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$\exists$ Kähler $\hat{\omega} \in -c_1(M)$
$\Uparrow$
$\Rightarrow -c_1(M) > 0$

## §4.2 Case of $c_1(M) < 0$

Let $(M, \omega_0)$ be a cpt Kähler mfd with $c_1(M) < 0$. By METT

$$T := \sup\{t > 0 \mid [\omega_0] - 2\pi t c_1(M) > 0\} = +\infty.$$

hence $\exists$ a soln. of KRF for all time. $\underline{\underline{[\omega(t)]}} \nearrow \to +\infty$.

To see the convergence, we need to rescale the follow as follows.

Assume $\tilde{\omega}(s)$ is a soln. of $\frac{\partial \tilde{\omega}(s)}{\partial s} = -\mathrm{Ric}(\tilde{\omega}(s))$, w/ $\tilde{\omega}(0) = \omega_0$

Define
$$\omega(t) = \frac{1}{1+s}\tilde{\omega}(s), \quad t = \log(1+s)$$
$\Rightarrow s = e^t - 1$
$\Rightarrow \frac{ds}{dt} = e^t = 1+s.$

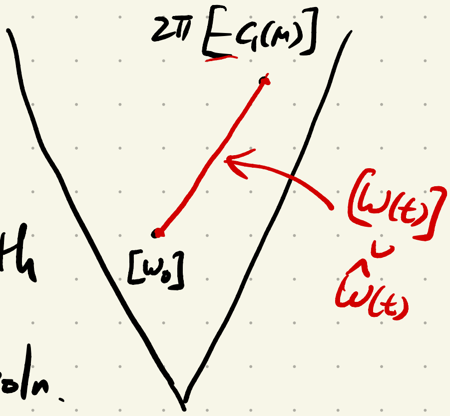
Then (NKRF)
$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) - \omega(t) \quad \text{----} (4.1)$$

- $$\frac{\partial \omega(t)}{\partial t} = \frac{\partial}{\partial s}\left(\frac{1}{1+s}\tilde{\omega}(s)\right) \cdot \frac{ds}{dt}$$
$$= \left[-\frac{1}{(1+s)^2}\tilde{\omega}(s) - \frac{1}{1+s}\mathrm{Ric}(\tilde{\omega}(s))\right] \cdot (1+s)$$
$$= -\frac{1}{1+s}\tilde{\omega}(s) - \mathrm{Ric}(\tilde{\omega}(s))$$
$$= -\omega(t) - \mathrm{Ric}(\omega(t))$$

By (4.1),
$$\frac{d}{dt}[\omega(t)] = -2\pi c_1(M) - [\omega(t)], \quad [\omega(0)] = [\omega_0]$$

$$\Rightarrow \quad [\omega(t)] = e^{-t}[\omega_0] + (1 - e^{-t})\, 2\pi [-c_1(M)] \quad \text{---} (4.2)$$

First main theorem of this subsection.




**Thm 4.1:**
Assume $(M, \omega_0)$ be a closed Kähler mfd with $c_1(M) < 0$. Then (NKRF) has a long time soln $\omega(t)$, and

$$\omega(t) \xrightarrow{C^\infty} \omega_{KE}, \quad \text{which satisfies}$$

$$Ric(\omega_{KE}) = -\omega_{KE} \quad \text{- - - -} \quad (4.3)$$

Moreover, $\omega_{KE}$ is the unique Kähler metric solving (4.3).

Rk: Tsuji, Tian-Zhang, Zhou ~ more generally.

Pf of Thm 4.1:

(1) Uniqueness. Suppose $\omega_{KE}$ and $\tilde{\omega}_{KE}$ are two solns. of (4.3)

then $[\omega_{KE}] = [\tilde{\omega}_{KE}] = -2\pi c_1(M)$

By $\partial\bar\partial$-Lemma, $\tilde{\omega}_{KE} = \omega_{KE} + \sqrt{-1}\partial\bar\partial\psi$, $\psi \in C^\infty(M)$.

