天元西南中心·短课 2022.12.20 — 2023.2.16

Introduction to the Kähler-Ricci flow

§1 Quick introduction to the Kähler geometry

§2 Definition of KRF and some examples

§3 The maximal existence time theorem

§4 Covergence theorem with $C_1(M)<0$ and $C_1(M)=0$.

§5 KRF on Fano mfds, Perelman's μ-entropy and Perelman's estimate

§6 A convergence theorem for KRF on Fano manifolds admits Kähler-Ricci soliton

§7 Brief introduction to Hamilton-Tian conjecture and Song-Tian's analytic minimal model program with KRF

Reference:

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(3) Tosatti, Valentino, KAWA lecture notes on the Kähler-Ricci flow

(4) Weinkove, Ben, The Kähler-Ricci flow on compact Kähler manifolds

(5) Chow, B., Lu, P. & Ni, L., Hamilton's Ricci flow.

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§1 Quick introduction to Kähler geometry

§1.1 Complex manifold

Def^n (Complex manifold)

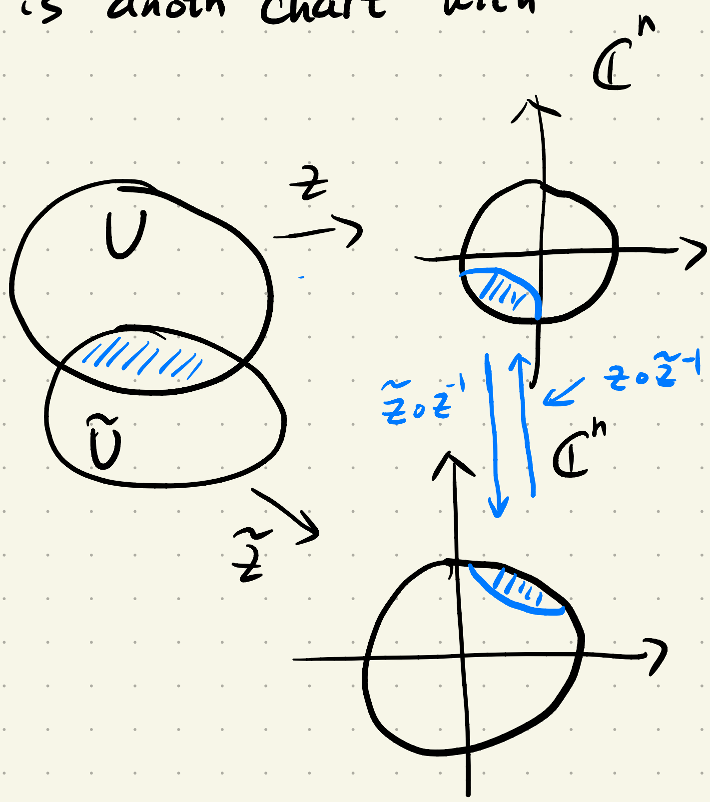
Let M be a smooth mfd with $\dim_{\mathbb{R}} M = 2n$. We say that M is a complex mfd w/ $\dim_{\mathbb{C}} M = n$ if M can be covered by chart (U, z) where $U \overset{\text{open}}{\subset} M$ and $z : U \to \mathbb{C}^n$ is a homemorphism onto an open subset $z(U) \subset \mathbb{C}^n$, with the following property: if $(\tilde{U}, \tilde{z})$ is anoth chart with $U \cap \tilde{U} \neq \phi$, then the transition maps

$$\tilde{z} \circ z^{-1} : z(U \cap \tilde{U}) \to \tilde{z}(U \cap \tilde{U})$$

and

$$z \circ \tilde{z}^{-1} : \tilde{z}(U \cap \tilde{U}) \to z(U \cap \tilde{U})$$

are holomorphic.



We write $z = (z^1, \cdots, z^n)$ and $(\tilde{z}^1, \cdots, \tilde{z}^n)$ are called complex coordinates.

