Since M is cpt, $\forall \tilde{\lambda}_i$ sufficiently close to λ_i, we have

$$\sum_{i=1}^{N} \tilde{\lambda}_i \alpha_i + \sqrt{-1}\partial\bar{\partial}\varphi > 0$$

$$\Longrightarrow \quad \sum_{i=1}^{N} \tilde{\lambda}_i [\alpha_i] \in \mathcal{K}_a(M)$$

i.e., $\exists$ neighborhood U of $[\alpha]$ s.t. $U \subset \mathcal{K}_a(M)$

$\Longrightarrow \mathcal{K}_a(M)$ is open

#.

2022.12.27

Def$\underline{n}$ (Nef) If $[\alpha] \in \overline{\mathcal{K}_a(M)}$, then $[\alpha]$ is called nef.

"numerally + effective".

Lemma 2.2. Assume (M, ω) is a cpt Kähler mfd.

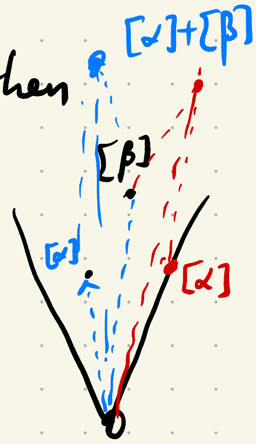
Then $[\alpha] \in H^{1,1}(M, \mathbb{R})$ is nef iff $\forall \varepsilon > 0$, $\exists \varphi_\varepsilon \in C^\infty(M)$, such that

$$\alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon > -\varepsilon\omega$$

Pf: Omit. Detail of pf, refer to Tosatti's KAWA lecture.

Cor 2.3: If $[\alpha] \in \overline{\mathcal{K}_a(M)}$, $[\beta] \in \mathcal{K}_a(M)$, then

$$[\alpha] + [\beta] \in \mathcal{K}_a(M).$$



Def$\underline{n}$ (Big): A nef class $[\alpha]$ is called big

if $$\int_M \alpha^n > 0.$$

Def$\underline{n}$ (First Chern Class) $\quad c_1(M) := \frac{1}{2\pi}[\mathrm{Ric}(\omega)]$

Prop. 2.4: $C_1(M)$ is indepent of choice of ω.

pf: Assume $\tilde{\omega} = \sqrt{-1}\,\tilde{g}_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ be any other Kähler metric.

Then $\exists F \in C^\infty(M)$ s.t.

$$\tilde{\omega}^n = e^F \cdot \omega^n$$

$$\Rightarrow \mathrm{Ric}(\tilde{\omega}) = -\sqrt{-1}\,\partial\bar{\partial} \log \tilde{\omega}^n = -\sqrt{-1}\,\partial\bar{\partial} \log(e^F \omega^n)$$

$$= -\sqrt{-1}\,\partial\bar{\partial} F + \mathrm{Ric}(\omega)$$

$$\Rightarrow [\mathrm{Ric}(\tilde{\omega})] = [\mathrm{Ric}(\omega)]$$ #

Example: Let $M = M_1 \times M_2$, (M_1, ω_1), (M_2, ω_2).

$\pi_1 : M \to M_1$, $\pi_2 : M \to M_2$ projection map

Define $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$.

Then $\mathrm{Ric}(\omega) = \pi_1^* \mathrm{Ric}(\omega_1) + \pi_2^* \mathrm{Ric}(\omega_2)$

If $\omega_1(t)$, $\omega_2(t)$ are soln. of (KRF) on M_1 and M_2, respectively.

w/ $\omega_1(0) = \omega_1$, $\omega_2(0) = \omega_2$

Then $\omega(t) = \pi_1^* \omega_1(t) + \pi_2^* \omega_2(t)$ is a soln. of (KRF)

on M with $\omega(0) = \omega$.

§3 The maximal existence time theorem

Assume $g(t)$ is a soln. of (RF) on $M\times[0,T)$.

T: the maximal time & $T<+\infty$.

- Hamilton, $\sup\limits_{M\times[0,T)}|Rm| = +\infty$.

- Sesum, $\sup\limits_{M\times[0,T)}|Ric| = +\infty$

Question: $\sup\limits_{M\times[0,T)}|R| = +\infty$? $\left(\Leftrightarrow \sup\limits_{M\times[0,T)} R = +\infty\right)$

Kahler case, $\checkmark$, due to Zhang, Zhou.

