frac{\partial}{\partial t}\dot{\phi}_t' = \Delta \dot{\phi}_t' + \dot{\phi}_t' + X(\dot{\phi}_t') = (\Delta + X)\dot{\phi}_t' + \dot{\phi}_t'$$

Denote $v = -\dot{\phi}_t'$, then

$$\nabla v = \nabla u_{X,\phi'}, \quad \Delta v = \Delta u_{X,\phi'}$$

(6.17): $\frac{\partial}{\partial t} v = (\Delta + X)v + v$

(6.18): $\frac{\partial}{\partial t}|\nabla v|^2 = (\Delta + X)|\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar{\nabla} v|^2 + |\nabla v|^2$

(6.19): $\frac{\partial}{\partial t}(\Delta + X)v = (\Delta + X)(\Delta + X)v + (\Delta + X)v + |\nabla\bar{\nabla} v|^2$.

<u>RK</u>: (6.17 − 6.19) $\Rightarrow$ To prove the Lem 6.9.

(6.20) $\frac{d}{dt}H(t) = \frac{d}{dt}\int_M |\nabla v|^2 e^{\theta_X(\omega_{\phi_t'})}\omega^n_{\phi_t'}$

Note:

$$\frac{d}{dt}\left(e^{\theta_X(\omega_{\phi_t'})}\underbrace{\omega^n_{\phi_t'}}\right) = \left(X(\dot{\phi}_t') - R + n + \Delta\theta_X\right) e^{\theta_X(\omega_{\phi_t'})}\omega^n_{\phi_t'}$$

$$= \int_M \frac{d}{dt}|\nabla v|^2 e^{\theta_X(\omega_{\phi_t'})}\omega^n_{\phi_t'} + \int_M |\nabla v|^2 \frac{d}{dt}\left(e^{\theta_X(\omega_{\phi_t'})}\omega^n_{\phi_t'}\right)$$

$$\overset{(6.18)}{=} \int_M \Big[\underbrace{(\Delta+X)|\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar\nabla v|^2 + |\nabla v|^2}_{=0}\Big] e^{\theta_X(\omega_{\varphi_t'})}\omega_{\varphi_t'}^n$$

$$+ \int_M |\nabla v|^2 \underbrace{\Big(-R+n+\underline{X(\dot\varphi_t')}+\Delta\theta_X\Big)}_{\leq C \text{ by Lem 6.9.}} e^{\theta_X(\omega_{\varphi_t'})}\omega_{\varphi_t'}^n$$

$$= -X(v) = -\langle\nabla\theta_X, \nabla v\rangle = -\langle\nabla\theta_X, \nabla u\rangle$$

$$\leq C \cdot H(t). \qquad \#$$

With the Claim, we have

$$\lim_{t\to\infty} H(t) = 0.$$

Let $\tilde{u}_t := u_{X,\phi'} - \frac{1}{V}\int_M u_{X,\phi'} e^{h_t'}\omega_{g_t'}^n \qquad \Big[\int_M e^{h_t'}\omega_{g_t'}^n = V\Big]$

- [PSSW, Prop 4]: $\exists\, \delta, K$ depending on n and $\sup_{t\in[0,\infty)} \|X\|_{C^0}$ so that, for any $\varepsilon>0$ with $0<\varepsilon\leq\delta$ and any $t_0\geq 0$, if

$$\|\tilde{u}_{t_0}\|_{C^0} \leq \varepsilon$$

$$\overset{(6.21)}{\Rightarrow} \|\nabla u_{X,\phi'}(t_0+2)\|_{g'_{t_0+2}} + \underbrace{\|(\Delta+X)u_{X,\phi'}(t_0+2)\|_{C^0}}_{\Rightarrow\ \|\Delta u_{X,\phi'}(t_0+2)\|_{C^0}\leq K'\varepsilon} \leq K\varepsilon.$$

- Poincaré-type inequality (weighted) [Tian-Zhu, JAMS, Lem 3.1]

$$(6.22)\quad \int_M \tilde{u}_t^2 e^{h_t'} \omega_{g_t'}^n \le \int_M |\nabla \tilde{u}_t|^2 e^{h_t'} \omega_{g_t'}^n$$

$$= \int_M |\nabla u_{X,\phi'}|^2 e^{h_t'} \omega_{g_t'}^n = H(t)$$

Since $H(t) \to 0$, $\displaystyle\lim_{t\to\infty} \int_M \tilde{u}_t^2 e^{h_t'} \omega_{g_t'}^n = 0$

$\Downarrow$ h_t' is uniformly bounded

$$\lim_{t\to\infty} \int_M \tilde{u}_t^2 \omega_{g_t'}^n = 0$$

Using an inequality:

$$\underline{(6.23)}\quad \|\tilde{u}_t\|_{C^0}^{n+1} \le C \|\nabla u_{X,\phi'}\|_{g_t'}^n \left[\int_M \tilde{u}_t^2 \omega_{g_t'}^n\right]^{1/2}.$$

pf of (6.23): Let $A := \|\tilde{u}_t\|_{C^0} = |\tilde{u}_t|(x_0) > 0$.

Set $r = \dfrac{A}{2\|\nabla \tilde{u}_t\|_{g_t'}}$, then on the ball $B_r(x_0)$, we have

$$|\tilde{u}_t|(x) \ge \frac{A}{2}, \quad \forall x \in B_r(x_0).$$

$[\because\ |\tilde{u}_t(x) - \tilde{u}_t(x_0)| \le \|\nabla \tilde{u}_t\| \cdot d_t(x, x_0) \le r \cdot \|\nabla \tilde{u}_t\| = \frac{A}{2}]$.

