$$\Delta_d = (d+d^*)^2, \quad \Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

$\Rightarrow \frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar\partial}$ for Kähler metric.

$$\Delta_{\bar\partial} := \frac{1}{2} g^{i\bar j}(\nabla_i \nabla_{\bar j} + \nabla_{\bar j}\nabla_i)$$

$$\Delta f := g^{i\bar j}\partial_i \partial_{\bar j} f$$

$$\Rightarrow \begin{cases} n\omega^{n-1}\wedge\beta = g^{i\bar j}\beta_{i\bar j}\,\omega^n = (\mathrm{tr}_\omega \beta)\cdot\omega^n \\ n\omega^{n-1}\wedge(\sqrt{-1}\partial f\wedge\bar\partial f) = |\partial f|_g^2\,\omega^n \end{cases}$$

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Defn (Ricci curvature)

$$R_{i\bar j} := g^{k\bar l} R_{i\bar j k\bar l} = g^{k\bar l} R_{k\bar l i\bar j}$$

Prop: $R_{i\bar j} = -\partial_i\partial_{\bar j}\log\det g$

Pf: For an invertible matrix A

$$\delta \det A = \mathrm{trace}(A^{-1}\delta A)\det A \qquad \left(\delta A = \frac{dA}{dt}\Big|_{t=0}\right)$$

$$\Rightarrow \delta \log A = \mathrm{trace}(A^{-1}\delta A)$$

By defn, $R_{i\bar j} = -g^{k\bar l} g_{p\bar l}\,\partial_{\bar j}\Gamma^p_{ik} = -\partial_{\bar j}\Gamma^p_{ip} = -\partial_{\bar j}(g^{p\bar q}\partial_i g_{p\bar q})$

$$= -\partial_{\bar j}(\partial_i \log\det g)$$

$$= -\partial_i\partial_{\bar j}\log\det g.$$

#

Def$^{\underline{n}}$ (Ricci form)

Ricci form of g is the $(1,1)$ form

$$\bar\partial f = \frac{\partial f}{\partial \bar z^i} d\bar z^i$$

$$\underline{\partial \bar\partial f} = \underline{\frac{\partial^2 f}{\partial z_j \partial \bar z^i} dz_j \wedge d\bar z_i}$$

$$\begin{aligned} Ric(\omega) &:= \sqrt{-1}\, R_{i\bar j}\, dz^i \wedge d\bar z^j \\ &= -\sqrt{-1}\, \partial_i \partial_{\bar j} \log \det g \; dz^i \wedge d\bar z^j \\ &= -\sqrt{-1}\, \partial \bar\partial \log \det g \longleftrightarrow (1,1) \text{ form.} \end{aligned}$$

$\because\ \partial_k R_{i\bar j} = \partial_i R_{k\bar j} \implies d\, Ric(\omega) = 0$ i.e, $Ric(\omega)$ is d-closed.

We often write

$$Ric(\omega) = -\sqrt{-1}\, \partial\bar\partial \log \omega^n$$

$$\left(\omega^n = n!\, \det g \cdot dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n,\ (n,n) \text{ form}\right)$$

If Ω is any smooth volume form, locally written as

$$\Omega = a(z)\, \underline{(\sqrt{-1})^n dz^1 \wedge d\bar z^n \wedge \cdots \wedge dz^n \wedge d\bar z^n} \quad \left(dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n\right)$$

then we define

$$\sqrt{-1}\, \partial\bar\partial \log \Omega := \sqrt{-1}\, \partial\bar\partial \log a.$$

Exercise: $\sqrt{-1}\,\partial\bar\partial \log \Omega$ is well defined, independent of choice of local coord.

Example: $(\mathbb{P}^n, \omega_{FS})$, then $Ric(\omega_{FS}) = (n+1)\, \omega_{FS}$

Pf: In local coord, $\omega_{FS} = \sqrt{-1}\, \partial\bar\partial \log(1+|z|^2)$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log(1+|z|^2) = \frac{1}{1+|z|^2}\left(\delta_{ij} - \frac{\bar{z}_i z_j}{1+|z|^2}\right)$$