Real coord. $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ by

$$z^i = x^i + \sqrt{-1}\, y^i$$

We define

$$\frac{\partial}{\partial z^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad \frac{\partial}{\partial \bar{z}^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right)$$

$$dz^i := dx^i + \sqrt{-1}\, dy^i, \quad d\bar{z}^i := dx^i - \sqrt{-1}\, dy^i$$

$\Rightarrow \quad dz^i\left(\frac{\partial}{\partial z^j}\right) = \delta^i_j \,, \quad dz^i\left(\frac{\partial}{\partial \bar{z}^j}\right) = 0, \quad d\bar{z}^i\left(\frac{\partial}{\partial \bar{z}^j}\right) = \delta^i_j$

Defn (Holomorphic fcn)

$f \in C^\infty(\mathbb{C}^n)$ is called a holomorphic fcn if $\frac{\partial f}{\partial \bar{z}^i} = 0, \forall i \in [1, \cdots, n]$

(This definition is well defined)

Example: (1) $\mathbb{C}^n \quad (\cong \mathbb{R}^{2n})$

(2) $T^{2n} = \mathbb{C}^n / \mathbb{Z}^{2n} = \underbrace{T \times \cdots \times T}_{n}, \quad T = S^1 \times S^1$

(3) $\mathbb{C}P^n := \left(\mathbb{C}^{n+1} - \{0\}\right)/\sim$, here $z \sim \tilde{z}$ iff $\exists \lambda \in \mathbb{C}^*$ s.t.

$\tilde{z} = \lambda z$, i.e., $\tilde{z}_i = \lambda z_i$

RK: $\mathbb{C}P^1 \overset{\text{diff}}{\cong} S^2$.

Basics of T_pM: $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\right\}$.

A linear map (almost complex structure)

$J : T_pM \longrightarrow T_pM \quad$ with $J^2 = -1$

$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i} \quad \Rightarrow J^2 = -1$

$\Rightarrow J\left(\frac{\partial}{\partial z^i}\right) = \sqrt{-1}\,\frac{\partial}{\partial z^i}, \quad J\left(\frac{\partial}{\partial \bar{z}^i}\right) = -\sqrt{-1}\,\frac{\partial}{\partial \bar{z}^i} \quad \begin{pmatrix} \partial_i := \frac{\partial}{\partial z^i} \\ \partial_{\bar{i}} := \frac{\partial}{\partial \bar{z}^i} \end{pmatrix}$

$\Rightarrow (T_pM)^{\mathbb{C}} = T_p^{1,0}M \oplus T_p^{0,1}M$.

$T_p^{1,0}M = \mathrm{Span}\left\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\right\}, \quad T_p^{0,1}M = \mathrm{Span}\left\{\frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}\right\}$

Notation: $C^\infty(M) := C^\infty(M; \mathbb{R})$.

Defn (holomorphic vector field)

A smooth complex-valued field X on M is called $T^{1,0}$ vector field on M if $X_p \in T_p^{1,0}M$, $\forall p \in M$. Locally, X can be written as

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial z^i}, \quad \underline{X^i \in C^\infty(M;\mathbb{C})} \text{ satisfying}$$

transformation rule: If $X = \sum_{i=1}^n \tilde{X}^i \frac{\partial}{\partial \tilde{z}^i}$ on $\tilde{U}$, then

$$X^i = \sum_{j=1}^n \tilde{X}^j \frac{\partial z^i}{\partial \tilde{z}^j} \quad \text{on } U \cap \tilde{U}.$$

A $T^{1,0}$ vector field $X = X^i \frac{\partial}{\partial z^i}$ is called a holomorphic vector field on M if

$$\frac{\partial X^i}{\partial \bar{z}^j} = 0, \quad \forall i,j \in \{1,\cdots,n\}$$

Similarly, $T^{0,1}$ vector field $Y = Y^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}$ w/

$$Y^{\bar{j}} = \tilde{Y}^{\ell} \overline{\frac{\partial z^j}{\partial \tilde{z}^\ell}} = \tilde{Y}^{\ell} \frac{\partial \bar{z}^j}{\partial \bar{\tilde{z}}^\ell} \quad \text{on } U \cup \tilde{U}$$

Cotangent space $(T_p^* M)^{\mathbb{C}} := \operatorname{span}\{dz^1, \cdots, dz^n, d\bar{z}^1, \cdots, d\bar{z}^n\}$

dz^i: (1,0)-forms; $\quad d\bar{z}^i$: (0,1)-forms.

