2022.12.29.

Under RF: $\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$

$$\Rightarrow \frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2 \Rightarrow \frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{n}R^2.$$

$$\Rightarrow \frac{dR_{min}}{dt} \geq \frac{2}{n}R_{min}^2 \geq 0$$

$$\Rightarrow R_{min}(t) \nearrow$$

- Calculate the evolution of $R(t)$ under KRF $\quad \frac{\partial \omega(t)}{\partial t} = -Ric(\omega)$

$$\because R = g^{i\bar{j}}R_{i\bar{j}} = -g^{i\bar{j}}\partial_i\partial_{\bar{j}}\log\det g$$

$$\therefore \frac{\partial R}{\partial t} = \frac{\partial g^{i\bar{j}}}{\partial t}R_{i\bar{j}} - g^{i\bar{j}}\partial_i\partial_{\bar{j}}\left(\frac{d}{dt}\log\det g\right)$$

$$= -g^{i\bar{q}}\underline{\frac{\partial g_{p\bar{q}}}{\partial t}}g^{p\bar{j}}R_{i\bar{j}} - g^{i\bar{j}}\partial_i\partial_{\bar{j}}\underbrace{tr\left(g^{-1}\frac{\partial g}{\partial t}\right)}_{g^{k\bar{l}}\frac{\partial g_{k\bar{l}}}{\partial t}}$$

$\frac{\partial g_{k\bar{l}}}{\partial t} = -R_{k\bar{l}}$

$$= \underline{g^{i\bar{q}}R_{p\bar{q}}g^{p\bar{j}}R_{i\bar{j}}} - g^{i\bar{j}}\partial_i\partial_{\bar{j}}(-R)$$

$$= \Delta R + |Ric|^2$$

- $\left(\frac{\partial}{\partial t} - \Delta\right)R_{i\bar{j}} = R_{i\bar{j}k\bar{l}}R_{l\bar{k}} - R_{i\bar{l}}R_{l\bar{j}}$

Some evolutions:

(1) $\frac{\partial}{\partial t}\dot{\varphi} = -R(t)$

(2) $\left(\frac{\partial}{\partial t}-\Delta\right)\varphi = \dot{\varphi} - n + \mathrm{tr}_{\omega(t)}\hat{\omega}_t$

(3) $\left(\frac{\partial}{\partial t}-\Delta\right)\dot{\varphi} = \mathrm{tr}_{\omega(t)}\chi$

(4) $\left(\frac{\partial}{\partial t}-\Delta\right)(t\dot{\varphi}) = \dot{\varphi} + \mathrm{tr}_{\omega(t)}(\hat{\omega}_t - \omega_0)$

(5) $\left(\frac{\partial}{\partial t}-\Delta\right)(t\dot{\varphi}-\varphi-nt) = -\mathrm{tr}_{\omega(t)}\omega_0 < 0$

(6) $\left(\frac{\partial}{\partial t}-\Delta\right)\left((T'-t)\dot{\varphi}+\varphi+nt\right) = \mathrm{tr}_{\omega(t)}(\hat{\omega}_{T'}) > 0$

<u>Thm 3.9</u>: $\exists\, C = C(\omega_0, T_{max}) > 0$ s.t.

$$\sup_M \mathrm{tr}_{\omega_0}\omega(t) \le C, \quad \forall t\in[0,T_{max})$$

$\Rightarrow \omega(t) \le C\omega_0$

Cor 3.10: $\exists$ a const $C>0$, $C = C(\omega_0, T_{max})$ s.t.

$$C^{-1}\omega_0 \le \omega(t) \le C\omega_0, \quad \forall t\in[0,T_{max})$$

pf of Cor 3.10: ① Thm 3.9 $\Rightarrow \omega(t) \le C\omega_0$.

② Lem 3.7, $|\dot{\varphi}| \le C \Leftrightarrow e^{-C}\Omega \le \omega^n \le e^{C}\cdot\Omega$

holo. normal coord. w/ ω_0, $\omega(t) \leftrightarrow (\lambda_1, \dots, \lambda_n)$

$$\frac{1}{C_1} \le \prod_{i=1}^n \lambda_i \le C_1 \qquad (C_1>0)$$

$$\Rightarrow \quad \lambda_i \geq \frac{1}{C_1} \cdot \frac{1}{\prod_{j \neq i} \lambda_j} \overset{\textcircled{D}}{\geq} \frac{1}{C_2}$$

#.

Thm 3.11 (Sherman-Weinkove, 2012, Pac. J. Math.)

