2023.1.5:

Pf of Lem 4.8:

Parabolic Schwartz Lemma:

$$\left(\frac{\partial}{\partial t}-\Delta\right)\log tr_{\omega_0}\omega(t) \le C_0\, tr_{\omega(t)}\omega_0, \quad \text{where } Bisec(g_0)\geq -C_0$$

$\because$ $\omega(t)=\omega_0+\sqrt{-1}\partial\bar\partial\varphi \Rightarrow \Delta\varphi = tr_{\omega(t)}(\omega(t)-\omega_0)$

$\therefore$ $tr_{\omega(t)}\omega_0 = n-\Delta\varphi$

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(\log tr_{\omega_0}\omega(t)-A\varphi\right) \le C_0\, tr_{\omega(t)}\omega_0 - A\dot\varphi + A\, tr_{\omega(t)}(\omega(t)-\omega_0)$$

$$= A(n-\dot\varphi) - tr_{\omega(t)}(A-C_0)\omega_0.$$

Set $A=C_0+1$

$$= A(n-\dot\varphi) - tr_{\omega(t)}\omega_0$$

Assume

$$\left(\log tr_{\omega_0}\omega - A\varphi\right)(x_0,t_0) = \max_{M\times[0,t]}\left(\log tr_{\omega_0}\omega - A\varphi\right)$$

$$\left(\log tr_{\omega_0}\omega\right)(x_1,t_1) = \max_{M\times[0,t]}\log tr_{\omega_0}\omega$$

- If $t_0=0$.

$$\Rightarrow \log tr_{\omega_0}\omega(x,t) \le \log tr_{\omega_0}\omega(x_0,0) \underline{-A\varphi(x_0,0)+A\varphi(x,t)}$$

$$\le \underbrace{\log tr_{\omega_0}\omega}_{\le C} + A\left(\varphi - \inf_{M\times[0,t]}\varphi\right)$$

- If $t_0>0$.

Since $|\dot\varphi|\le C$, $tr_{\omega}\omega_0(x_0,t_0)\le C \Rightarrow \omega(x_0,t_0)\geq C^{-1}\omega_0$

$\because$ $\omega^n(t)\le C\cdot\Omega \Rightarrow \omega(x_0,t_0)\le C\cdot\omega_0$

$$\Rightarrow \log tr_{\omega_0}\omega(x,t) \le \underline{\log tr_{\omega_0}\omega(x_0,t_0)} - A\varphi(x_0,t_0) + \underline{A\varphi(x,t)}$$

$$\le C - A \inf_{M\times[0,t]} \varphi + A\varphi(x,t)$$

$$\le C + A\left(\varphi(x,t) - \inf_{M\times[0,t]} \varphi\right).$$

$$\Rightarrow tr_{\omega_0}\omega(x,t) \le C\cdot \exp\left(A\left(\varphi(x,t) - \inf_{M\times[0,t]}\varphi\right)\right) \cdots\cdots (4.13)$$

We define $\tilde{\varphi} = \varphi - \frac{1}{V}\int \varphi\Omega$, where $V = \int_M \Omega = \int_M \omega_0^n$.

$\because |\operatorname{osc}\varphi| \le C \Rightarrow |\tilde{\varphi}| \le C.$

$\inf(f+g) \ge \inf f + \inf g.$

By (4.13),

$$tr_{\omega_0}\omega(x,t) \le C\exp\left[A\left(\tilde{\varphi} + \frac{1}{V}\int_M \varphi\Omega - \inf_{M\times[0,t]}\tilde{\varphi} - \inf_{[0,t]} \frac{1}{V}\int\varphi\Omega\right)\right]$$

$$\le C'\exp\left(\frac{A}{V}\left(\int_M \varphi\Omega - \inf_{[0,t]}\int\varphi\Omega\right)\right).$$

Since $\frac{d}{dt}\left(\int_M \varphi\Omega\right) = \int_M \frac{\partial\varphi}{\partial t}\cdot\Omega = \int \log\frac{\omega^n}{\Omega}\cdot\Omega$

$$= V\cdot \int_M \log\frac{\omega^n}{\Omega}\cdot\frac{\Omega}{V}$$

$$\overset{\text{Jensen's inequality}}{\le} V\cdot \log\int_M \frac{\omega^n}{\Omega}\cdot\frac{\Omega}{V}$$

$$= 0$$

$\Rightarrow \int_M \varphi \Omega \searrow \Rightarrow \inf_{[0,t]} \int_M \varphi \Omega = \int_M \varphi(\cdot,t)\Omega$