A (1,0) form a on M is written locally as $a = a_i dz^i$

(0,1) form b on M - - - - - - - - as $b = b_{\bar{j}} d\bar{z}^j$.

where a_i and $b_{\bar{j}}$ transform by

$$a_i = \tilde{a}_k \frac{\partial \tilde{z}^k}{\partial z^i}, \qquad b_{\bar{j}} = \tilde{b}_{\bar{l}} \overline{\frac{\partial \tilde{z}^l}{\partial z^j}} \quad \text{on } U \cap \tilde{U}$$

Defn (Hermitian metric)

We define a Hermitian metric g on M to be a Hermitian inner product on $T_p^{1,0}M$ for each p, which varies smoothly in p.

Locally, g is given by $(g_{i\bar{j}})$: positive definite Hermitian matrix which transforms according to

$$g_{i\bar{j}} = \tilde{g}_{k\bar{l}} \frac{\partial \tilde{z}^k}{\partial z^i} \cdot \overline{\frac{\partial \tilde{z}^l}{\partial z^j}} \quad \text{on } U \cap \tilde{U}.$$

(Rk: $\overline{g_{i\bar{j}}} = g_{\bar{i}j} = g_{j\bar{i}}$)

$\Rightarrow g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j$ defines a tensor $g: T^{1,0}M \times T^{0,1}M \to \mathbb{C}$.

$$g(X, Y) = g_{i\bar{j}} X^i \overline{Y^j} \quad \text{for } X = X^i \frac{\partial}{\partial z^i}, \; Y = \overline{Y^j} \frac{\partial}{\partial \bar{z}^j}$$

or $\forall X, Y \in T_p^{1,0}M$, $X = X^i \partial_i$, $Y = Y^j \partial_j$

$$\langle X, Y \rangle_g := g(X, \overline{Y}) = g_{i\bar{j}} X^i \overline{Y^j}$$

$$|X|_g := \sqrt{\langle X, X \rangle_g}$$

Rk: (1) Riem. metric reduces a hermitian metric $(g_{i\bar{j}})$

J: complex structure, g: Riem. metric, satisfying

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in T_pM, \; \forall p \in M.$$

$\Longrightarrow \quad g(\partial_i, \partial_j) = 0$

Then define $\quad g_{i\bar{j}} := g(\partial_i, \partial_{\bar{j}})$ is a Hermitian metric.

(2) A Hermitian metric g defines a Riem. metric g_R by

$$g_R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) := 2\mathrm{Re}(g_{i\bar{j}}) =: g_R\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$$

$$g_R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) := 2\mathrm{Im}(g_{i\bar{j}})$$

(3) Define $g^{i\bar{j}}$ to be (i,j)-th component of $(g_{k\bar{l}})^{-1}$, short for g^{-1}, i.e.

$$g^{i\bar{j}} g_{k\bar{j}} = \delta^i_k \quad \left(\text{or } g^{i\bar{j}} g_{i\bar{k}} = \delta^j_k\right)$$

(4) If $S = S^{ik}_{\bar{j}} \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^k} \otimes d\bar{z}^j \in \Gamma(M, T^{1,0}M \otimes T^{1,0}M \otimes (T^{0,1})^* M)$

Then $\quad |S|^2_g := g^{l\bar{j}} g_{i\bar{p}} g_{k\bar{q}} S^{ik}_{\bar{j}} \overline{S^{pq}_{\bar{l}}}$.

§1.2 Kähler metric

Defn (Kähler metric)

We say that a Hermitian metric $g = (g_{i\bar{j}})$ is Kähler if

(1.1) $\qquad \partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}} \quad$ for all $i, j, k \in \{1, 2, \cdots, n\}$

(1.1) $\Leftrightarrow d\omega = 0 \qquad$ (Prove this later)

Example: (1) $\mathbb{C}^n \quad g_{i\bar{j}} = \delta_{i\bar{j}} \quad \left(\Rightarrow g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 2\delta_{ij}\right)$

(2) $\mathbb{C}P^n$ Fubini-Study metric

$$\mathbb{P}^n = U_0 \cup U_1 \cup \cdots \cup U_n, \text{ where}$$

$$U_i = \{[z_0, \cdots, z_n] \mid z_i \neq 0\}$$ homogeneous coord.