Assume U is an open set of M, $\omega(t)$ is soln. of KRF on $U \times [0,T)$ w/ $\omega(0) = \omega_0$. Assume $\exists\, C_0 > 0$ s.t.

$$C_0^{-1} \omega_0 \leq \omega(t) \leq C_0 \omega_0 \quad \text{on } U \times [0,T)$$

for some Kähler metric ω_0 on M. Then given $\forall$ compact set $K \subset U$ and $\exists\, C_k = C_k(K, U, k, \omega_0, C_0)$ s.t.

$$\|\omega(t)\|_{C^k(K,\omega_0)} \leq C, \quad \forall t \in [0,T).$$

pf of Thm 3.11 omit.

pf of Thm 3.6: By Cor 3.10 + Thm 3.11

$$\Rightarrow \|\omega(t)\|_{C^k(M,\omega_0)} \leq C_k, \quad \forall t \in [0, T_{max})$$

$\because \sqrt{-1}\partial\bar{\partial}\varphi(t) = \omega(t) - \hat{\omega}_t$

$\hat{\omega}_t = \omega_0 + t\chi$

$\Rightarrow tr_{\omega_0} \hat{\omega}_t = n + t\, \underline{tr_{\omega_0}\chi}$

$$\Rightarrow \Delta_{\omega_0} \varphi = tr_{\omega_0}(\omega(t) - \hat{\omega}_t) = tr_{\omega_0}\omega(t) - tr_{\omega_0}\hat{\omega}_t =: f$$

$\because \|\omega(t)\|_{C^k(M,\omega_0)} \leq C_k \quad \Rightarrow \|f\|_{C^k(M,\omega_0)} \leq C_k.$

$\forall$ fixed $\alpha \in (0,1)$, we have the elliptic estimates

$$\|\varphi(t)\|_{C^k(M,g_0)} = \|\varphi(t)\|_{C^{k,\alpha}(M,g_0)}$$

$$\|f\|_{C^{0,\alpha}} := \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

Schauder interior estimates

Gilbarg-Trudinger Chapter 6.

$$\leq C_k\left(\|\Delta_{\omega_0}\varphi(t)\|_{C^{k-2,\alpha}(M,g_0)} + \|\varphi(t)\|_{C^0(M)}\right)$$

$$\leq C_k\left(\|\underbrace{\Delta_{\omega_0}\varphi(t)}_{\substack{\shortparallel \\ f}}\|_{C^{k-1}(M,g_0)} + \|\varphi(t)\|_{C^0(M)}\right)$$

#

Now we begin to prove the Thm 3.9.

Lem 3.12 (Parabolic Schwartz Lemma)

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)\log \operatorname{tr}_{\omega_0}\omega(t) \leq C_0 \operatorname{tr}_{\omega(t)}\omega_0$$

Where $-C_0$ is a lower bound for the bisectional curvature of ω_0, i.e,

$$\mathring{R}_{k\bar{k}i\bar{i}} = \mathring{R}_m(\partial_k, \bar{\partial}_k, \partial_i, \bar{\partial}_i) \geq -C_0$$

$\mathring{R}$, $\mathring{R}_m$ with ω_0.

Pf of Lem 3.12: Set $u = \operatorname{tr}_{\omega_0}\omega(t) = (g_0)^{i\bar{j}} g_{i\bar{j}} > 0$

$$\frac{\partial u}{\partial t} = g_0^{i\bar{j}} \frac{\partial g_{i\bar{j}}}{\partial t} = -g_0^{i\bar{j}} R_{i\bar{j}} = -\operatorname{tr}_{\omega_0}(Ric(\omega(t)))$$

$$\Delta u = g^{k\bar{l}} \partial_k \partial_{\bar{l}} u = g^{k\bar{l}} \partial_k \partial_{\bar{l}} (g_0^{i\bar{j}} g_{i\bar{j}})$$

At point p, we calculate with holo. normal coord. for ω_0 where $\omega(t)$ is diagonal, i.e.,

$$(g_0)_{i\bar j}(p)=\delta_{ij}, \quad \partial_k (g_0)_{i\bar j}(p)=0, \quad g_{i\bar j}(p)=\lambda_i\delta_{ij}$$

$$\Delta u(p) = g^{k\bar l}\partial_k\partial_{\bar l}(g_0^{i\bar j})\, g_{i\bar j} + g^{k\bar l} g_0^{i\bar j}\partial_k\partial_{\bar l} g_{i\bar j}$$

$$= -g^{k\bar l} g_0^{i\bar q} g_0^{p\bar j}\,\partial_k\partial_{\bar l}(g_0)_{p\bar q}\cdot g_{i\bar j}\Big|_p + g^{k\bar l}g_0^{i\bar j}\partial_k\partial_{\bar l} g_{i\bar j}\Big|_p$$

