$\because |\nabla\bar\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2$

$$\Rightarrow -\frac{|\nabla\bar\nabla u|^2}{2B-u} \leq -\frac{1}{n}\left(\frac{\Delta u}{2B-u}\right)^2 \cdot (2B-u)$$

$$= -\frac{1}{n}(G-2H)^2(2B-u) \quad \cdots \quad (5.27)$$

Take (5.27) into (5.26), since $0 \leq H \leq C$.

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)G \leq -C_1 G^2 + C_2 G - \frac{2\mathrm{Re}(\bar\nabla u \cdot \nabla G)}{(2B-u)^2}$$

Using the maximum principle,

$$\Rightarrow \quad G \leq C \quad \Rightarrow \quad K \leq C$$

#

Cor 5.8: $\exists$ a uniform const. C, s.t.

$$-u(y,t) \leq C d^2_{g(t)}(x,y) + C$$

$$R(y,t) \leq C d^2_{g(t)}(x,y) + C$$

$$|\nabla u| \leq C d_{g(t)}(x,y) + C$$

$d_{g(t)}(x,y) \leq \mathrm{diam}_{g(t)}(M)$

where $u(x,t) = \max\limits_{y\in M} u(y,t)$.

2023.1.12

Pf: $\int_M e^u dV_{g(t)} = (2\pi)^n \Rightarrow u(x,t) \geq \ln\frac{(2\pi)^n}{V}$

$\leq e^{u(x,t)} \cdot \int_M dV_{g(t)} = e^{u(x,t)} \cdot V \Rightarrow (2\pi)^n \leq e^{u(x,t)} \cdot V$

By Prop 5.7, $|\nabla\sqrt{C-u}| \leq C$.

$$\Rightarrow \left|\sqrt{c-u}(y,t)-\sqrt{c-u}(x,t)\right| \le C\, d_{g(t)}(x,y)$$

$$\Rightarrow \sqrt{c-u}(y) \le \sqrt{c-u}(x) + c\, d_{g(t)}(x,y)$$

$$\Rightarrow c-\underbrace{u(y)} \le 2\left[(c-u(x)) + c^2 d^2_{g(t)}(x,y)\right]$$

$$\Rightarrow -u(y,t) \le c\, d^2_{g(t)}(x,y) + C. \qquad \#$$

RK: Shrinker $\quad Ric + \nabla^2 f = \frac{1}{2} g$

$$\Rightarrow \begin{cases} R + \Delta f = \frac{n}{2} \\ R + |\nabla f|^2 = f \end{cases}$$

Fact: $R \geq 0 \quad \Rightarrow \underline{|\nabla f|^2 \le f}$

$\Rightarrow M$ is noncpt, $\qquad f \sim \frac{1}{4} d^2(0,x)$.

- A uniform bound on diameters.

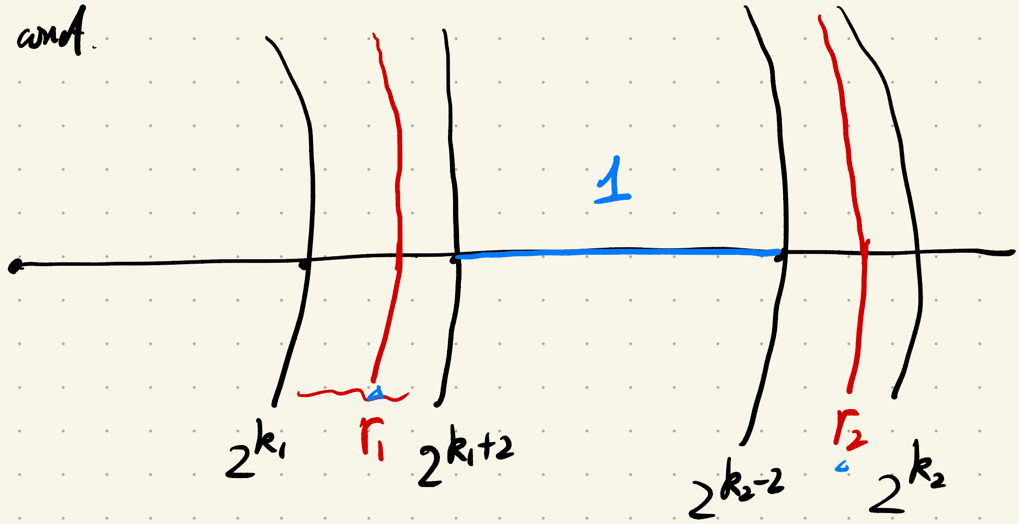
Prop 5.9: $\exists$ a uniform const. C such that

$$\operatorname{diam}(M, g(t)) \le C.$$

- Outline. Argument by contradiction!