$\Rightarrow \mathrm{tr}_{\omega_0}\omega(x,t) \le C'$

$$\left.\begin{aligned} \Rightarrow \quad & \omega(t) \le C'\omega_0 \\ + \quad & \omega^n(t) \ge C^{-1}\Omega \end{aligned}\right\} \Rightarrow \omega(t) \ge C^{-1}\omega_0$$

#

Same argument as in the proof of METT and the case of $C_1(M)<0$, we have

Prop: Let $\varphi = \varphi(t)$ solve (4.11) for $t \in [0,+\infty)$. Then for each $k = 0,1,2,\cdots$, $\exists A_k > 0$ s.t. on $[0,+\infty)$

$+\boxed{\int_M \varphi\,\omega^n = 0}$

$\underset{?}{\|\varphi\|_{C^k(M)} \le A_k}$, $\quad \omega_0 + \sqrt{-1}\partial\bar\partial\varphi \ge \frac{1}{A_0}\omega_0$

Hence we get subseq. $\varphi(t_i) \xrightarrow{C^\infty} \varphi_\infty$. To obtain the global convergence, we need a <u>furthur argument</u>.

Reference: Phong-Sturm, "On stability and the convergence of the Kähler-Ricci flow", JDG, 2006.

Define $\quad P(t) := \int_M \dot\varphi\,\omega^n \quad$ --- (4.14)

$\Rightarrow \quad |P(t)| \le C.$

$$\frac{d}{dt}P(t) = \int_M \left[\ddot{\varphi} + \dot{\varphi}(-R)\right]\omega^n \qquad \frac{\partial \omega^n}{\partial t} = -R\cdot\omega^n$$

$$= \Delta\dot{\varphi}\cdot\omega^n$$

$$= \int_M (\Delta\dot{\varphi} + \dot{\varphi}\Delta\dot{\varphi})\omega^n \qquad \Rightarrow \quad R = -\Delta\dot{\varphi} = -\ddot{\varphi}$$

$$= -\int_M |\nabla\dot{\varphi}|^2\omega^n \leq 0 \qquad \Rightarrow \ddot{\varphi} = \frac{\partial}{\partial t}\left(\log\frac{\omega^n}{\Omega}\right)$$

$$= \Delta\dot{\varphi}$$

$\Rightarrow \quad P(t)\downarrow$

$$\Rightarrow \quad \underline{P(t) - P(0)} = \int_0^t \left(\frac{d}{ds}P(s)\right) ds$$

$$= -\int_0^t \left(\underline{\int_M |\nabla\dot{\varphi}|^2_{\omega(s)}}\,\omega^n_{(s)}\right) ds$$

Then $\exists \{t_i\}$, $t_i \in [i, i+1]$, s.t.

$$\left(\int_M |\nabla\dot{\varphi}|^2\omega^n\right)(t_i) = \left(\int_M \left|\nabla \log\frac{\omega^n}{\Omega}\right|^2\omega^n\right)(t_i) \to 0$$

$$\Rightarrow \quad \nabla \log\frac{\omega^n_\infty}{\Omega} = 0 \Rightarrow \frac{\omega^n_\infty}{\Omega} = \text{const.}$$

$$\Rightarrow \quad \mathrm{Ric}(\omega_\infty) = 0.$$

Lem 4.10: For flow: $\frac{\partial g(t)}{\partial t} = -\mathrm{Ric}(g(t)) + \mu g(t)$, and

$$\left(\frac{\partial}{\partial t} - \Delta\right)u = f.$$

Then $\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|^2 = 2\,\mathrm{Re}(\nabla_i u \nabla_{\bar{i}} f) - \mu|\nabla u|^2 - |\nabla\bar{\nabla}u|^2 - |\nabla\nabla u|^2$

Pf: $|\nabla u|^2 = g^{i\bar j}\partial_i u \cdot \partial_{\bar j} u$

$$\Rightarrow \frac{\partial}{\partial t}|\nabla u|^2 = \frac{\partial g^{i\bar j}}{\partial t}\partial_i u\, \partial_{\bar j} u + \underline{g^{i\bar j}\partial_i \frac{\partial u}{\partial t}\cdot\partial_{\bar j} u + g^{i\bar j}\partial_i u\cdot \partial_{\bar j}\frac{\partial u}{\partial t}}$$

$$= -g^{i\bar q} g^{p\bar j} \frac{\partial g_{p\bar q}}{\partial t} \partial_i u\, \partial_{\bar j} u +$$

$$= -g^{i\bar q}g^{p\bar j}(-R_{p\bar q}+\mu g_{p\bar q})\partial_i u\,\partial_{\bar j}u +$$

$$= g^{i\bar q}g^{p\bar j}R_{p\bar q}\partial_i u\,\partial_{\bar j}u - \mu|\nabla u|^2 +$$

holo. normal coord.