Hence $Ric(\tilde{\omega}_{KE}) = -\tilde{\omega}_{KE} = -\omega_{KE} - \sqrt{-1}\partial\bar\partial\psi$

$$= -Ric(\omega_{KE}) - \sqrt{-1}\partial\bar\partial\psi$$

$$\Rightarrow -\sqrt{-1}\partial\bar\partial \log \tilde{\omega}_{KE}^n = +\sqrt{-1}\partial\bar\partial \log \omega_{KE}^n - \sqrt{-1}\partial\bar\partial\psi$$

$$\Rightarrow \log \frac{\tilde{\omega}_{KE}^n}{\omega_{KE}^n} = \psi + C \quad \text{for some const. } C$$

$$\log \frac{(\omega_{KE} + \sqrt{-1}\partial\bar\partial\psi)^n}{\omega_{KE}^n} = \psi + C.$$

$$\Rightarrow \quad \left.\begin{array}{l}\psi_{max} + C \leq 0 \\ \psi_{min} + C \geq 0\end{array}\right\} \Rightarrow \psi_{max} = \psi_{min}$$

$$\Rightarrow \psi \text{ is const.}$$

$$\Rightarrow \tilde{\omega}_{KE} = \omega_{KE}.$$

(2) Convergence ($\Rightarrow$ Existence)

We define the refence metric $\hat{\omega}_t$

$$\hat{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\hat{\omega}_\infty$$

here $\hat{\omega}_\infty \in -2\pi c_1(M)$ is a fixed kähler metric.

$$\Rightarrow \quad \hat{\omega}_t \in [\omega(t)]$$

$$\Rightarrow \exists\, \varphi(t) \in C^\infty(M) \quad \text{s.t.}$$

$$\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$$

Fixed a smooth volume form $\Omega$ satisfying

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = \hat{\omega}_\infty, \quad \int_M \Omega = \int_M \omega_0^n$$

Then NKRF (4.1) is equivalent to the following parabolic complex Monge-Ampère (MA) eqn (or $(2)'_{-1}$)

$$(4.4)\quad \begin{cases} \frac{\partial}{\partial t}\varphi = \log\dfrac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} - \varphi \\ \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \\ \varphi(\cdot, 0) = 0 \end{cases}$$

Prop. 4.2: Let $\varphi = \varphi(t)$ solve (4.4) for $t \in [0, +\infty)$. Then $\forall k \in \mathbb{N} \cup \{0\}$, $\exists A_k > 0$ s.t. $\forall t \in [0, +\infty)$

$$\|\varphi\|_{C^k(M,\omega_0)} \le A_k, \quad \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi \ge \frac{1}{A_0}\omega_0.$$

Lem 4.3: $\exists\, C>0$ (uniform) s.t. on $M\times[0,+\infty)$

(i) $|\varphi(t)| \le C$

(ii) $|\dot{\varphi}(t)| < C(1+t)e^{-t}$

(iii) $C^{-1}\Omega \le \omega^n \le C\Omega$

(iv) $\exists\, \varphi_\infty \in C^0(M)$ s.t.

$$|\varphi(t) - \varphi_\infty| \le Ce^{-\frac{t}{2}}.$$

Pf: (i) Since $0 < C_1 \le \dfrac{\omega_t^n}{\Omega} \le C_2$.

$$\Rightarrow \frac{\partial}{\partial t}\varphi_{max} \le \log C_2 - \varphi_{max}$$

$$\frac{\partial}{\partial t}\left(e^t \varphi_{max}\right) \le e^t \log C_2$$

$$\Rightarrow \varphi_{max} \le C.$$

$$\frac{\partial}{\partial t}\varphi_{min} \ge \log C_1 - \varphi_{min}$$

$$\Rightarrow \varphi_{min} \ge -C.$$

(ii) From the complex MA.

$$\begin{aligned}
\frac{\partial}{\partial t}\dot{\varphi} &= tr_{\omega(t)}\left(\frac{\partial}{\partial t}(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)\right) - \dot{\varphi} \\
&= \Delta\dot{\varphi} + tr_{\omega(t)}\left(e^{-t}(\hat{\omega}_\infty - \omega_0)\right) - \dot{\varphi} \\
&= \Delta\dot{\varphi} + tr_{\omega(t)}\hat{\omega}_\infty - n + \Delta\varphi - \dot{\varphi}
\end{aligned}$$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)(\dot{\varphi}+\varphi) = tr_{\omega(t)}\hat{\omega}_\infty - n$$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)(\dot{\varphi}+\varphi+nt) = tr_{\omega(t)}\hat{\omega}_\infty > 0 \quad \cdots\cdots (4.5)$$