Calculate at pt $(z, 0, \cdots, 0)$, i.e, $z^1 = z$, $z^i = 0$ $(i \geq 2)$

$$g_{1\bar{1}} = \frac{1}{(1+|z|^2)^2}, \quad g_{i\bar{i}} = \frac{1}{1+|z|^2} \quad (i \geq 2)$$

$$\Rightarrow \det g = \frac{1}{(1+|z|^2)^{n+1}} \Rightarrow \log \det(g) = -(n+1)\log(1+|z|^2)$$

$$\therefore \mathrm{Ric}(\omega_{FS}) = -\sqrt{-1}\,\partial\bar{\partial} \log \det g = (n+1)\,\underbrace{\sqrt{-1}\,\partial\bar{\partial}\log(1+|z|^2)}$$

$$= (n+1)\,\omega_{FS}.$$

In fact,

$$R_{i\bar{j}k\bar{l}} = g_{p\bar{l}} R_{i\bar{j}k}{}^{p} = -g_{p\bar{l}}\,\partial_{\bar{j}} \Gamma^{p}_{ik} = -g_{p\bar{l}}\,\partial_{\bar{j}}\left(g^{p\bar{q}}\,\partial_i g_{k\bar{q}}\right)$$

$$= -g_{p\bar{l}}\,\partial_{\bar{j}} g^{p\bar{q}} \cdot \partial_i g_{k\bar{q}} - g_{p\bar{l}}\, g^{p\bar{q}}\,\partial_{\bar{j}}\partial_i g_{k\bar{q}}$$

$$= -\partial_i \partial_{\bar{j}} g_{k\bar{l}} + g^{p\bar{q}}\,\partial_i g_{k\bar{q}}\,\partial_{\bar{j}} g_{p\bar{l}}$$

$$\left(\begin{aligned} &\because \delta_l^q = g_{p\bar{l}}\, g^{p\bar{q}} \\ &\therefore 0 = \partial_{\bar{j}}\left(g_{p\bar{l}}\, g^{p\bar{q}}\right) \\ &= \partial_{\bar{j}} g_{p\bar{l}} \cdot g^{p\bar{q}} \\ &\quad + g_{p\bar{l}} \cdot \partial_{\bar{j}} g^{p\bar{q}} \end{aligned}\right)$$

In general,

$$R_{i\bar{j}k\bar{l}}(0) = -\left.\frac{\partial^4 \log(1+|z|^2)}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l}\right|_{z=0}$$

$$\frac{\partial^2}{\partial z^k \partial \bar{z}^l} \log(1+|z|^2) = \frac{1}{(1+|z|^2)^2}\left((1+|z|^2)\delta_{kl} - \bar{z}_k z_l\right)$$

$$\Rightarrow \partial_i \partial_{\bar{j}} \partial_k \partial_{\bar{l}} \log(1+|z|^2)\Big|_{z=0} = \partial_i \partial_{\bar{j}}\left(\frac{1}{(1+|z|^2)^2}\left((1+|z|^2)\delta_{kl} - \bar{z}_k z_l\right)\right)\Big|_{z=0}$$

$$= \partial_i \partial_{\bar{j}} \left(\frac{1}{(1+|z|^2)^2} \right) \left((1+|z|^2)\delta_{k\ell} - \bar{z}_k z_\ell \right) + \frac{1}{(1+|z|^2)^2} \partial_i \partial_{\bar{j}} \begin{pmatrix} (1+|z|^2)\delta_{k\ell} \\ - \bar{z}_k z_\ell \end{pmatrix} \Bigg|_{z=0}$$

$$= -2\, \partial_i \partial_{\bar{j}} |z|^2 \cdot \delta_{k\ell} + (\delta_{ij}\delta_{k\ell} - \delta_{i\ell}\delta_{jk})$$

$$= -2\, \delta_{ij} \cdot \delta_{k\ell} + (\delta_{ij}\delta_{k\ell} - \delta_{i\ell}\delta_{jk})$$

$$= -(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk})$$

$$\Rightarrow R_{i\bar{j}k\bar{\ell}}\Big|_{z=0} = \delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}$$

$\Rightarrow$ Bisectional curvature of $g = 1$

i.e, $\mathrm{Bisec}(g) = 1$.

§2 The Kähler–Ricci flow and the Kähler cone.

§2.1 Brief induction to the Ricci flow

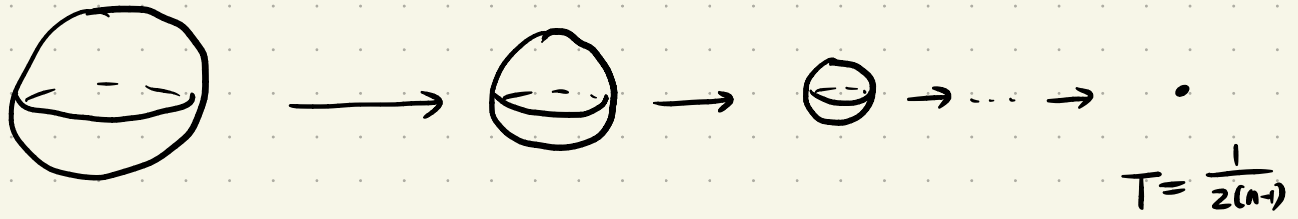
Assume (M, g_0) is a Riem. mfd. If $\exists$ a smooth family of $g(t)$ satisfying

$$(RF) \quad \begin{cases} \dfrac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(g(t)) \\ g(0) = g_0 \end{cases}$$

1982', Hamilton

Example: S^n, g_0: standard metric, i.e $\sec(g_0) = 1$.