$$\Delta|\nabla u|^2 = \nabla_p\nabla_{\bar p}(\nabla_i u \cdot \nabla_{\bar i} u)$$

$$= \nabla_p(\nabla_{\bar p}\nabla_i u\cdot\nabla_{\bar i}u + \nabla_i u\cdot\nabla_{\bar p}\nabla_{\bar i}u)$$

$$= \nabla_p\nabla_{\bar p}\nabla_i u\cdot\nabla_{\bar i}u + \underbrace{\nabla_{\bar p}\nabla_i u\cdot\nabla_p\nabla_{\bar i}u}_{=|\nabla\bar\nabla u|^2} + \underbrace{\nabla_p\nabla_i u\cdot\nabla_{\bar p}\nabla_{\bar i}u}_{=|\nabla\nabla u|^2}$$

$$+ \nabla_i u\cdot\nabla_p\nabla_{\bar p}\nabla_{\bar i}u$$

$$\because \nabla_p\nabla_{\bar p}\nabla_i u = \nabla_p\nabla_i\nabla_{\bar p}u = \nabla_i\nabla_p\nabla_{\bar p}u = \nabla_i\Delta u.$$

$$\nabla_p\nabla_{\bar p}\nabla_{\bar i}u = \nabla_p\nabla_{\bar i}\nabla_{\bar p}u = \nabla_{\bar i}\nabla_p\nabla_{\bar p}u + R_{p\bar i\ell\bar p}\nabla_{\bar\ell}u$$

Refer: Chow, Ben... (I).

$$= \nabla_{\bar i}\Delta u + \underline{R_{p\bar p}}{}_{\ell\bar i}\nabla_{\bar\ell}u$$

$$= \nabla_{\bar i}\Delta u + R_{\ell\bar i}\nabla_{\bar\ell}u.$$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)|\nabla u|^2 = 2\mathrm{Re}(\nabla_i f\,\nabla_{\bar i}u) - \mu|\nabla u|^2 - |\nabla\nabla u|^2 - |\nabla\bar\nabla u|^2$$

#.

By Lem 4.10, for (KRF) and $\dot{\varphi}$, $\left(\frac{\partial}{\partial t}-\Delta\right)\dot{\varphi}=0$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)|\nabla\dot{\varphi}|^2 = -|\nabla\bar{\nabla}\dot{\varphi}|^2-|\nabla\nabla\dot{\varphi}|^2 \quad \text{--- (4.16)}$$

$$\Rightarrow \frac{d}{dt}\dot{p}(t) = -\frac{d}{dt}\int_M |\nabla\dot{\varphi}|^2\omega^n$$

$$= -\int_M \left[\left(\frac{d}{dt}-\Delta\right)|\nabla\dot{\varphi}|^2+|\nabla\dot{\varphi}|^2(-R)\right]\omega^n$$

$$= \int_M \left(|\nabla\nabla\dot{\varphi}|^2+|\nabla\bar{\nabla}\dot{\varphi}|^2\right)\omega^n + \int_M R|\nabla\dot{\varphi}|^2\omega^n$$

$\because R(t)\geq -C$

$$\Rightarrow \frac{d}{dt}\dot{p}(t) \geq -C\cdot\int|\nabla\dot{\varphi}|^2\omega^n$$

$$= C\cdot\dot{p}(t).$$

$$\Rightarrow \frac{d}{dt}\left(e^{-Ct}\dot{p}(t)\right)\geq 0$$

$$\Rightarrow \forall t>s,\quad e^{-Ct}\dot{p}(t)\geq e^{-Cs}\dot{p}(s)$$

$$\Rightarrow \dot{p}(t)\geq e^{C(t-s)}\dot{p}(s)$$

Since $t_i \in [i, i+1]$, $\dot{\rho}(t_i) \to 0$. $\forall t \in [i, i+1]$

$$\Rightarrow \dot{\rho}(t) \geq e^{-C}\dot{\rho}(t_i) \quad (\dot{\rho}(t) \leq 0)$$

$$\Rightarrow \dot{\rho}(t) \to 0.$$

$$\Rightarrow \omega(t) \xrightarrow{C^\infty} \omega_\infty.$$

Otherwise, $\exists \{s_i\}$ $s_i \to +\infty$, s.t.

$$\omega(t_i) \xrightarrow{C^\infty} \tilde{\omega}_\infty \quad \text{s.t. } \tilde{\omega}_\infty \neq \omega_\infty$$

Since $\frac{d\rho}{dt} \to 0 \Rightarrow \log \frac{\tilde{\omega}_\infty^n}{\Omega} = \text{const} \Rightarrow \mathrm{Ric}(\tilde{\omega}_\infty) = 0$.