From $\frac{\partial}{\partial t}\dot{\varphi} = \Delta\dot{\varphi} + e^{-t} tr_{\omega(t)}(\hat{\omega}_\infty - \omega_0) - \dot{\varphi}$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)(e^t\dot{\varphi}) = tr_{\omega(t)}(\hat{\omega}_\infty - \omega_0) \quad \cdots (4.6)$$

(4.6) − (4.5),

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left((e^t-1)\dot{\varphi} - \varphi - nt\right) = -tr_{\omega(t)}\omega_0 < 0 \quad \cdots (4.7)$$

maximum principle

$$\Longrightarrow \quad (e^t-1)\dot{\varphi} - \varphi - nt \le 0 \quad (t=0)$$

$\because$ (i) $|\varphi| \le C$.

For $t \ge 1 \Rightarrow \dot{\varphi} \le \frac{\varphi + nt}{e^t - 1} \le C(1+t)e^{-t}$.

For lower bound of $\dot{\varphi}$, (4.6) + (4.5) × A, we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left((e^t+A)\dot{\varphi} + A\varphi + Ant\right) = tr_{\omega(t)}(\hat{\omega}_\infty + A\hat{\omega}_\infty - \omega_0)$$

$\because$ $\hat{\omega}_\infty, \omega_0$ are fixed Kähler metrics, we can choose A s.t.

$$A\hat{\omega}_\infty - \omega_0 > 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)\left((e^t+A)\dot{\varphi} + A\varphi + Ant\right) \ge tr_{\omega(t)}\hat{\omega}_\infty > 0 \quad — (4.8)$$

$\because$ $\varphi$ is bounded

$$\Longrightarrow \quad \dot{\varphi} \ge -C(1+t)e^{-t}.$$

(iii) follows from (i) + (ii), $\because$ $\log\frac{\hat{\omega}(t)}{\Omega} = e^{\varphi+\dot{\varphi}}$

(iv) Fix $x \in M$, $\forall s > t$

$$|\varphi(x,s) - \varphi(x,t)| = \left| \int_t^s \dot{\varphi}(x,u)\,du \right| \le \int_t^s |\dot{\varphi}(x,u)|\,du$$

$$\overset{(ii)}{\le} C \int_t^s e^{-\frac{u}{2}}\,du$$

$$= 2C\left(e^{-\frac{t}{2}} - e^{-\frac{s}{2}}\right)$$

Hence $\varphi(t)$ converges uniformly to continuous fcn $\varphi_\infty$.

Letting $s \to +\infty$, we get

$$|\varphi(x,t) - \varphi_\infty| \le C \cdot e^{-\frac{t}{2}}.$$

#.

Lem 4.4: $\exists$ a uniform const. $C$ s.t. on $M \times [0, +\infty)$

$$C^{-1}\omega_0 \le \omega(t) \le C\,\omega_0.$$

Pf: Parabolic Schwartz Lem. (refer Lem 3.12)

Set $u = tr_{\omega_0}(\omega(t))$. $\left(\frac{\partial u}{\partial t} = -tr_{\omega_0}(Ric(\omega(t)) \underline{\underline{- u}}\right)$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log tr_{\omega_0}\omega(t) \le C_0\, tr_{\omega(t)}\omega_0 - 1 \quad \cdots (4.9)$$

here. $-C_0 = \inf_M Bisec(\omega_0)$.

Now define $Q = \log tr_{\omega_0}\omega(t) - A\varphi$, $A$: to be determined.

Then $\left(\frac{\partial}{\partial t}-\Delta\right) Q \leq C_0 \operatorname{tr}_{\omega(t)} \omega_0 - 1 - A\left(\frac{\partial}{\partial t}-\Delta\right)\varphi$

Since $\Delta\varphi = \operatorname{tr}_{\omega(t)}(\sqrt{-1}\partial\bar\partial\varphi) = \operatorname{tr}_{\omega(t)}(\omega_t - \hat\omega_t) = n - \operatorname{tr}_{\omega(t)}\hat\omega_t$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right) Q \leq An - 1 - A\dot\varphi + \operatorname{tr}_{\omega(t)}(C_0\omega_0 - A\hat\omega_t)$$

$\because$ $\hat\omega_t \geqslant \min\{\omega_0, \hat\omega_\infty\}$

$\therefore$ We can choose $A$ large enough, s.t.

$$A\hat\omega_t \geqslant (1+C_0)\omega_0$$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right) Q \leq C_1 - \operatorname{tr}_{\omega(t)}\omega_0$$

At the maximum pt $(x_0, t_0)$ of $Q$.

