

$$(5.5) \quad \tau(2\Delta f - |\nabla f|^2 + R) + f - n = \mu(g, \tau).$$

Reference: Rothaus, Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators, JFA, 42(1), 1981, 110-120.

2022.1.10

Lem 5.3: (Perelman) Let $g(t)$ be a soln. of RF: $\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t))$ on $M^n \times [0, T)$ w/ $0 < T < \infty$, then $\mu(g(t), T_0 - t)$ is non-decreasing along the RF for any $T_0 \geq T$. Moreover, the monotonicity is strict unless $g(t)$ is a shrinking gradient soliton:

$$\text{Ric}(g(t)) + \nabla^2 f - \frac{1}{2\tau} g(t) = 0.$$

Pf: Fix any time t_0 , let f_0 be a minimizer of $\mu(g(t_0), T_0 - t_0)$ i.e.,

$$\mu(g(t_0), T_0 - t_0) = \mathcal{W}(g(t_0), f_0, T_0 - t_0)$$

Note that the backward heat eqn:

$$(5.6) \quad \frac{\partial f}{\partial \tau} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}$$

$$\text{Set } u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$$

$$\Leftrightarrow \frac{\partial u}{\partial \tau} = -\Delta u + Ru.$$

$$\Leftrightarrow \frac{\partial u}{\partial \tau} = \Delta u + Ru$$

Thus we can solve the backward heat eqn of f w/ $f|_{t=t_0} = f_0$ to obtain $f(t)$ for $t \in [0, t_0]$ satisfying (5.6).

Then, for $t \leq t_0$, it follows from Lem 5.2 that

$$\mu(g(t), T_0 - t) \leq \mathcal{W}(g(t), f(t), T_0 - t)$$

$$\leq \mathcal{W}(g(t_0), f(t_0), T_0 - t_0)$$

$$= \mu(g(t_0), T_0 - t_0)$$

#

Defn 5.1 (k -noncollapsed)

Let $g(t)$, $t \in [0, T)$, be a soln. of RF on M , let $k, r > 0$

We say $g(t)$ is k -noncollapsed at $(x_0, t_0) \in M \times [0, T)$ on the scale r if we have

$$\text{Vol}_{g(t_0)}(B(x_0, r)) \geq k r^n$$

whenever

$$|R_m|(x, t_0) \leq r^{-2}, \quad \forall x \in B(x_0, r)$$

$t_0 \leftarrow g(t_0)$

Thm 5.4 (Strong no local collapsing theorem)

Let M be a cpt Riem. mfd, and let $g(t)$ be a soln. of RF on $M^n \times [0, T)$ w/ $T \in (0, +\infty)$. Then $\exists k > 0$ $k = k(g_0, T)$, s.t., for any $(x_0, t_0) \in M \times [0, T)$ and

$0 < r \leq \sqrt{t}$ we have

$$\text{Vol}_{g(t_0)}(B(x_0, r)) \geq K r^n$$

whenever

$$|R(x, t_0)| \leq r^{-2}, \quad \forall x \in B_{t_0}(x_0, r).$$

The pt refer to Klein-Lott, Cao-Zhu, or Sesum-Tian's paper "Bounding scalar curvature and diameter along the Kähler-Ricci flow (after Perelman).", Prop 2.6.

- The μ -entropy and NKRF.

local holo. coords. $(z^1, \dots, z^n)$, with $z^i = x^i + \sqrt{-1} y^i$

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial y^i} \right)$$

Riem. metric ds^2 , we have

$$ds^2 \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = ds^2 \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 2 \operatorname{Re}(g_{i\bar{j}})$$

$$ds^2 \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right) = 2 \operatorname{Im}(g_{i\bar{j}})$$