$g(t) = r^2(t)\, g_0$, $r^2(t) = 1 - 2(n-1)t$, singular time $T = \frac{1}{2(n-1)}$



Normalized Ricci flow

$$(NRF) \begin{cases} \frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + \frac{2r}{n} g(t), \\ g(0) = g_0 \end{cases}$$

here $r = \dfrac{\int_M R(g(t))\, dV_{g(t)}}{Vol_{g(t)}(M)}$

Thm (Hamilton, 82', short-time existence and uniqueness)

If (M^n, g) is a closed smooth complete, Riem mfd, then $\exists!$ smooth soln. $g(t)$ to (RF) defined on some interval $[0, \delta)$, $\delta > 0$ w/ $g(0) = g$.

- Hamilton's original pf relied on Nash-Moser inverse fcn. thm.
- DeTurck provided a simplified pf by showing that Ricci flow is equivalent to a strictly parabolic system.

 Fixed a background metric $\tilde{g}$, and the Levi-Civita

Connection $\tilde{\Gamma}$ w/ $\tilde{g}$. Ricci-DeTurck flow

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i \quad \text{----} (2.1) \\ g(0) = g_0 \end{cases}$$

$W = W(g)$ is defined by $W_j := g_{jk} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq})$

Assume $\frac{\partial}{\partial s}\big|_{s=0} g_{ij} = v_{ij}$, $V := g^{ij} v_{ij}$, then

$$\frac{\partial}{\partial s} R_{ij} = -\frac{1}{2}\left(\Delta_L v_{ij} + \nabla_i \nabla_j V - \nabla_i (\operatorname{div} v)_j - \nabla_j (\operatorname{div} v)_i\right)$$

here $\Delta_L v_{ij} := \Delta v_{ij} + 2R_{kijl} v_{kl} - R_{ik} v_{jk} - R_{jk} v_{ik}$.

If we define $X := \frac{1}{2} \nabla V - \operatorname{div}(v)$. Then

$$\frac{\partial}{\partial s}(-2R_{ij}) = \Delta_L v_{ij} + \nabla_i X_j + \nabla_j X_i$$

On the other hand,

$$\frac{\partial}{\partial s}\Big|_{s=0} W(g(s)) = -X_j + 0^{th} \text{ order terms in } v.$$

$$\Rightarrow \frac{\partial}{\partial t} v_{ij} = \Delta_L v_{ij} + \text{first order terms in } v. \quad \text{---} (2.2)$$

Then the Ricci-DeTurck flow is strictly parabolic.

$\exists!\ g(t)$ of (2.1) w/ $g(0) = g_0$.

ODE at each pt in M

$$\frac{\partial}{\partial t}\varphi_t = -W^* \quad \text{---} \quad (2.3)$$

w/ $\varphi_0 = id$.

Here $W^*(t)$ is the vector field dual to $W(t)$ w.r.t. $g(t)$.

Define $\bar{g}(t) := \varphi_t^* g(t)$, then $\bar{g}(0) = g(0)$, and

$$\frac{\partial}{\partial t}\bar{g}(t) = -2Ric(\bar{g}(t))$$

$\Rightarrow$ Short time existence of RF.

- Uniqueness of RF

(2.3) $\Leftrightarrow$ harmonic map heat flow

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{\underline{\bar{g}(t)}, \tilde{g}}\varphi_t \quad \text{----} \quad (2.4)$$

Here $\bar{g}(t)$ is a soln. of RF, and $\varphi_t : M \xrightarrow{\text{diff.}} M$ is a soln. of (2.4). Then

$$g(t) := (\varphi_t^{-1})^* \bar{g}(t)$$

$\Rightarrow$ $g(t)$ is a soln. of the Ricci-DeTurck flow.