回顾: $R_{i\bar j k\bar l} = -\partial_i\partial_{\bar j} g_{k\bar l} + g^{p\bar q}\partial_i g_{k\bar q}\cdot\partial_{\bar j} g_{p\bar l}$

$$= \underline{g^{k\bar l}g_0^{i\bar q}g_0^{p\bar j}}\,\mathring{R}_{k\bar l p\bar q} + g^{k\bar l}g_0^{i\bar j}\left(-R_{k\bar l i\bar j} + g^{p\bar q}\partial_i g_{k\bar q}\partial_{\bar j} g_{p\bar l}\right)$$

$$= g^{k\bar k} g_{i\bar i}\,\underline{\mathring{R}_{k\bar k i\bar i}} - \underbrace{g_0^{i\bar j}R_{i\bar j}}_{\neq R} + g_0^{i\bar j}g^{k\bar l}g^{p\bar q}\,\mathring\nabla_i g_{k\bar q}\,\mathring\nabla_{\bar j} g_{p\bar l}$$

$$\Rightarrow \left(\frac{\partial}{\partial t}-\Delta\right)\log u = \frac{1}{u}(\partial_t-\Delta)u + \frac{|\nabla u|^2}{u^2}$$

$$= -\frac{1}{u}\, g^{k\bar k}g_{i\bar i}\,\mathring{R}_{k\bar k i\bar i} - \frac{1}{u}\left(g_0^{i\bar j}g^{k\bar l}g^{p\bar q}\,\mathring\nabla_i g_{k\bar q}\,\mathring\nabla_{\bar j}g_{p\bar l} - \frac{|\nabla u|^2}{u}\right)$$

$$\le C_0\cdot\frac{1}{u}(\mathrm{tr}_{\omega(t)}\omega_0)(\mathrm{tr}_{\omega_0}\omega(t)) - \frac{A}{u}$$

$$A := g_0^{i\bar j}g^{k\bar l}g^{p\bar q}\,\mathring\nabla_i g_{k\bar q}\,\mathring\nabla_{\bar j}g_{p\bar l} - \frac{|\nabla u|^2}{u}$$

$$\leq C_0 \operatorname{tr}_{\omega(t)}\omega_0 - \frac{A}{u}$$

Define: $B_{ki\bar{q}} := \mathring{\nabla}_k g_{i\bar{q}} - \frac{\partial_k u}{u} g_{i\bar{q}}$

Then

$$0 \leq g_0^{i\bar{j}} g^{k\bar{\ell}} g^{p\bar{q}} B_{ki\bar{q}} \overline{B_{\ell j\bar{p}}} = A$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right) \log u \leq C_0 \operatorname{tr}_{\omega(t)}\omega_0 \qquad \#$$

<u>pf of Thm 3.9</u>: $\quad u \leq C.$

Set $Q := \log \operatorname{tr}_{\omega_0}\omega(t) + C_0(t\dot{\varphi} - \varphi - nt)$

here C_0 is in the Lem 3.12.

$\because$ (5) $\left(\frac{\partial}{\partial t} - \Delta\right)(t\dot{\varphi} - \varphi - nt) = -\operatorname{tr}_{\omega(t)}\omega_0$

$\therefore \left(\frac{\partial}{\partial t} - \Delta\right) Q \leq 0$

Using the maximum principle,

$$\frac{\partial}{\partial t} Q_{max} \leq 0 \Rightarrow Q_{max}(t) \leq Q_{max}(0) \leq C.$$

$\because |\dot{\varphi}| \leq C, \ |\varphi| \leq C$

$$\Rightarrow \operatorname{tr}_{\omega_0}\omega(t) \leq C. \qquad \#$$

§4 Convergence Theorems with $c_1(M)<0$ or $c_1(M)=0$

§4.1 Introduction to Calabi conjecture.

$\because$ $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det g$

and $Ric(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j \in 2\pi c_1(M)$

Thm (Calabi-Yau) Let (M, ω) be a Kähler mfd. Let $T \in 2\pi c_1(M)$ be a closed $(1,1)$ form. Then $\exists!$ Kähler metric $\tilde{g}_0$ on M with Kähler form $\tilde{\omega}_0$ s.t.

$$[\tilde{\omega}_0] = [\omega] \quad \text{and} \quad Ric(\tilde{\omega}_0) = T.$$

RK: Calabi conj. 1954, "The space of Kähler metrics", Proc. ICM, 1954.

Calabi proved the uniqueness.

Yau (1976) proved the existence.

<u>Cor</u>: If $c_1(M)=0$, then M has a unique Ricci-flat Kähler metric $\tilde{g}$ w/ Kähler form $\tilde{\omega}$ s.t. $[\tilde{\omega}]=[\omega]$.

- The mfd w/ $c_1(M)=0$ is called the Calabi-Yau mfd.