- If $R_{i\bar{j}} = \lambda \delta_{i\bar{j}}$ ($R_{i\bar{j}} = \lambda g_{i\bar{j}}$) $\rightarrow$ KE metric)

$$\Rightarrow \operatorname{Ric}_{ds^2} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \operatorname{Ric}_{ds^2} \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 2\lambda \delta_{ij}$$

$$\operatorname{Ric}_{ds^2} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right) = 0.$$

$$\Rightarrow \bullet \operatorname{Ric}_{ds^2} = \lambda ds^2$$

$$\bullet R_{ds^2} = 2R$$

}

$$\left. \begin{aligned} \cdot \Delta ds^2 &= 2\Delta \\ \cdot |\nabla f|_{ds^2}^2 &= 2|\nabla f|^2 \end{aligned} \right\} \Rightarrow R_{ds^2} + |\nabla f|_{ds^2}^2 = 2(R + |\nabla f|^2)$$

Then for Kähler mfd (M^n, g) , we write Perelman's $\mathcal{W}$ -func.

as

$$(5.7) \quad \mathcal{W}(g, f, \tau) = \frac{1}{(4\pi\tau)^n} \int_M [2\tau(R + |\nabla f|^2) + f - 2n] e^{-f} dV_g$$

Set $u = e^{-\frac{f}{2}}$.

$$\mathcal{W}(g, u, \tau) = \frac{1}{(4\pi\tau)^n} \int_M [2\tau(Ru^2 + |\nabla u|^2) - u^2 \log u^2 - 2nu^2] dV_g.$$

For any soln. $\hat{g}(s)$ to the KRF: $\frac{\partial \hat{g}(s)}{\partial s} = -\text{Ric}(\hat{g}(s))$, $s \in [0, 1)$.

If we define $g(t) := \frac{1}{1-s} \hat{g}(s)$, $t = -\log(1-s)$

then
(NKRF) $\frac{\partial g(t)}{\partial t} = -\text{Ric}(g(t)) + g(t)$, $t \in [0, +\infty)$.

Hence

$$\begin{aligned} \mu(g(t), 1) &= \mu\left(\frac{1}{1-s} \hat{g}(s), \frac{1}{1-s} (1-s)\right) \\ &= \mu(\hat{g}(s), 1-s) \end{aligned}$$

$\Rightarrow \mu(g(t), 1)$ is non-decreasing along (NKRF).

Fix $\tau = \frac{1}{2}$. By Perelman's first variation for $\mathcal{W}(f, g, \tau)$.
we have

$$(5.8) \quad \delta \mu(g, \frac{1}{2}) = -\frac{1}{2} (2\pi)^{-n} \int_M \langle \delta g, \text{Ric}(g) - g + \nabla^2 f \rangle e^{-f} dV_g.$$

for $\int_M (2\pi)^{-\frac{n}{2}} e^{-f} dV_g = 1.$

$\Rightarrow \mu(g(t), \frac{1}{2})$ is non-decreasing along (NKRF).

- $\mu(g, \frac{1}{2})$ can be obtain by some smooth fcn f .
 f satisfies the Euler-Lagrange eqn.

$$\Delta f + f + \frac{1}{2}(R - |df|^2) = (2\pi)^{\frac{n}{2}} V^{-1} \mu(g, \frac{1}{2}).$$

here $V = \int_M dV_g.$

- Now we consider NKRF on Fano mfd M , w/ $g_0 \in 2\pi C_1(M).$

$$(NKRF) \begin{cases} \frac{\partial g(t)}{\partial t} = -Ric(g(t)) + g(t) \\ g(0) = g_0 \end{cases}$$

$\Rightarrow g(t) \in 2\pi C_1(M).$ Hence $\exists u(x,t) \in C^\infty(M)$, s.t.

$$Ric_{i\bar{j}} - g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u, \quad \Rightarrow \quad R - n = \Delta u$$

and $\exists \varphi(x,t) \in C^\infty(M)$ s.t. $\varphi(\cdot, 0) = 0$

$$g_{i\bar{j}}(t) - g_{i\bar{j}}(0) = \partial_i \partial_{\bar{j}} \varphi, \quad \varphi \text{ is called a Kähler potential.}$$