Assume $\operatorname{diam}(M, g(t)) \to +\infty$.

$\mathrm{Vol}(M, g(t)) = V = \text{const.}$



① $\mathrm{Vol}\left(B(2^{k_1+1}, 2^{k_2-2})\right) \geq C(n) \cdot \mathrm{Vol}\left(B(2^{k_1}, 2^{k_2})\right)$

② $\mathrm{Vol}\left(B(2^{k_1}, 2^{k_2})\right) < \varepsilon$.

Pf of Prop 5.9:

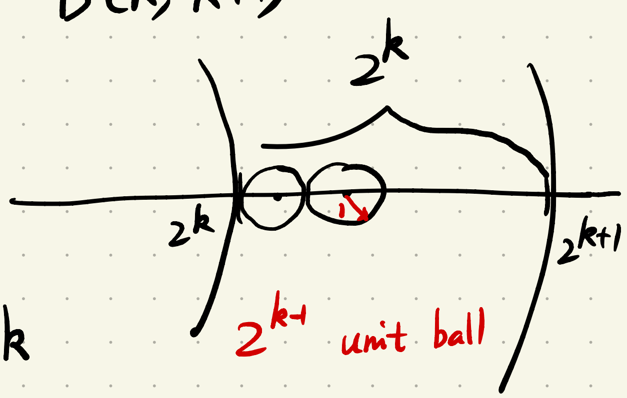
Assume that the diameters are unbounded in time.

Denote: $d_t(z) := d_{g(t)}(z, x)$, where $u(x,t) = \max_{y \in M} u(y,t)$.

$B(k_1, k_2) := \{ z \mid 2^{k_1} \leq d_t(z) \leq 2^{k_2} \}$

Consider an annulus $B(k, k+1)$. By Cor. 5.8, we have

$$R \leq C\, 2^{2k} \quad \text{on} \quad B(k, k+1)$$



Take $r = 2^{-k}$, then $B(k, k+1)$ contains 2^{2k-1} balls of radii $r = 2^{-k}$.

$$\mathrm{Vol}(B(y, \underset{r}{\underset{\shortparallel}{2^{-k}}})) \geq c\, r^{2n} = c \cdot 2^{-2nk}$$

Hence $V \geq \mathrm{Vol}(B(k, k+1)) \geq \sum_{i=1}^{2^{2k-1}} \mathrm{Vol}(B(y_i, 2^{-k})) \geq 2^{2k-1}(c \cdot 2^{-2nk})$

$$= C \cdot 2^{(2-2n)k}$$

$$\Rightarrow \quad Vol(B(k,k+1)) \geqslant C \cdot 2^{(2-2n)k} \quad \text{--- (5.28)}$$

Claim 5.10: $\forall \varepsilon > 0$, we can find $B(k_1, k_2)$ with $k_1 < k_2$ s.t. if $\operatorname{diam}(M, g(t))$ is large enough, then

(a) $Vol(B(k_1,k_2)) < \varepsilon$ and

(b) $Vol(B(k_1,k_2)) \leq 2^{10n} Vol(B(k_1+2, k_2-2))$

pf: $\because$ $Vol_{g(t)}(M) = Vol_{g(0)}(M)$.

If $\operatorname{diam}_{g(t)} M$ is sufficiently large, $\exists k_0 > 0$. s.t. $\forall k_2 \geqslant k_1 \geqslant k_0$, we have

$$Vol(B(k_1,k_2)) < \varepsilon.$$

If the estimate (b) did not hold, that is if

$$Vol(B(k_1,k_2)) \geqslant 2^{10n} Vol(B(k_1+2, k_2-2))$$

We would consider $B(k_1+2, k_2-2)$ instead. (a) ✓

- If (b) ✓ for $B(k_1+2, k_2-2)$. Done.