Using Perelman's non-collapsing (Lem 6.8, (b)), then

$$\int_M \tilde{u}_t^2 \omega_{g_t'}^n \ge \int_{B_r(x_0)} \frac{A^2}{4} \omega_{g_t'}^n = \frac{A^2}{4} \mathrm{Vol}_{g_t'}(B_r(x_0))$$

Lem 6.8, (b)

$$\geq \frac{A^2}{4} \cdot c \cdot r^{2n} = c \cdot \frac{A^2}{4} \left(\frac{A}{2\|\nabla \tilde{u}_t\|} \right)^{2n}$$

$$= C_n \cdot A^{2n+2} / \|\nabla u_{X,\phi'}\|^{2n}$$

$\Rightarrow$ We get (6.23).

$\left(\lim_{t\to\infty} \|\dot{\phi}_t\|_{C^0} = 0 \right)$ #

$$(6.23) \Rightarrow \lim_{t\to\infty} \|\tilde{u}_t\|_{C^0} = 0.$$

$+ (6.21) \Rightarrow$ (b) & (c) of Prop 6.10.

With the normalization conditions

$$\int_M e^{\theta_X(\omega_{\phi_t'})} \omega_{g_t'}^n = \int_M e^{h_t'} \omega_{g_t'}^n = V.$$

$$\Rightarrow \lim_{t\to+\infty} \|u_{X,\phi'}\|_{C^0} = 0.$$

#.

[Since $u_{X,\phi'} = -\dot{\phi}_t' + c_t$,

$$\int_M u_{X,\phi'} e^{\theta_X(\omega_{\phi_t'})} \omega_{g_t'}^n = -\int_M \dot{\phi}_t' e^{\theta_X(\omega_{\phi_t'})} \omega_{g_t'}^n + c_t \cdot V$$

If $(*)$: $\lim \int_M \dot{\phi}_t' e^{\theta_X(\omega_{\phi_t'})} \omega_{g_t'}^n = 0$, then $\lim_{t\to+\infty} c_t = 0$

$\Downarrow$

$\lim_{t\to+\infty} \|\dot{\phi}_t'\|_{C^0} = 0$

Now we prove $(*)$. We assume modified K-energy $\mu(\cdot)$ is bounded from below.

Define $$c := \frac{1}{V}\int_0^{+\infty} e^{-t} \underbrace{\int |\nabla u_{X,\phi'_t}|^2 e^{\theta_X(\omega_{\phi'_t})}\omega^n_{\phi'_t}}_{H(t)}\, dt - \frac{1}{V}\int_M u_{X,\omega_0}\, e^{\theta_X(\omega_0)}\omega_0^n$$

Consider the eqn (6.14) with initial data, $\phi'(0,\cdot) = c$.

Define $\alpha(t) := \frac{1}{V}\int_M \dot{\phi}'_t\, e^{\theta_X(\omega_{\phi'_t})}\omega^n_{g'_t}$. Then

$$\frac{d\alpha}{dt} = \alpha - \frac{1}{V}\int_M |\nabla u_{X,\phi'_t}|^2 e^{\theta_X(\omega_{\phi'_t})}\omega^n_{g'_t}$$

$$= \alpha - \frac{1}{V}H(t)$$

$$\Rightarrow \frac{d}{dt}\left(e^{-t}\alpha\right) = -\frac{e^{-t}}{V}H(t)$$

$$\Rightarrow (6.25) \quad \alpha(t) = e^t\alpha_0 - \frac{e^t}{V}\int_0^t e^{-s}H(s)\,ds$$

$$\alpha_0 = \frac{1}{V}\int_M \dot{\phi}'_0\, e^{\theta_X(\omega_0)}\omega_0^n$$

$$\overset{(6.14)}{=\!=} \frac{1}{V}\int_M (c + u_{X,\omega_0})e^{\theta_X(\omega_0)}\omega_0^n$$

$$= \frac{1}{V}\int_0^{+\infty} e^{-t}H(t)\,dt$$

$$\Rightarrow \alpha(t) = e^t\cdot\frac{1}{V}\int_t^{+\infty} e^{-s}H(s)\,ds$$

$$= \frac{1}{V}\int_t^{+\infty} e^{t-s} H(s)ds$$

$$\leq \frac{1}{V}\int_t^{+\infty} H(s)ds \xrightarrow{t\to\infty} 0 \quad \left(\because \int_0^{+\infty} H(s)ds < +\infty\right).$$

Get. (*). #

§6.3 Estimate for $L(\cdot)$

By the monotonicity of $\lambda(g_t)$, and $\lambda(g_t)$ is bounded from above

We know $\lim\limits_{t\to\infty} \lambda(g_t)$ exists and is finite.

We define the energy level $L(g)$ as

$$(6.26) \quad L(g) := \lim_{t\to\infty} \lambda(g_t)$$

Prop 6.11: Suppose the the modified Mabuchi's K-energy is bounded from below in K_X. Then for any $g \in K_X$,

$$L(g) = (2\pi)^{-n}\left(nV - N_X(G(M))\right)$$

Cor 6.12: Suppose the the modified Mabuchi's K-energy is bounded from below in K_X. Then

$$\sup\{\lambda(g') \mid g' \in K_X\} = (2\pi)^{-n}\left(nV - N_X(G(M))\right).$$

Conj: Suppose that the modified Mabuchi's K-energy is bounded from below. Then

$$\sup_{g' \in 2\pi c_1(M)} \lambda(g') = (2\pi)^{-n} \left[nV - N_X(c_1(M)) \right].$$

✓ It is true, was proved by Dervan-Szekelyhidi.