Assume $f \in C^\infty(M)$, $Ric(g_0) - g_0 = \sqrt{-1} \partial \bar{\partial} f.$

Then (NKRF) $\Leftrightarrow$ Complex MA eqn as following

$$(MA) \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_0^n} + \varphi + f \\ \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \\ \varphi(\cdot, 0) = 0. \end{cases}$$

By the eqn of (NKRF),

$$\Rightarrow u(t) = -\dot{\varphi}(t) + c(t) \quad \dots (5.9)$$

Normalizing so that

$$\int_M e^u dV_{g(t)} = (2\pi)^n.$$

Take $f = -u$, then

$$\begin{aligned} \mu(g_0, \tfrac{1}{2}) &\leq \mu(g(t), \tfrac{1}{2}) \\ &\leq W(g(t), f(t), \tfrac{1}{2}) = W(g(t), -u(t), \tfrac{1}{2}) \\ &= (2\pi)^{-n} \int_M (\underbrace{R + 10|u|^2}_{\Delta u + n} - u - 2n) e^u dV_{g(t)} \end{aligned}$$

$$\begin{aligned} (5.10) \quad &= (2\pi)^{-n} \int_M (\Delta u + 10|u|^2 - u - n) e^u dV_g \\ &= (2\pi)^{-n} \left[\underbrace{\int_M (\Delta e^u) dV_g}_{=0} - \int_M u e^u dV_g - n \right] \\ &= -(2\pi)^{-n} \int_M u e^u dV_g - n. \end{aligned}$$

Lem 5.5: $\exists$ a uniform const. $C = C(\mu(g_0, \tfrac{1}{2}), n)$ s.t.

$$C' \leq \int_M u e^u dV_g \leq C.$$

By Jensen's inequality,

$$\begin{aligned} (2\pi)^{-n} \int_M (-u) e^u dV_g &= \int_M \log(e^{-u}) \frac{e^u dV_g}{(2\pi)^n} \\ &\leq \log \int_M e^{-u} \frac{e^u dV_g}{(2\pi)^n} = \log \left[\frac{V}{(2\pi)^n} \right] \neq \end{aligned}$$

$$V(g(t)) = \text{const} = \text{Vol}(M, g_0).$$

From (MA), we have

$$\frac{\partial}{\partial t} \dot{\varphi} = \Delta \dot{\varphi} + \dot{\varphi}$$

$$\therefore \dot{\varphi} = -u + c(t).$$

$$\Rightarrow \frac{\partial}{\partial t} (-u + c(t)) = \Delta(-u + c(t)) + (-u + c(t))$$

$$\Rightarrow \frac{\partial}{\partial t} u = \Delta u + u - c(t) + \dot{c}(t)$$

$$\text{Set } a(t) = \dot{c}(t) - c(t) = e^t (e^{-t} c(t))'$$

$$\text{Since } \int_M e^u dV_{g(t)} = (2\pi)^n,$$

$$\begin{aligned} 0 &= \frac{d}{dt} \int_M e^u dV_{g(t)} = \int_M \left[\frac{du}{dt} - (R-n) \right] e^u dV_{g(t)} \\ &= \int_M \left(\frac{du}{dt} - \Delta u \right) e^u dV_{g(t)} \\ &= \int_M (u + a(t)) e^u dV_{g(t)} \\ &= (2\pi)^n \cdot a(t) + \int_M u e^u dV_{g(t)} \end{aligned}$$

$$\Rightarrow a(t) = -(2\pi)^{-n} \int_M u e^u dV_{g(t)}.$$

$$\text{By Lem 5.5, } |a(t)| \leq C. \quad \dots \quad (5.12)$$

$$\frac{\partial}{\partial t} u = \Delta u + u + a \quad \dots \quad (5.13)$$

Lem 5.6: $u(t)$ is uniformly bounded from above.