- If $\exists \tilde{\omega}_0$ s.t. $Ric(\tilde{\omega}_0)=T$ & $[\tilde{\omega}_0]=[\omega]$

Then $\exists \varphi \in C^\infty(M)$ s.t. $\tilde{\omega}_0 = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$.

$T \in 2\pi c_1(M) \Rightarrow \exists F \in C^\infty(M)$ s.t. $T = Ric(\omega) + \sqrt{-1}\partial\bar{\partial}F$

$$= -\sqrt{-1}\partial\bar{\partial}\log\omega^n + \sqrt{-1}\partial\bar{\partial}F$$

$$= -\sqrt{-1}\partial\bar{\partial}\log e^{-F}\omega^n$$

On the other hand,

$$T = Ric(\tilde{\omega}_0) = -\sqrt{-1}\partial\bar{\partial}\log\tilde{\omega}_0^n$$

$$\Rightarrow \log\tilde{\omega}_0^n = \log e^{-F}\omega^n + C$$

$$\Rightarrow \tilde{\omega}_0^n = e^{-F+C}\omega^n \qquad \text{Set } f = -F+C.$$

$$(*) \quad (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^f\omega^n \quad \cdots \text{ Monge-Ampère equation}$$

Calabi Conj. $\Longleftrightarrow$ To solve MA equ $(*)$.

$$(*)_t: \quad (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{tf}\omega^n$$

$$E := \{t \mid (*)_t \text{ has soln. } \varphi_t,\ t\in[0,1]\}$$

Easy to see $0\in E$. $\Rightarrow E \neq \phi$.

To prove $\underbrace{\text{① } E \text{ is open} \quad \text{② } E \text{ is closed}}$

$\Rightarrow E = [0,1]$. $\Rightarrow (*)$ has a soln. φ.

Thm (Aubin-Calabi-Yau) If $C_1(M)<0$, then M has a unique KE metric ω_0, s.t.

$$Ric(\omega_0) = -\omega_0$$

KE eqn: $\quad Ric(\tilde{\omega}) = \lambda\tilde{\omega} \in 2\pi C_1(M)$ & $[\tilde{\omega}] = [\omega]$

$$\Rightarrow \text{KE eqn} \Longleftrightarrow (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{f-\lambda\varphi}\omega^n$$

here $\quad Ric(\omega) - \lambda\omega = \sqrt{-1}\partial\bar{\partial}f.$

1985, Cao introduced the Kähler-Ricci flow

$$(1) \quad \frac{\partial \tilde{g}_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}}(t) + T_{i\bar{j}}, \quad \tilde{g}_{i\bar{j}}(0) = g_{i\bar{j}}$$

$$\Rightarrow \frac{d}{dt}[\tilde{\omega}] = 0 \Rightarrow [\tilde{\omega}(t)] = [\omega]$$

$$\Rightarrow \exists\, \varphi(t) \in C^\infty(M) \text{ s.t. } \tilde{\omega}(t) = \omega + \sqrt{-1}\partial\bar{\partial}\varphi(t)$$

$$\because [T] = [Ric(\omega)]$$

$$\therefore \exists\, f \in C^\infty(M) \text{ s.t. } T_{i\bar{j}} - R_{i\bar{j}} = \sqrt{-1}\partial_i \partial_{\bar{j}} f$$

$$\Rightarrow T_{i\bar{j}} = -\sqrt{-1}\partial_i\partial_{\bar{j}} \log(e^{-f}\omega^n).$$

Hence $(1) \Leftrightarrow$

$$(1)' \quad \begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n} + f \\ \varphi(x,0) = 0 \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0. \end{cases}$$

- For Kähler-Einstein problem, consider the following KRF

$$(2)_\lambda \quad \frac{\partial \tilde{g}_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} + \lambda \tilde{g}_{i\bar{j}}, \quad \tilde{g}_{i\bar{j}}(0) = g_{i\bar{j}} \quad \text{w/ } \lambda g \in 2\pi C_1(M)$$

$$\Leftrightarrow (2)'_\lambda \quad \begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n} + f + \lambda\varphi \\ \varphi(x,0) = 0 \\ \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \end{cases}$$

- $\frac{\partial}{\partial t}\dot{\varphi} = \Delta_{\omega(t)}\dot{\varphi} + \lambda\dot{\varphi} \Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right)\dot{\varphi} = \lambda\dot{\varphi}$

$\rightsquigarrow \dot{\varphi}(t) \sim e^{\lambda t}\dot{\varphi}(0)$

- In fact, there exist closed Kähler mfd with $C_1(M)>0$. but M does not have KE metric. e.g., $\mathbb{C}P^2$ blow up at one pt, $\mathbb{C}P^2$ blow up at two pts, refer Tian's paper.

- Fact. $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ is Fano, if $0<k\leq 8$