If (b) not hold for $B(k_1+2, k_2-2)$, we repeat again

$$Vol(B(k_1+2, k_2-2)) \geqslant 2^{10n} Vol(B(k_1+4, k_2-4))$$

$\vdots$

Assume that for every p, at the pth step, we are still not able to find the ball that (b). are satisfied.

$$\Rightarrow \quad Vol(B(k_1,k_2)) \geq 2^{10np}\, Vol(B(k_1+2p,\ k_2-2p))$$

- Take $k_1=2p$, $k_2=6p+1$, then

$$\varepsilon > Vol(B(k_1,k_2)) \geq 2^{10np}\, Vol(B(4p,4p+1))$$

$$\overset{(5.28)\ k=4p}{\geq} C\cdot 2^{10np}\, 2^{(2-2n)\cdot 4p}$$

$$= C\cdot 2^{(2n+8)p} \geq 2\varepsilon. \quad \text{Contradiction!}$$

If we choose p large enough, s.t. $C\cdot 2^{(2n+8)p} \geq 2\varepsilon$.

$\#$

Lem 5.11: $\exists\, r_1, r_2$ and a uniform const. C such that

$$r_1 \in [2^{k_1}, 2^{k_1+1}],\quad r_2 \in [2^{k_2-1}, 2^{k_2}]$$

and

$$\int_{T(r_1,r_2)} R\, dV_{g(t)} \leq C\cdot V$$

where $T(r_1,r_2) := \{z\in M \mid r_1 \leq d_t(z) \leq r_2\}$

$$V := Vol_{g(t)}(B(k_1,k_2))$$

Pf: Since $Vol(B(k_1,k_1+1)) = \int_{2^{k_1}}^{2^{k_1+1}} \underline{\underline{Vol(S(r))}}\, dr \leq V$

$$\left(= \int_{2^{k_1}}^{2^{k_1+1}} \left(\frac{d}{dr} Vol(B(r))\right) dr \right)$$

$$\therefore \exists\, r_1 \in [2^{k_1}, 2^{k_1+1}] \text{ s.t. } Vol(S(r_1)) \leq 2\cdot \frac{V}{2^{k_1}} \quad \cdots (5.29)$$

Same argument, $\exists\, r_2 \in [2^{k_2-1}, 2^{k_2}]$ s.t.

$$\mathrm{Vol}(S(r_2)) \le 2 \cdot \frac{V}{2^{k_2}} \quad \cdots \quad (5.30)$$

Hence

$$\int_{T(r_1,r_2)} R \, dV_{g(t)} = \int_{T(r_1,r_2)} (\underline{\Delta u} + n) \, dV_{g(t)}$$

$$\le \int_{S(r_1)} |\nabla u| \, dA_{g(t)} + \int_{S(r_2)} |\nabla u| \, dA_{g(t)} + n \underbrace{\mathrm{Vol}(T(r_1,r_2))}_{\le V}$$

By Cor 5.8, on $B(k_1, k_1+1)$, $|\nabla u| \le C \cdot 2^{k_1+1}$

on $B(k_2-1, k_2)$, $|\nabla u| \le C \cdot 2^{k_2}$

$+$ (5.29), (5.30)

$$\Rightarrow \int_{T(r_1,r_2)} R \le 2 \cdot \frac{V}{2^{k_1}} \cdot \left(C \cdot 2^{k_1+1}\right) + 2 \cdot \frac{V}{2^{k_2}} \left(C \cdot 2^{k_2}\right) + nV$$

$$\le \tilde{C} V \quad (< \tilde{C} \varepsilon)$$

#

Pf of Prop. 5.9: Assume diam $(M, g(t))$ is not bound uniformly.

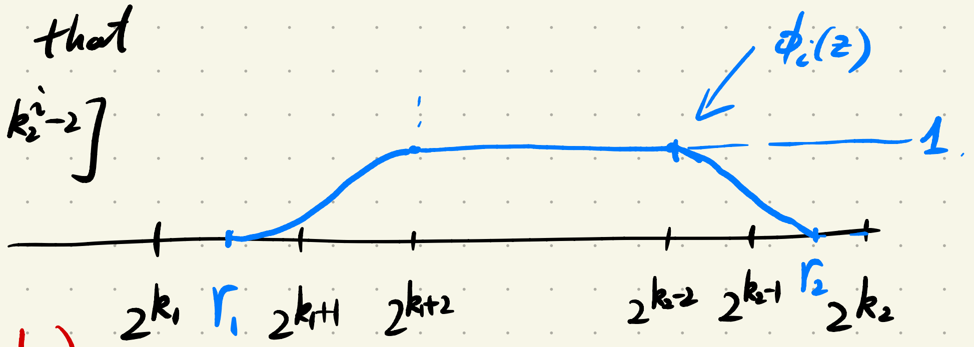
$\exists$ a seq. $t_i \to \infty$ s.t. diam $(M, g(t_i)) \to \infty$.