If $t_0 = 0 \Rightarrow Q(x,t) \leq Q(x_0, 0) \leq C$

$$\Rightarrow \operatorname{tr}_{\omega_0}\omega(t) \leq C$$

If $t_0 > 0 \Rightarrow \operatorname{tr}_{\omega}\omega_0(x_0, t_0) \leq C_1$

$$\left.\begin{aligned} &\Rightarrow \omega(x_0,t_0) \geqslant C_1^{-1}\omega_0 \\ &\because \omega^n \leq C\Omega \end{aligned}\right\} \Rightarrow \omega(x_0,t_0) \leq C\omega_0$$

$$\Rightarrow \operatorname{tr}_{\omega_0}\omega(x_0,t_0) \leq C$$

$\Rightarrow Q(x,t) \leq Q(x_0, t_0)$

$\Rightarrow \log \operatorname{tr}_{\omega_0}\omega(x,t) \leq \log \operatorname{tr}_{\omega_0}\omega(x_0,t_0) - A\varphi(x_0,t_0) + A\varphi(x,t)$

$$\Rightarrow \operatorname{tr}_{\omega_0}\omega(x,t) \leq C.$$

$$\left.\begin{aligned} \Rightarrow \quad & \omega(t) \leq C\omega_0 \\ \because \quad & \omega^n(t) \geq C^{-1}\Omega \end{aligned}\right\} \Rightarrow \omega(t) \geq C^{-1}\omega_0$$

#

Same argument as in the proof of Thm 3.6, we obtain the Prop 4.2.

From part (iv) of Lem 4.3

$$|\varphi(t) - \varphi_\infty| \leq C e^{-\frac{t}{2}}$$

$+$ Prop 4.2 $\Rightarrow$ $\varphi(t)$ converges uniformly exponentially fast to $\varphi_\infty$ in $C^\infty$

$\Rightarrow$ Hence $\varphi_\infty$ is smooth.

By Lem 4.3 (ii), we obtain

$$0 = \log \frac{(\hat{\omega}_\infty + \sqrt{-1}\partial\bar{\partial}\varphi_\infty)^n}{\Omega} - \varphi_\infty$$

$$\Rightarrow \quad \mathrm{Ric}(\omega_\infty) = -\omega_\infty.$$

here $\omega_\infty := \hat{\omega}_\infty + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$. #

## §4.3 Case of $C_1(M)=0$

For any Kähler metric $\omega_0$, by METT, we have

$$T = \sup\{t>0 \mid [\omega_0] - t\, C_1(M) > 0\} = \infty.$$

Thm 4.5: Suppose $C_1(M)=0$. Then for any Kähler metric $\omega_0$ on $M$, the KRF

$$(\text{KRF}) \quad \begin{cases} \dfrac{\partial \omega(t)}{\partial t} = -\,Ric(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$$

exists for all time and converges in $C^\infty$ to KE metrics $\omega_{KE}$ satisfying

$$Ric(\omega_{KE}) = 0 \quad \cdots \quad (4.10)$$

Moreover, $\omega_{KE}$ is the unique Kähler metric in $[\omega_0]$ satisfying (4.10).

<u>RK</u>: Uniqueness: due to Calabi

Existence: Yau

Cao provided a parabolic proof of Thm 4.5.

For prove Thm 4.5, we need the following (important)

Yau's $L^\infty$ estimate for the complex MA eqn.

Thm 4.6 (Yau, Prop. 2.1)

Let $(M,\omega_0)$ be a cpt Kähler mfd and let $F\in C^\infty(M)$.
Suppose that $\theta$ satisfies the complex MA eqn

$$(\omega_0+\sqrt{-1}\partial\bar\partial\theta)^n=e^F\omega_0^n,\quad \omega_0+\sqrt{-1}\partial\bar\partial\theta>0$$

Then

$$\operatorname{osc}(\theta):=\sup_M\theta-\inf_M\theta\le C=C(\omega_0,\sup_M F)$$

Pf: Omit.