Assume $g_1(t), g_2(t)$ are soln. of RF w/ $g_1(0)=g_2(0)=g_0$.

then by the existence thm of the harmonic map heat flow, $\exists\ \varphi_1(t), \varphi_2(t)$ of (2.4) w/ $g_1(t), g_2(t)$

s.t. $\hat{g}_i(t) := \left(\varphi_i^{-1}(t)\right)^* g_i(t)$

are soln. of Ricci-DeTurck flow w/ $\hat{g}_1(0)=\hat{g}_2(0)=g_0$

By the uniqueness thm of the Ricci-DeTurck flow

$\Rightarrow \hat{g}_1(t)=\hat{g}_2(t)=:\hat{g}(t)$

$\Rightarrow \varphi_1(t), \varphi_2(t)$ are solns. of ODE (2.3)

$$\frac{d}{dt}\varphi_i(t) = -W(t)\circ\varphi_i(t) \quad \text{w/ } \varphi_i(0)=id$$

where $W(t) := \hat{g}^{pq}(t)\left(\Gamma(\hat{g}(t))_{pq}^{k} - \Gamma(g_0)_{pq}^{k}\right)$.

By Uniqueness thm of ODE $\Rightarrow \varphi_1(t)=\varphi_2(t)$.

$\Rightarrow g_1(t)=g_2(t)$.

This completes the proof of the uniqueness thm of RF. #

Note: For $f \in C^\infty((M^n,g);(N^m,h))$, we define the map Laplacian of f as

$$(\Delta_{g,h} f)^\gamma := g^{ij}\left(\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} + (\Gamma(h))^\gamma_{\alpha\beta} \circ f \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\right) \quad \text{---- } (2.5)$$

- Some classical theorems for RF.

Thm 1 (Hamilton, 1982, JDG)

Let (M^3, g_0) be a closed mf w/ $Ric(g_0) > 0$. Then $\exists!$ $g(t)$ of NRF w/ $g(0) = g_0$ for all $t > 0$. Furthermore, as $t \to +\infty$,

$$g(t) \xrightarrow[\text{exponentially}]{C^\infty} g_\infty, \quad \text{where } sec(g_\infty) \equiv const. > 0.$$

Thm 2 (Hamilton, 1986, JDG)

Thm 1 holds for closed mfd (M^4, g_0) w/ $Rm(g_0) > 0$.

$\Rightarrow M^4 \overset{\text{diff}}{\simeq} S^4$ or $\mathbb{RP}^4$.

Thm 3 (Böhm – Wilking, 2008, Ann. Math.)

Same results hold for closed mfd (M^n, g_0) $(n \geq 4)$ and $Rm(g_0)$ is two-positive.

Thm 4 (Brendle – Schoen, JAMS 2009) $\frac{1}{4}$-pinching differential sphere theorem.

Let (M^n, g) be a compact Riem. mfd which is strictly $1/4$-pinched in the pointsense. Then $M \overset{\text{diff}}{\cong}$ a spherical space form.

Thm 5 (Perelman, 2002 – 2003)

Poincaré conjecture is true. That is, if M^3 is closed and $\pi_1(M) = \{0\}$, then $M^3 \overset{\text{diff}}{\cong} S^3$.

Thurston conjecture.

- Bamler

§2.2 The Kähler-Ricci flow and simple examples

Let (M, ω_0) be a cpt Kähler mfd. If $\omega = \omega(t)$ is a smooth family of Kähler metrics on M satisfying the equation

$$(\text{KRF}) \qquad \frac{\partial}{\partial t}\omega = -Ric(\omega), \quad \omega|_{t=0} = \omega_0 \quad \text{- - - -} (2.6)$$

then we say that $\omega(t)$ is a soln. of the Kähler-Ricci flow starting at ω_0.

Cao, Huai-Dong, 1985, provide a new proof of Calabi Conj.

Defn (Kähler-Einstein metric)

A Kähler metric ω_{KE} is called a Kähler-Einstein metric (short for KE) if $\exists\ \lambda \in \mathbb{R}$ s.t.

$$Ric(\omega_{KE}) = \lambda\, \omega_{KE}$$

After rescaling, we usually assume $\lambda = 1, 0,$ or -1.

[Fact: $\forall \lambda \in \mathbb{R}$, $Ric(\lambda g) = Ric(g)$]

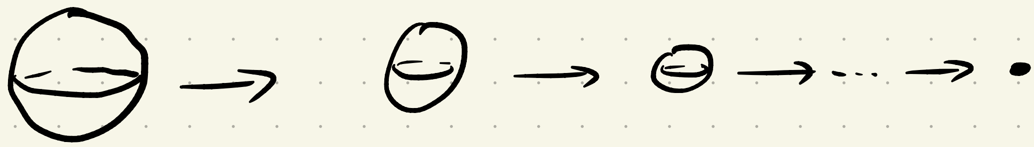
If $\omega_0 = \omega_{KE}$, set $\omega(t) = (1-\lambda t)\omega_{KE} = (1-\lambda t)\omega_0$

$$\Rightarrow \frac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) \quad w/ \ \omega(0) = \omega_0.$$

$\Rightarrow$ Hence $\omega(t)$ is a soln. of KRF.