Pf: Since $\Delta u = R - n$, $R \geq -C$ for $R \in \mathbb{R}$.

$$\Rightarrow \frac{\partial}{\partial t} u = R - n + u - a \geq u - C_1 \quad (C_1 > 0) \dots (5.14)$$

If $u(y_0, t_0) \gg 1$, say $u(y_0, t_0) \geq 2C_1 > 0$, then $\exists$ a neighborhood

U of y_0 s.t. $\forall y \in U, u(y, t_0) \gg 1$. Then by (5.14)

$$u(y, t) \gg 1, \quad \forall (y, t) \in U \times [t_0, +\infty).$$

Furthermore, $\frac{d}{dt} u \geq u - C_1$

$$\Rightarrow u(z, t) \geq e^{t-t_0} (u(z, t_0) - C_1)$$

$$\Rightarrow (\text{If } u(z, t_0) \geq 2C_1, \text{ then } u(z, t) \geq e^{t-t_0} C_1 = \tilde{C} e^t.$$

Set $\tilde{\varphi}(t) = \varphi(t) - \int_0^t c(s) ds$, then $u(t) = -\dot{\tilde{\varphi}}(t)$

$$\Rightarrow -\dot{\tilde{\varphi}}(t) \geq \tilde{C} e^t$$

$$\Rightarrow \tilde{\varphi}(z, t) \leq -\tilde{C} e^t.$$

$$\because \int_M e^u dV_{g(t)} = (2\pi)^n$$

$$\Rightarrow \exists \text{ uniform const. } C_2 \text{ s.t.}$$

$$u(\tilde{x}_t, t) = \min_{y \in M} u(y, t) \leq C_2.$$

From (5.13),

$$\frac{\partial}{\partial t} (u + \tilde{\varphi}) = \Delta u + a = R - n + a \geq -\tilde{C}$$

$$\Rightarrow u(\tilde{x}_t, t) + \tilde{\varphi}(\tilde{x}_t, t) \geq \min_M (u(\cdot, 0) + \tilde{\varphi}(\cdot, 0)) - \tilde{C} t$$

$$\Rightarrow \tilde{\varphi}(\tilde{x}_t, t) \geq -C_3 - \tilde{C} t \dots (5.15)$$

$$\therefore \int_M \partial \bar{\partial} \varphi(t) = \omega(t) - \omega_0$$

$$\Rightarrow \Delta_0 \varphi = \text{tr}_{\omega_0}(\omega(t) - \omega_0) = \text{tr}_{\omega_0} \omega(t) - n \geq -n.$$

Using Green's formula on closed mfd.

Refer Aubin's book, Thm 4.13.

$$\varphi(x, t) = \frac{1}{\text{Vol}(M)} \int_M \varphi(y, t) dV(y) + \int_M \Delta \varphi(y, t) \cdot G(x, y) dV(y)$$

$$\underline{\Delta = -g^{i\bar{j}} \nabla_i \nabla_{\bar{j}}}$$

$$\tilde{\varphi}(x_t, t) = \frac{1}{\text{Vol}_0(M)} \int_M \tilde{\varphi}(y, t) dV_0(y) - \frac{1}{\text{Vol}_0(M)} \int_M \Delta_0 \tilde{\varphi}(y, t) G_0(x_t, y) dV_0(y)$$

$$= \frac{\text{Vol}_0(M|U)}{\text{Vol}_0(M)} \sup \tilde{\varphi}(\cdot, t) + \frac{\text{Vol}_0(U)}{\text{Vol}_0(M)} \int_U \tilde{\varphi}(y, t) dV_0(y) + C$$

$$\leq \frac{\text{Vol}_0(M|U)}{\text{Vol}_0(M)} \sup \tilde{\varphi}(\cdot, t) - C_4 e^t + C$$

$$\text{for } t \geq t_0, \text{ where } \tilde{\varphi}(x_t, t) = \max_{x \in M} \tilde{\varphi}(x, t)$$

$$\Rightarrow \underline{\tilde{\varphi}(x_t, t)} \leq -C_5 e^t + C_6 \quad \dots (5.16)$$

By (5.15) and (5.16), for big enough t

$$-C_3 - \tilde{C}t \leq -C_5 e^t + C_6$$

It is a contradiction.