RK: If $\theta$ in Thm 4.6 is normalized by $\int_M\theta\omega^n=0$, then

$$\operatorname{osc}(\theta)\le C\Rightarrow \|\theta\|_{L^\infty}=\sup_M|\theta|\le M$$

Now we begin to prove Thm 4.5.

- Uniqueness. Suppose $\omega_{KE}$ and $\tilde\omega_{KE}$ are solns. of (4.10)

and $[\omega_{KE}]=[\tilde\omega_{KE}]=[\omega_0]$

$$\Rightarrow \tilde\omega_{KE}=\omega_{KE}+\sqrt{-1}\partial\bar\partial\varphi,\ \text{for } \varphi\in C^\infty(M)$$

$$\because\ Ric(\tilde\omega_{KE})=Ric(\omega_{KE})=0$$

$$\Rightarrow \sqrt{-1}\partial\bar\partial\log\frac{\tilde\omega_{KE}^n}{\omega_{KE}^n}=0$$

$$\Rightarrow \frac{\tilde\omega_{KE}^n}{\omega_{KE}^n}=C>0,\quad C \text{ is a const.}$$

On the other hand, $\int_M\tilde\omega_{KE}^n=\int\omega_{KE}^n$

$$\Rightarrow C=1.$$

Then using Stoke's Thm,

$$0 = \int_M \varphi (\omega_{KE}^n - \tilde{\omega}_{KE}^n)$$

$$= \int_M \varphi (\omega_{KE} - \tilde{\omega}_{KE}) \wedge \left( \sum_{i=0}^{n-1} \omega_{KE}^i \wedge \tilde{\omega}_{KE}^{n-1-i} \right)$$

$$= -\int_M \varphi \sqrt{-1} \partial \bar{\partial} \varphi \wedge \left[ \sum_{i=0}^{n-1} \omega_{KE}^i \wedge \tilde{\omega}_{KE}^{n-1-i} \right]$$

$$= \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \left[ \sum_{i=0}^{n-1} \omega_{KE}^i \wedge \tilde{\omega}_{KE}^{n-1-i} \right]$$

$$\geq \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_{KE}^{n-1}$$

$$= \frac{1}{n} \int_M |\partial \varphi|^2_{\omega_{KE}} \omega_{KE}^n \qquad \left( |\partial \varphi|^2 = |\bar{\partial} \varphi|^2 = ? \tfrac{1}{2} |\nabla \varphi|^2 \right)$$

$\Rightarrow$ $\varphi$ is const.

$\Rightarrow$ $\omega_{KE} = \tilde{\omega}_{KE}$.

- Convergence.

$\because$ Since $c_1(M) = 0$, $\exists$ a smooth volume form $\Omega$ with

$$\sqrt{-1} \partial \bar{\partial} \log \Omega = 0 \quad \text{and} \quad \int_M \Omega = \int_M \omega_0^n.$$

Then (KRF) $\Leftrightarrow$

$$(4.11) \quad \begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} \\ \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \\ \varphi(0) = 0 \end{cases}$$

Then we have

Lem 4.7: $\exists$ a uniform const. $C>0$ s.t. on $M\times[0,\infty)$

(i) $|\dot{\varphi}|\leq C$

(ii) $C^{-1}\Omega\leq\omega^n\leq C\Omega$

(iii) $\operatorname{osc}(\varphi)\leq C$.

pf: (i) by (4.11), $\frac{\partial}{\partial t}\dot{\varphi}=\Delta\dot{\varphi}$ i.e., $(\frac{\partial}{\partial t}-\Delta)\dot{\varphi}=0$.

Then (i) follows from the maximum principle.

(i) $\Rightarrow$ (ii)

(iii) By (4.11), $(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi)^n=e^{\dot{\varphi}}\Omega=e^{\dot{\varphi}+f}\omega_0^n$

By (i) $|\dot{\varphi}|\leq C$, then using Yau's $L^\infty$-estimate,

i.e., Thm 4.6, we have

$$\operatorname{osc}(\varphi)\leq C.$$

Lem 4.8 $\exists$ a uniform const. $C$ s.t. on $M\times[0,\infty)$

$$C^{-1}\omega_0\leq\omega(t)\leq C\omega_0 \quad --- \quad (4.12)$$

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