(1) If $\lambda = 1$, $\omega(t) = (1-t)\omega_0 \Rightarrow t \in [0,1)$, $T = 1$

- $n=1$, $M = \mathbb{P}^1 = S^2$



(2) If $\lambda = 0$, $\omega(t) = \omega_0$, $t \to +\infty$ $(T = +\infty)$

(2) If $\lambda < 0$, $\omega(t) = (1+t)\omega_0$, $t \to +\infty$ $(T = +\infty)$

$$\lim_{t \to +\infty} \frac{\omega(t)}{t} = \omega_0$$

- The Kähler cone and the first Chern class.

We define

$$H^{1,1}_{\bar\partial}(M, \mathbb{R}) = \frac{\{\bar\partial\text{-closed real } (1,1) \text{ form}\}}{\operatorname{Im} \bar\partial}$$

By Hodge theory, $\dim H^{1,1}_{\bar\partial}(M, \mathbb{R}) < +\infty$.

$\partial\bar\partial$-Lemma: If M is a cpt Kähler mfd, an exact real $(1,1)$ form α on M, that is $\exists \eta$, s.t. $\alpha = \bar\partial \eta$. Then $\exists \varphi \in C^\infty(M)$, s.t.

$$\alpha = \sqrt{-1}\partial\bar\partial\varphi.$$

By $\partial\bar\partial$-Lemma,

$$H^{1,1}_{\bar\partial}(M, \mathbb{R}) = \frac{\{\bar\partial\text{-closed real } (1,1) \text{ forms}\}}{\operatorname{Im} \partial\bar\partial} \quad \cdots\cdots (2.7)$$

If $[\omega]=[\omega'] \in H^{1,1}_{\bar\partial}(M,\mathbb{R})$, then $\exists\, \varphi \in C^\infty(M)$, s.t.

$$\omega' = \omega + \sqrt{-1}\,\partial\bar\partial\varphi, \quad \varphi \text{ is unique up to a const.}$$

Defn (Kähler class)

$[\alpha] \in H^{1,1}_{\bar\partial}(M,\mathbb{R})$ is called a Kähler class, if $\exists$ a Kähler metric ω s.t.

$$[\omega] = [\alpha]$$

write it as $[\alpha] > 0$.

- If $-[\alpha] > 0$, then we write it as $[\alpha] < 0$.

- Kähler cone of M.

$$Ka(M) := \left\{ [\alpha] \in H^{1,1}_{\bar\partial}(M,\mathbb{R}) \;\middle|\; [\alpha] > 0 \right\}$$

Lemma 2.1: $Ka(M)$ is an open, convex cone.

Pf: (1) Cone. If $[\alpha] \in Ka(M)$, then $\forall \lambda > 0$, $\lambda[\alpha] \in Ka(M)$.

(2) Convex. If ω_1, ω_2 is Kähler, then $\forall \lambda \in (0,1)$, $\lambda\omega_1 + (1-\lambda)\omega_2$ is Kähler.

(3) Open. Assume $\{[\alpha_1], \dots, [\alpha_N]\}$ is a basis of $H^{1,1}(M,\mathbb{R})$

Assume $[\alpha] \in Ka(M)$, then $\exists\, \lambda_1, \dots, \lambda_N \in \mathbb{R}$, s.t.

$$[\alpha] = \sum_{i=1}^N \lambda_i [\alpha_i] = \left[\sum_{i=1}^N \lambda_i \alpha_i \right]$$

$\therefore \exists\, \varphi \in C^\infty(M)$ s.t.
$$\sum_{i=1}^N \lambda_i \alpha_i + \sqrt{-1}\,\partial\bar\partial\varphi > 0$$

Since M is cpt, $\forall \tilde{\lambda}_i$ sufficiently close to λ_i, we have

$$\sum_{i=1}^{N} \tilde{\lambda}_i \alpha_i + \sqrt{-1}\,\partial\bar{\partial}\varphi > 0$$

$$\Longrightarrow \quad \sum_{i=1}^{N} \tilde{\lambda}_i [\alpha_i] \in K_a(M)$$

i.e., $\exists$ neighborhood U of $[\alpha]$ s.t. $U \subset K_a(M)$

$\Longrightarrow$ $K_a(M)$ is open

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