Hence $\Rightarrow$ a uniform const. B s.t.

$u(x,t) \leq B$ for all $(x,t) \in M \times [0, +\infty)$. #

Recall:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\Delta u) = |\nabla \bar{\nabla} u|^2 + \Delta u \quad \dots \quad (5.17)$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 = |\nabla u|^2 - |\nabla \nabla u|^2 - |\nabla \bar{\nabla} u|^2 \quad \dots \quad (5.18)$$

Prop. 5.7: $\exists$ a uniform const. C s.t.

$$|\nabla u|^2 \leq C(C - u) \quad \dots \quad (5.19)$$

$$\because \Delta u = R - u \quad R \leq C(C - u) \quad \dots \quad (5.20)$$

$$\Leftrightarrow \Delta u \leq C(C - u)$$

pf: Set $H = \frac{|\nabla u|^2}{2B - u}$. Then

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)H &= \frac{\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2}{2B - u} + |\nabla u|^2 \left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{1}{2B - u}\right) \\ &\quad - 2\operatorname{Re}\left(\partial_i |\nabla u|^2 \partial_{\bar{i}} \left(\frac{1}{2B - u}\right)\right). \end{aligned}$$

$$\begin{aligned} \underline{\underline{(5.17) + (5.18)}} & \quad \frac{|\nabla u|^2 - |\nabla \nabla u|^2 - |\nabla \bar{\nabla} u|^2}{2B - u} + \frac{|\nabla u|^2(u + a)}{(2B - u)^2} - \frac{2\operatorname{Re}(\bar{\nabla} u \cdot \nabla |\nabla u|^2)}{(2B - u)^2} \\ & \quad - \frac{2|\nabla u|^4}{(2B - u)^3} \quad \dots \quad (5.21) \end{aligned}$$

Write

$$-\frac{2\operatorname{Re}(\bar{\nabla}u \cdot \nabla|u|^2)}{(2B-u)^2} - \frac{2|u|^4}{(2B-u)^3}$$

$$= -(2-\varepsilon) \frac{\operatorname{Re}(\bar{\nabla}u \cdot \nabla|u|^2)}{(2B-u)^2} - \varepsilon \frac{\operatorname{Re}(\bar{\nabla}u \cdot \nabla|u|^2)}{(2B-u)^2} - 2 \frac{|u|^4}{(2B-u)^3}$$

$$= -(2-\varepsilon) \frac{\operatorname{Re}(\bar{\nabla}u \cdot \frac{\nabla|u|^2}{2B-u})}{2B-u} - \varepsilon \frac{\operatorname{Re}(\bar{\nabla}u \cdot \nabla|u|^2)}{(2B-u)^2} - \frac{2|u|^4}{(2B-u)^3}$$

$$\therefore \nabla H = \frac{\nabla|u|^2}{2B-u} + \frac{|u|^2}{(2B-u)^2} \nabla u \Rightarrow \frac{\nabla|u|^2}{2B-u} = \nabla H - \frac{|u|^2}{(2B-u)^2} \nabla u$$

$$= -(2-\varepsilon) \frac{\operatorname{Re}(\bar{\nabla}u \cdot \nabla H)}{2B-u} - \varepsilon \frac{|u|^4}{(2B-u)^3} - \varepsilon \frac{\operatorname{Re}(\bar{\nabla}u \cdot \nabla|u|^2)}{(2B-u)^2}$$

for small $\varepsilon > 0$. Since

$$\varepsilon |\operatorname{Re}(\bar{\nabla}u \cdot \nabla|u|^2)| \leq \varepsilon |\nabla_i u \cdot \nabla_i \bar{\nabla}_j u \cdot \nabla_j u + \nabla_i u \cdot \nabla_j u \cdot \nabla_i \bar{\nabla}_j u|$$

$$\leq \varepsilon |u|^2 (|\nabla u| + |\nabla \bar{u}|)$$

$$\leq \frac{\varepsilon}{2} \frac{|u|^4}{2B-u} + \frac{(2B-u)^\varepsilon}{2} (|\nabla u| + |\nabla \bar{u}|)^2$$

$$\leq \frac{\varepsilon}{2} \frac{|u|^4}{2B-u} + \varepsilon (2B-u) (|\nabla u|^2 + |\nabla \bar{u}|^2)$$