Take $\varepsilon_i > 0$, $\varepsilon_i \to 0$. By Claim 5.10, we find seqs. k_1^i and k_2^i s.t.

$$\mathrm{Vol}_{t_i}(B(k_1^i, k_2^i)) < \varepsilon_i \quad \cdots \quad (5.31)$$

$$\mathrm{Vol}_{t_i}(B(k_1^i, k_1^i)) \le 2^{10n} \mathrm{Vol}(B(k_1^i+2, k_2^i-2)) \quad \cdots \quad (5.32)$$

For each i, find r_1^i, r_2^i in Lem 5.11. Let ϕ_i be seq. of cut-off function such that



$$\begin{cases}\phi_i(z)=1, & \forall z\in[2^{k_1^i+2}, 2^{k_2^i-2}]\\ \phi_i(z)=0, & z\notin[r_1,r_2].\end{cases}$$

and $\phi_i\geqslant 0$, $|\phi_i'|\le 1 \cdot \left(\frac{1}{2^{k_1}}\right)$

Let $u_i(x)=e^{C_i}\phi_i(d_{t_i}(x,p_i))$ s.t. $(2\pi)^{-n}\int_M u_i^2\, dV_{g(t_i)}=1$.

If we take $e^{-f_i}=u_i^2$ i.e. $f_i=-2\ln u_i$, we can write Perelman's $\mathcal{W}$-functional

$$A\le \mathcal{W}(g,f_i,\tfrac12)=\mathcal{W}(g,u_i,\tfrac12)=(2\pi)^{-n}\int_M\left((Ru_i^2+4|\nabla u_i|^2)-2u_i^2\ln u_i\right)dV_{g_i} -2n$$

$A:=\mu(g(0),\frac12)$

(5.33)
$$=\underbrace{(2\pi)^{-n}e^{2C_i}\int_{T_{t_i}(r_1^i,r_2^i)}\left(4|\phi_i'(d_{t_i}(y))|^2-2\phi_i^2\ln\phi_i\right)dV_{g_i}}_{\mathrm{I}} + \underbrace{(2\pi)^{-n}\int_{T_{t_i}(r_1^i,r_2^i)} R\, u_i^2\, dV_{g_i}}_{\mathrm{II}} -2n-2C_i$$

Since $(2\pi)^n=\int_M u_i^2\, dV_{g_i} = e^{2C_i}\int_M\phi_i^2\,dV_{g_i}\le e^{2C_i}\,Vol(B(k_1^i,k_2^i))$
$\le e^{2C_i}\varepsilon_i$

$\varepsilon_i\to 0 \Rightarrow C_i\to+\infty$.

Estimate (I): $|\phi_i'|\le 1$, and $\forall x>0$, $-x\ln x\le e^{-1}<1$

$$\Rightarrow \mathrm{I} \leq C \cdot e^{2C_i} \, Vol_{t_i}(B(k_1^i, k_2^i)) \cdot (2\pi)^{-n}$$
$$\leq C \cdot e^{2C_i} \cdot 2^{10n} \cdot Vol_{t_i}(B(k_1^i+2, k_2^i-2)) \cdot (2\pi)^{-n}$$
$$\leq C \cdot 2^{10n} \int_M u_i^2 \, dV_{g_i} \cdot (2\pi)^{-n} \quad \text{--- (5.34)}$$
$$= C \cdot 2^{10n}.$$

$$\mathrm{II} \leq e^{2C_i} \underline{(2\pi)^{-n}} \int_{T_{t_i}(r_1^i, r_2^i)} R \, dV_{g_i}$$
$$\overset{\text{Lem 5.11}}{\leq} C e^{2C_i} (2\pi)^{-n} \, Vol_{t_i}(B_{t_i}(k_1^i, k_2^i)) \quad \text{--- (5.35)}$$
$$\leq C \cdot 2^{10n}$$

$$A := \mu(g(0), \tfrac{1}{2}) \leq \mu(g(t_i), \tfrac{1}{2}) \leq \mathcal{W}(g, u_i, \tfrac{1}{2}) \quad \text{--- (5.36)}$$

By (5.33 – 5.36), we have

$$A \leq C - 2C_i \xrightarrow{i \to +\infty} -\infty.$$

Contradiction! $\#$

<u>Rk:</u> Collins – Szekelyhidi, "The twisted Kähler–Ricci flow".