For (KRF):
$$\begin{cases} \dfrac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$$

$$\underset{\text{ODE}}{\Rightarrow} \begin{cases} \dfrac{d}{dt}[\omega(t)] = -2\pi C_1(M) \\ [\omega(0)] = [\omega_0] \end{cases}$$

$$\Rightarrow [\omega(t)] = [\omega_0] - 2\pi t\, C_1(M)$$

If $\omega(t)$ is kähler $\Rightarrow [\omega(t)]>0$ i.e., $[\omega_0]-2\pi t\, C_1(M)>0$.

- If $C_1(M)<0 \Rightarrow T > +\infty$
- If $C_1(M)=0 \Rightarrow T = +\infty$

Thm 3.1 (Tian-Zhang, Zhou, 2006)

Let (M,ω_0) be a closed Kähler mfd. Then (KRF) has a unique smooth soln. $\omega(t)$ defined on the maximal time interval $[0, T)$ $(0 < T \leq +\infty)$, where T is given by

$$T := \sup\{t > 0 \mid [\omega_0] - 2\pi t\, C_1(M) \in \underbrace{K_a(M)}_{>0}\} \quad \text{---}(3.1)$$

Cor 3.2: $T = +\infty$ iff $-C_1(M) \in \overline{K_a(M)}$ i.e., $-C_1(M)$ is nef.

- Parabolic complex Monge-Ampère equation.

Given a Kähler metric $\hat{\omega}$ (as a background metric) s.t. $[\omega(t)] = [\hat{\omega}]$, then by $\partial\bar{\partial}$-Lemma, $\exists\, \varphi(t) \in C^\infty(M)$ s.t.

$$\omega(t) = \hat{\omega} + \sqrt{-1}\,\partial\bar{\partial}\varphi.$$

Fix a smooth volume form Ω' on M, then $\exists\, F \in C^\infty(M)$, s.t.

$$\Omega' = e^F \cdot \omega_0^n$$

Then $\exists\, C \in \mathbb{R}$ s.t. $\displaystyle\int e^{F+C}\omega_0^n = \int \omega_0^n$

Set $\Omega = e^{F+C} \cdot \omega_0^n = e^C \cdot \Omega'$ satisfying

$$\int_M \Omega = \int_M \omega_0^n$$

& $[\mathrm{Ric}(\Omega)] = [\mathrm{Ric}(\Omega')] = [\mathrm{Ric}(\omega_0^n)] = [\mathrm{Ric}(\omega_0)] = 2\pi C_1(M)$.

Now we begin to prove Thm 3.1.

Fix any $T' \in (0, T)$ $(\Rightarrow T' < +\infty)$, and by definition, we have

$$[\omega_0] - 2\pi T' C_1(M) \in K_a(M)$$

$\therefore \exists$ Kähler metric η s.t.

$$[\eta] = [\omega_0] - 2\pi T' c_1(M)$$

Set $\chi := \frac{1}{T'}(\eta - \omega_0)$ ······ (3.2)

$\Rightarrow [\chi] = -2\pi c_1(M) = -[Ric(\omega)]$

and $\hat{\omega}_t := \omega_0 + t\chi = \frac{1}{T'}\left((T'-t)\omega_0 + t\eta\right)$ ······ (3.3)

$\Rightarrow \hat{\omega}_t$ is a Kähler metric for $t \in [0, T']$.

We choose a smooth volume form Ω s.t.

$$Ric(\Omega) = -\chi \quad \text{and} \quad \int_M \Omega = \int_M \omega_0^n \quad \text{---- (3.4)}$$

$\because [\omega(t)] = [\hat{\omega}_t]$

$\therefore \exists\ \varphi(t) \in C^\infty(M)$ s.t. $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t)$

Lem 3.3: The KRF equation is equivalent to the following
if and only if
parabolic complex Monge-Ampère equation

$$(KRF) \Longleftrightarrow \begin{cases} \frac{\partial}{\partial t}\varphi(t) = \log \dfrac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\Omega} \\ \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t) > 0 \\ \varphi(0) = 0 \end{cases} \quad \text{--- (3.5)}$$

Pf. "$\Leftarrow$" ("if" direction)

Direct calculation.
$$\frac{\partial\omega(t)}{\partial t} = \frac{\partial\hat{\omega}_t}{\partial t} + \sqrt{-1}\partial\bar{\partial}\frac{\partial\varphi(t)}{\partial t}$$
$$= \chi + \sqrt{-1}\partial\bar{\partial}\log\frac{\omega^n(t)}{\Omega}$$

$$= \chi - \mathrm{Ric}(\omega(t)) + \mathrm{Ric}(\Omega)$$
$$= \chi - \mathrm{Ric}(\omega(t)) - \chi$$
$$= -\mathrm{Ric}(\omega(t)).$$