On U_0, we define $z^i = \dfrac{z_i}{z_0}$, $1 \le i \le n$. and

$$g_{i\bar{j}} := \partial_i \partial_{\bar{j}} \log(1 + |z^1|^2 + \cdots + |z^n|^2)$$ 小写的 z.

Then $(g_{i\bar{j}})$ defines a Kähler metric on $\mathbb{P}^n$. This metric is called a Fubini-Study metric.

On U_1, we define $\tilde{z}^1 = \dfrac{z_0}{z_1} = \dfrac{1}{z^1}$, $\tilde{z}^k = \dfrac{z_k}{z_1} = \dfrac{z^k}{z^1}$ $(k \ge 2)$

$$\tilde{g}_{i\bar{j}} = \tilde{\partial}_i \tilde{\partial}_{\bar{j}} \log(1 + |\tilde{z}^1|^2 + \cdots + |\tilde{z}^n|^2)$$

Then on $U_0 \cap U_1$, we have

$$g_{i\bar{j}} = \tilde{g}_{k\bar{l}} \frac{\partial \tilde{z}^k}{\partial z^i} \cdot \overline{\frac{\partial \tilde{z}^l}{\partial z^j}}$$

$$\Rightarrow \quad g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j = \tilde{g}_{k\bar{l}}\, d\tilde{z}^k \otimes d\bar{\tilde{z}}^l.$$

Defn (Kähler form)

Given a Kähler metric g, we define its Kähler form to be

$$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j.$$

Since $\overline{g_{i\bar{j}}} = g_{\bar{i}j} = g_{j\bar{i}}$,

$$\bar{\omega} = -\sqrt{-1}\, \overline{g_{i\bar{j}}}\, d\bar{z}^i \wedge dz^j = -\sqrt{-1}\, g_{j\bar{i}}\, d\bar{z}^i \wedge dz^j = \sqrt{-1}\, g_{j\bar{i}}\, dz^j \wedge d\bar{z}^i = \omega.$$

$\Rightarrow$ ω is a real $(1,1)$-form.

Given a (p,q) form $a = a_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}$

We define

$\partial a := \partial_k(a_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}) dz^k\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q} \leftrightarrow (p+1,q)$ form

$\bar\partial a := \partial_{\bar{k}}(a_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}) d\bar{z}^k\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}$

$\leftrightarrow (p,q+1)$ form

$d := \partial + \bar\partial$

Prop 1: g is a Kähler metric $\Leftrightarrow d\omega=0 \Leftrightarrow \partial\omega=0 \Leftrightarrow \bar\partial\omega=0$

Pf: $\Leftrightarrow \partial\omega=0$.

$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i\wedge d\bar{z}^j$

$\Rightarrow \partial\omega = \sqrt{-1}\,\partial_k g_{i\bar{j}}\, dz^k\wedge dz^i\wedge d\bar{z}^j$

$= \sqrt{-1} \sum_{k<i} \qquad + \sum_{k>i}$ $(k\leftrightarrow i)$

$= \sqrt{-1} \sum_{k<i} \partial_k g_{i\bar{j}}\, dz^k\wedge dz^i\wedge d\bar{z}^j + \sum_{k<i} \partial_i g_{k\bar{j}}\, \underline{dz^i\wedge dz^k}\wedge d\bar{z}^j$

$= \sqrt{-1} \sum_{k<i} (\partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}})\, \underline{dz^k\wedge dz^i\wedge d\bar{z}^j}$

$\Rightarrow$ $\left(\partial\omega = 0 \Leftrightarrow \partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}\right.$, i.e., g is Kähler$\left.\right)$

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Prop 2: If (M,g) is a Kähler mfd, and $N\subset M$ a complex submfd.

Then $g|_N$ is a Kähler metric on N.

Pf: $\iota: N \to M$ inclusion map s.t. $\omega|_N = \iota^*\omega$

$$\Rightarrow d(\omega|_N) = d i^*\omega = i^* d\omega = 0$$

$\Rightarrow \omega|_N$ is Kähler.

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Cor: Every smooth projective variety admits a Kähler metric.

Indeed, a smooth projective variety can be defined to be a complex submfd of $\mathbb{P}^N$ for some N.