By (5.21), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)H \leq -(1-\varepsilon) \frac{|\nabla u|^2 + |\nabla \bar{u}|^2}{2B-u} + \frac{2B+a}{2B-u} \cdot H - (2-\varepsilon) \frac{\operatorname{Re}(\bar{\nabla}u \cdot \nabla H)}{2B-u}$$

$$-\frac{\varepsilon}{2} H^2$$

Take $\varepsilon \in (0, 1)$, e.g. $\varepsilon = \frac{1}{2}$.

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right) H \leq \frac{2B+a}{2B-u} H - \frac{1}{4} H^2 - \frac{3}{2} \frac{\operatorname{Re}(\bar{\nabla} u \cdot \nabla H)}{2B-u}.$$

Using the maximum principle,

$$0 \leq \frac{d}{dt} H_{\max} \leq H_{\max} \left(\frac{2B+a}{2B-u} - \frac{1}{4} H_{\max} \right).$$

$$\Rightarrow H_{\max} \leq 4 \cdot \frac{2B+a}{2B-u} \leq C.$$

$$\Rightarrow |\nabla u|^2 \leq C(C-u).$$

$$\text{Set } K := \frac{\Delta u}{2B-u}.$$

$$\left(\frac{\partial}{\partial t} - \Delta\right) K = \frac{|\nabla \bar{\nabla} u|^2}{2B-u} + \frac{2B+a}{2B-u} K - \frac{2\operatorname{Re}(\nabla_i K \cdot \nabla_{\bar{i}} u)}{2B-u} \quad \dots (5.24)$$

Set $G = K + bH$, $b > 1$ to be determined.

$$\left(\frac{\partial}{\partial t} - \Delta\right) G = \frac{-b|\nabla \nabla u|^2 - (b-1)|\nabla \bar{\nabla} u|^2}{2B-u} + \frac{2B+a}{2B-u} G - \frac{2\operatorname{Re}(\nabla G \cdot \bar{\nabla} u)}{(2B-u)^2} \quad \dots (5.26)$$

take $b=2$.

$$\leq -\frac{|\nabla \bar{\nabla} u|^2}{2B-u} + \frac{2B+a}{2B-u} G - \frac{2\operatorname{Re}(\nabla G \cdot \bar{\nabla} u)}{(2B-u)^2}.$$

$$\therefore |\nabla \bar{\partial} u|^2 \geq \frac{1}{h} (\Delta u)^2$$

$$\Rightarrow -\frac{|\nabla \bar{\partial} u|^2}{2B-u} \leq -\frac{1}{h} \left(\frac{\Delta u}{2B-u} \right)^2 (2B-u)$$

$$= -\frac{1}{h} (G-2H)^2 (2B-u) \quad \dots (5.27)$$

Take (5.27) into (5.26), since $0 \leq H \leq C$.

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta \right) G \leq -C_1 G^2 + C_2 G - \frac{2\operatorname{Re}(\bar{\partial} u \cdot \nabla G)}{(2B-u)^2}$$

Using the maximum principle,

$$\Rightarrow G \leq C \Rightarrow K \leq C$$

#

Cor 5.8. $\exists$ a uniform const. C , s.t.

$$-u(y, t) \leq C \underline{d_{g(t)}^2(x, y)} + C$$

$$R(y, t) \leq C d_{g(t)}^2(x, y) + C$$

$$|ou| \leq C d_{g(t)}(x, y) + C$$

$$d_{g(t)}(x, y) \leq \operatorname{diam}_{g(t)}(M)$$

where

$$u(x, t) = \max_{y \in M} u(y, t).$$