"$\Longrightarrow$" ("Only if")

$\because \dfrac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t))$ and $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t)$

$$\therefore \frac{\partial \hat{\omega}_t}{\partial t} + \sqrt{-1}\partial\bar{\partial}\frac{\partial \varphi}{\partial t} = \sqrt{-1}\partial\bar{\partial}\log \omega^n(t)$$

$$\chi + \sqrt{-1}\partial\bar{\partial}\frac{\partial \varphi}{\partial t} = \sqrt{-1}\partial\bar{\partial}\log \omega^n(t)$$

$\because \chi = -\mathrm{Ric}(\Omega)$

$$\Longrightarrow \quad \sqrt{-1}\partial\bar{\partial}\frac{\partial \varphi}{\partial t} = \sqrt{-1}\partial\bar{\partial}\log \omega^n(t) - \sqrt{-1}\partial\bar{\partial}\log \Omega$$
$$= \sqrt{-1}\partial\bar{\partial}\log \frac{\omega^n(t)}{\Omega}$$

$\therefore \exists\, C(t) \in \mathbb{R}$ s.t.

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega^n(t)}{\Omega} + C(t)$$
$$= \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\Omega} + C(t)$$

Normalizing for φ s.t.

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\Omega} \\ \varphi(0) = 0 \\ \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi(t) > 0. \end{cases}$$

#

Short-time existence and uniqueness due to Hamilton 1982.

Here we provide a pf for uniqueness theorem of KRF.

Thm 3.4: The uniqueness thm of KRF.

pf: Assume $\omega_1(t)$ and $\omega_2(t)$ are soln. of (KRF) w/ $\omega_1(0)=\omega_2(0)=\omega_0$

$\Rightarrow \exists\, \varphi_1(t), \varphi_2(t) \in C^\infty(M)$ s.t.

$$\omega_1(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_1(t), \quad \omega_2(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_2(t)$$

and $\varphi_1(t)$, $\varphi_2(t)$ are soln. of (3.5) for $t\in[0,T')$.

Goal: $\varphi_1(t)=\varphi_2(t)$.

Let $\psi(t)=\varphi_1(t)-\varphi_2(t)$. Then

$$\omega_2(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_2 = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}(\varphi_1-\psi)$$
$$= \omega_1(t) - \sqrt{-1}\partial\bar{\partial}\psi$$

$$\therefore \left(\omega_2(t)+\sqrt{-1}\partial\bar{\partial}\psi(t)\right)^n = \omega_1^n(t) = e^{\dot{\varphi}_1(t)}\Omega = e^{\dot{\psi}+\dot{\varphi}_2(t)}\Omega$$
$$= e^{\dot{\psi}}\omega_2^n(t)$$

$$\Rightarrow \begin{cases} \dfrac{\partial\psi}{\partial t} = \log \dfrac{\left(\omega_2(t)+\sqrt{-1}\partial\bar{\partial}\psi(t)\right)^n}{\omega_2^n(t)} \\ \psi(0)=0 \\ \omega_2(t)+\sqrt{-1}\partial\bar{\partial}\psi(t)>0 \end{cases}$$

pf of the maximum principle.

For any given $\varepsilon>0$, set $\tilde{\psi}(t) := \psi(t)-\varepsilon t \Rightarrow \tilde{\psi}\in C^\infty(M\times[0,T'))$

and
$$\frac{\partial\tilde{\psi}}{\partial t} = \log\frac{\left(\omega_2(t)+\sqrt{-1}\partial\bar{\partial}\tilde{\psi}(t)\right)^n}{\omega_2^n(t)} - \varepsilon$$

~~Using the maximum principle~~, At the maximum pt (x_0, t_0) of $\tilde{\psi}$

$$\sqrt{-1}\,\partial\bar{\partial}\tilde{\psi}(x_0, t_0) \leq 0.$$

$$\left.\begin{array}{l} \text{If } t_0 > 0, \Rightarrow \dfrac{\partial\tilde{\psi}}{\partial t}(x_0, t_0) \geq 0 \\ \text{On the other hand, } \dfrac{\partial\tilde{\psi}}{\partial t}(x_0, t_0) \leq -\varepsilon \end{array}\right\} \text{contradition.}$$