And since $\tilde{\omega}_0 \in [\omega_0]$, by the uniqueness, we have

$$\tilde{\omega}_\infty = \omega_\infty. \quad \text{Contradiction!}$$

#

§5 Perelman's estimate for KRF on Fano mfds

$C_1(M) > 0$

§5.1 Perelman $\mathcal{W}$-functional and μ-entropy for RF.

Reference: Perelman's paper I or Kleiner-Lott's note on Perelman's paper

Let (M^n, g) be a cpt mfd, $f \in C^\infty(M)$. Perelman's $\mathcal{W}$-functional

$$(5.1) \quad \mathcal{W}(g, f, \tau) := \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g$$

$\tau \in \mathbb{R}$, $\tau > 0$.

For any $\varphi: M^n \to M^n$ is a diffeomorphism.

$$\Rightarrow \mathcal{W}(\varphi^* g, \varphi^* f, \tau) = \mathcal{W}(g, f, \tau).$$

and
$$\mathcal{W}(ag, f, a\tau) = \mathcal{W}(g, f, \tau)$$

Perelman's first variation

Lem 5.1 (Perelman) If $v = \delta g$, $h = \delta f$, and $\eta = \delta\tau$, then

$$\delta \mathcal{W}(v, h, \eta) = \int_M -\tau \langle v, Ric + \nabla^2 f - \frac{1}{2\tau} g\rangle (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g$$
$$+ \int_M \left(\frac{V}{2} - h - \frac{n}{2\tau}\eta\right)\left[\tau(R + 2\Delta f - |\nabla f|^2) + f - n - 1\right](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g$$
$$+ \int_M \eta\left(R + |\nabla f|^2 - \frac{n}{2\tau}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g.$$

Here $V = g^{ij} v_{ij} = tr_g(v)$.

Lem 5.2 (Perelman) If $g(t)$, $f(t)$ and $\tau(t)$ evolve according to the following system

$$\begin{cases} \frac{\partial g}{\partial t} = -2Ric \\ \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{1}{2\tau} \\ \frac{\partial \tau}{\partial t} = -1 \end{cases} \quad \Longleftrightarrow \quad \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g = \text{const}$$

then

$$(5.3) \quad \frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_M \left|Ric + \nabla^2 f - \frac{1}{2\tau} g\right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_{g(t)}.$$

and $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g = \text{const}$.

$\Rightarrow \mathcal{W}(g(t), f(t), \tau(t)) \nearrow$ along (RF) is strict unless $g(t)$ is a shrinking gradient Ricci soliton

$$Ric(g(t)) + \nabla^2 f - \frac{1}{2\tau} g(t) = 0.$$

Pf: If $\frac{\partial g(t)}{\partial t} = -2(Ric + \nabla^2 f)$,

$$\frac{\partial f}{\partial t} = -\Delta f - R + \frac{n}{2\tau}$$

$$\frac{\partial \tau}{\partial t} = -1$$

$$\Rightarrow \frac{d}{dt} \mathcal{W}(g(t), f(t), \tau(t)) = 2\tau \int_M \left|Ric + \nabla^2 f - \frac{1}{2\tau} g\right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dv_g$$

Let $\Phi(t): M^n \to M^n$ a flow is generated by ∇f.

Define $\tilde{g}(t) = \Phi^* g(t)$, $\tilde{f}(t) = \Phi^* f(t)$.

$$\frac{\partial \tilde{g}(t)}{\partial t} = \Phi^* \frac{\partial g(t)}{\partial t} + \Phi^* \underbrace{(\mathcal{L}_{\nabla f} g(t))}_{= 2\nabla^2 f}$$

$$= -2Ric(\tilde{g}(t))$$

$$\frac{\partial \tilde{f}(t)}{\partial t} = -\Delta \tilde{f} + |\nabla \tilde{f}|^2 - R + \frac{1}{2\tau}$$

$$\because \mathcal{W}(\tilde{g}(t), \tilde{f}(t), \tau) = \mathcal{W}(\Phi^* g(t), \Phi^* f(t), \tau)$$

$$= \mathcal{W}(g(t), f(t), \tau)$$

#

Define Perelman's μ-entropy

$$(5.4) \quad \mu(g,f) := \inf_f \left\{ \mathcal{W}(g,f,\tau) \,\middle|\, f \in C^\infty(M), \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M e^{-f} dv_g = 1 \right\}$$

For closed mf, $\mu(g,\tau)$ is achieved by a minimizer of satisfying the nonlinear eq$\underline{n}$.

(5.5) $$\tau(2\Delta f - |\nabla f|^2 + R) + f - n = \mu(g, \tau).$$

Reference: Rothaus, Logarithmic Sobolev inequalities and the specturum of Schrödinger operators, JFA, 42(1), 1981, 110-120.