$\Longrightarrow$ $-u$ is bound from above uniformly.

§6 A convergence theorem for KRF on a Fano mfd.

Thm 6.1 (Main Thm) (Tian-Zhu, 2007, JAMS; Tian-Z.-Zhang-Zhu, 2013)
Let (M, J) be a cpt Kähler mfd which admits a Kähler-Ricci soliton (g_{KS}, X). Then the KRF

$$(6.1) \qquad \frac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) + \omega(t), \quad \omega(0) = \omega_0 \in K_X$$

will converges to a Kähler-Ricci soliton in C^∞ in the sense of Kähler potentials. Moreover, the convergence can be made fast exp.

§6.1 Modified Futaki invariant and Modified Mabuchi's K-energy.

Defn. (Kähler-Ricci soliton)
For Kähler metric ω on M, if $\exists$ holo. v. f. X on M s.t.

$$(6.2) \qquad Ric(\omega) - \lambda\omega = L_X \omega$$

where $L_X\omega$ denotes the Lie derivative along X, then (M, ω, X) is called a Kähler-Ricci soliton.

$\lambda > 0$, shrinking;
$\lambda = 0$, steady;
$\lambda < 0$, expanding.

Here X is a real vector field, X is called vector field if its $(1,0)$-part is a holomorphic vector field.

For Kähler-Ricci soliton, we have $L_X J = 0$, hence (6.2)

$$\Leftrightarrow \quad (6.2)' \qquad Ric(g) - \lambda g = L_X g, \qquad \omega \text{ is the Kähler form of } g.$$

• If X is a gradient vector field, that is, $\exists f \in C^\infty(M)$ s.t.

$$X = \nabla_{\mathbb{R}} f$$

Then the Kähler-Ricci soliton is called a gradient Ricci soliton.

Thm 6.2 (Uniqueness of Kähler-Ricci soliton) (Tian-Zhu, 2000 Acta, 2002 Comment. Math. Helv.)

If g and g' are two Kähler-Ricci solitons with respect to two h.v.f.s X and X' on M, respectively, then

$\exists\, \sigma \in \mathrm{Aut}^\circ(M)$ ($\mathrm{Aut}^\circ(M) :=$ identity component of $\mathrm{Aut}(M)$)

s.t.

$$\omega_g = \sigma^* \omega_{g'} \quad \text{and} \quad X = (\sigma^{-1})_*(X').$$

• For h.v.f. X, define a $(0,1)$-form $i_X \omega_g$ by

$$(6.3) \qquad i_X \omega_g(Y) := \omega_g(X, Y), \quad \forall \text{ smooth complex-valued v.f. } Y \text{ on } M.$$

$\Rightarrow \bar\partial i_X \omega_g = 0$

$\because C_1(M) > 0 \xRightarrow{\text{Calabi-Yau Thm}} \exists$ Kähler metric $\tilde g$ s.t. $Ric(\tilde g) > 0$.

$+$ Bochner formular $\Rightarrow \nexists$ nontrivial harmonic $(0,1)$-form.

Then using Hodge Thm, $\exists!$ smooth complex-valued fcn. $\hat\theta_X(g)$ on M

s.t.

$$(6.4) \quad \begin{cases} i_X \omega_g = \sqrt{-1}\,\bar\partial \hat\theta_X(g) \Rightarrow L_X \omega_g = \sqrt{-1}\,\partial\bar\partial \hat\theta_X(g) \\ \int_M \hat\theta_X(g) e^{h_g} \omega_g^n = 0 \end{cases}$$

where h_g is a Ricci potential of g, i.e., $Ric(\omega_g) - \omega_g = \sqrt{-1}\,\partial\bar\partial h_g$

$\hat\theta_X(g)$ is called a potential of X associated to g.