$\Rightarrow t_0 = 0.$

$$\Rightarrow \tilde{\psi}(t) \leq \tilde{\psi}(0) = 0 \Rightarrow \psi(t) \leq \varepsilon t \leq \varepsilon T'.$$

By the arbitrary of $\varepsilon \Rightarrow \psi(t) \leq 0.$

Similarly, at the minimum pt $(\tilde{x}_0, \tilde{t}_0)$ of $\psi(t) + \varepsilon t$.

$$\Rightarrow \tilde{t}_0 = 0 \Rightarrow \psi(t) \geq -\varepsilon t \geq -\varepsilon T'$$

$$\Rightarrow \psi(t) \geq 0$$

Hence $\psi(t) \equiv 0$, i.e. $\varphi_1(t) \equiv \varphi_2(t)$.

#.

The maximum principle:

Let M be a closed mfd. Let $f = f(x,t)$ be a smooth fcn on $M \times [0,a]$ $(a > 0)$. Then f achieves a global maximum (minimum) at some pt $(x_0, t_0) \in M \times [0,a]$. Then at (x_0, t_0), we have

$$\sqrt{-1}\,\partial\bar{\partial} f(x_0, x_0) \leq 0 \quad (\geq 0)$$

and (1) if $t_0 = 0$, then $\frac{\partial}{\partial t} f(x_0, t_0) \leq 0$ (≥ 0)

(2) if $t_0 \in (0,a)$, then $\frac{\partial}{\partial t} f(x_0,t_0)=0$ $(=0)$ $\Rightarrow t_0>0 \Rightarrow \frac{\partial f}{\partial t}(x_0,t_0)\geq 0$

(3) if $t_0=a$, then $\frac{\partial}{\partial t} f(x_0,t_0)\geq 0$ (≤ 0) $(t_0>0 \Rightarrow \frac{\partial f}{\partial t}(x_0,t_0)\leq 0)$

Thm 3.5 (Short-time existence theorem, Hamilton)

Let (M^n,ω_0) be a closed Kähler mfd. Then $\exists\, \varepsilon>0$ and a unique smooth soln. $\omega(t)$ of (KRF) defined on $[0,\varepsilon)$

Denote $T_{max} := \sup\{\varepsilon>0 \mid \varepsilon \text{ in Thm 3.5}\}$

The main goal of this section is to prove

$$T_{max}=T, \qquad T:=\sup\{t>0 \mid [\omega_0]-t\cdot 2\pi C_1(M)>0\}$$

Eeasy to see $T_{max}\leq T$.

It suffices to prove $T\leq T_{max}$. We will prove for any $T'<T$, if $T'>T_{max}$, we will obtain a contradition. $(\Rightarrow T'\leq T_{max})$

Thm 3.6 (Key Thm)

$\forall k\geq 0$, $\exists\, C_k = C(k,\omega_0,T_{max},T')$ s.t.

$$\|\varphi(t)\|_{C^k(M,g_0)} \leq C_k$$

and

$$\omega(t)\geq C_0^{-1}\omega_0$$

Given Thm 3.6, we will prove Thm 3.1.

$\forall k\geq 0$, $\|\varphi\|_{C^k}\leq C_k \xRightarrow{\text{Arzela-Ascoli}}$ $\exists\, t_i \to T_{max}$ s.t.
① $\varphi(t_i)\to \varphi_{T_{max}}$ in C^∞ as $i\to+\infty$
② $\varphi_{T_{max}}$ is smooth
$\omega(t)\geq C_0^{-1}\omega_0 \Longrightarrow$ ③ $\omega_{T_{max}} := \omega_0+\sqrt{-1}\partial\bar\partial\varphi_{T_{max}} \geq C_0^{-1}\omega_0>0$

And we have

$$\boxed{\text{Lem 3.7: } |\dot{\varphi}| \le C.}$$

By Lem 3.7, we have $\varphi(t) \longrightarrow \varphi_{T_{max}}$ in C^∞ as $t \to T_{max}$.

$\exists$ soln. $\omega(t)$ s.t. $\begin{cases} \dfrac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) \\ \omega(T_{max}) = \omega_{T_{max}} \end{cases}$ on $t \in [T_{max}, T_{max}+\varepsilon)$

$\Rightarrow$ $\omega(t)$ $(t \in [0, T_{max}+\varepsilon))$ is a soln. of

KRF: $\begin{cases} \dfrac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$

Contradict with the definition of T_{max}.

$\Rightarrow T_{max} \geq T'$

$\Rightarrow T_{max} \geq T.$

#

Next, we will prove Thm 3.6 and Lem 3.7

- Estimate on φ and $\dot{\varphi}$

$\dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}$

$\dfrac{\partial \varphi_{max}}{\partial t} \le \log \dfrac{\hat{\omega}_t^n}{\Omega}$

$\Rightarrow \dfrac{\partial \varphi_{max}}{\partial t} \le \log \dfrac{\hat{\omega}_t^n}{\Omega} = \sup\limits_{M \times [0, T_{max}]} \log \dfrac{\hat{\omega}_t^n}{\Omega} =: A$

$\Rightarrow \dfrac{\partial}{\partial t}(\varphi_{max} - At) \le 0$

$\Rightarrow \varphi_{max}(t) - \varphi(0) \le At \le A T_{max}$

$\Rightarrow \varphi(x,t) \le C.$

Similarly, $\frac{\partial \varphi_{min}}{\partial t} \geq \log \frac{\hat{\omega}_t^n}{\Omega} \geq \inf_{M\times[0,T_{max}]} \frac{\hat{\omega}_t^n}{\Omega} =: B$

$$\Rightarrow \varphi(x,t) \geq Bt \geq -|B| T_{max} \geq -C.$$

We get

Lem 3.8: $\exists\, C = C(\omega_0, T_{max}) > 0$ s.t. on $M\times[0, T_{max})$

$$|\varphi| \leq C.$$

Next, we prove Lem 3.7.

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \\ \varphi(0) = 0 \end{cases} \qquad \begin{aligned} \hat{\omega}_t &= \omega_0 + t\chi \\ &= \frac{1}{T'}((T'-t)\,\omega_0 + t\eta) \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial t}\dot{\varphi} = tr_{\omega(t)}\left(\frac{\partial \hat{\omega}_t}{\partial t} + \sqrt{-1}\partial\bar{\partial}\dot{\varphi}\right)$$

$$= \Delta\dot{\varphi} + tr_{\omega(t)}\chi$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right)\dot{\varphi} = tr_{\omega(t)}\chi = tr_{\omega(t)}\left(\frac{\omega(t) - \sqrt{-1}\partial\bar{\partial}\varphi - \omega_0}{t}\right)$$

$$= \frac{1}{t}(n - \Delta\varphi - tr_{\omega(t)}\omega_0)$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right)(t\dot{\varphi}) = \dot{\varphi} + t\left(\frac{\partial}{\partial t} - \Delta\right)\dot{\varphi}$$

$$= \left(\frac{\partial}{\partial t} - \Delta\right)\varphi + n - tr_{\omega(t)}\omega_0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right)(t\dot{\varphi} - \varphi - nt) = -tr_{\omega(t)}\omega_0 < 0$$

And $\left(\frac{\partial}{\partial t} - \Delta\right)\left((T'-t)\,\dot{\varphi}(t) + \varphi(t) + nt\right) = tr_{\omega(t)}\hat{\omega}_{T'} > 0$

$$\Rightarrow \left[(T'-t)\dot{\varphi}(t)+\varphi(t)+nt\right]_{min} \geq T'[\dot{\varphi}(0)]_{min}$$

$$\geq T' \inf_{M} \log \frac{\omega_0^n}{\Omega} \geq -C$$

Since $T'-t \geq T'-T_{max} > 0$

$$\Rightarrow \quad (\dot{\varphi})_{min}(t) \geq -C$$

$$\Rightarrow \quad \dot{\varphi} \geq -C$$

- $$\frac{\partial \dot{\varphi}}{\partial t} = \frac{\partial}{\partial t} \log \frac{\omega^n(t)}{\Omega} = \frac{\frac{\partial}{\partial t}\omega^n(t)}{\omega^n(t)}$$

$$= \frac{n \frac{\partial \omega}{\partial t} \wedge \omega^{n-1}(t)}{\omega^n(t)}$$

$$= \frac{-n\, Ric(\omega(t)) \wedge \omega^{n-1}(t)}{\omega^n(t)} = -R(t).$$

Along RF, $R_{min}(t) \geq R_{min}(0) \geq -C$

$$\Rightarrow \quad \frac{\partial}{\partial t}\dot{\varphi} \leq C$$

$$\Rightarrow \quad \dot{\varphi} \leq C$$

$$\Rightarrow \quad |\dot{\varphi}| \leq C \qquad C = C(\omega_0, T_{max}, T')$$

#