- Covariant differentiation

Define the Christoffel symbols of g on the chart (U, z) to be the fcns

$$\Gamma^i_{kp} : U \longrightarrow \mathbb{C}$$

defined by

$$\Gamma^i_{kp} := g^{i\bar{q}} \partial_k g_{p\bar{q}}$$

Kähler codition (1.1) $\Rightarrow \Gamma^i_{kp} = \Gamma^i_{pk}$.

Rk: The christoffel symbols do not define a tensor.

Defn (Covariant derivative)

Given a $T^{1,0}$ vector field X, we define the covariant derivative $\nabla_k X^i$ by

$$\nabla_k X^i := \partial_k X^i + \Gamma^i_{kp} X^p$$

$$\left(\nabla_k X = \nabla_k (X^i \partial_i) \right.$$
$$= \nabla_k X^i \cdot \partial_i + X^i \nabla_k \partial_i$$
$$= \nabla_k X^i \cdot \partial_i + X^i \Gamma^p_{ki} \partial_p$$
$$\left. = (\partial_k X^i + X^p \Gamma^i_{kp}) \partial_i \right)$$

Then $\nabla_k X^i$ define a tensor

$$\nabla X = (\nabla_k X^i)\, \partial_i \otimes dz^k$$

$$\nabla_{\bar{\ell}} X^i := \partial_{\bar{\ell}} X^i$$

Similarly, for a $T^{0,1}$ vector field $Y = Y^{\bar j}\partial_{\bar j}$,

a $(1,0)$ form $a = a_i dz^i$, $(1,0)$ form $b = b_{\bar j} d\bar z^{j}$, we define

$$\nabla_k Y^{\bar j} = \partial_k Y^{\bar j}, \quad \nabla_{\bar l} Y^{\bar j} = \partial_{\bar l} Y^{\bar j} + \overline{\Gamma^{j}_{lq}}\, Y^{\bar q}$$

$$\nabla_k a_i = \partial_k a - \Gamma^p_{ki} a_p, \quad \nabla_{\bar l} a_i = \partial_{\bar l} a_i$$

$$\nabla_k b_{\bar j} = \partial_k b_{\bar j}, \quad \nabla_{\bar l} b_{\bar j} = \partial_{\bar l} b_{\bar j} - \overline{\Gamma^{q}_{lj}}\, b_{\bar q}.$$

Exercise: Show all of above terms define tensors.

$$\nabla_k S^{ab}_{\bar c} = \partial_k S^{ab}_{\bar c} + \Gamma^a_{kp} S^{pb}_{\bar c} + \Gamma^b_{kp} S^{ap}_{\bar c}$$

$$\nabla_{\bar l} S^{ab}_{\bar c} = \partial_{\bar l} S^{ab}_{\bar c} - \overline{\Gamma^{q}_{lc}}\, S^{ab}_{\bar q}$$

$$\nabla_l g_{i\bar j} = 0 \quad \big(\Leftarrow \nabla_k g_{i\bar j} = \partial_k g_{i\bar j} - \Gamma^p_{ki} g_{p\bar j} = \underbrace{\partial_k g_{i\bar j} - g^{p\bar q}\partial_k g_{i\bar q} g_{p\bar j}}_{=0}\big)$$

§1.3 Curvature

For kähler metric g. We define the curvature tensor $R_{i\bar j k}{}^p$ by

$$R_{i\bar j k}{}^p := -\partial_{\bar j}\Gamma^p_{ik}$$

$$R_{i\bar j k\bar l} := R_{i\bar j k}{}^p g_{p\bar l} = -g_{p\bar l}\,\partial_{\bar j}\Gamma^p_{ik}$$

Prop. $R_{i\bar j k\bar l} = R_{k\bar j i\bar l} = R_{i\bar l k\bar j} = R_{k\bar l i\bar j}$

and

$$\overline{R_{i\bar j k\bar l}} = R_{j\bar i l\bar k}.$$

Prop. For $X = X^p \partial_p$, $Y = Y^{\bar q}\partial_{\bar q}$, $a = a_p dz^p$, $b = b_{\bar q} d\bar z^q$, we have the following commutation formulae:

$$[\nabla_i, \nabla_{\bar j}] X^p = R_{i\bar j k}{}^p X^k$$

$$[\nabla_i, \nabla_{\bar j}] Y^{\bar q} = -R_{i\bar j}{}^{\bar q}{}_{\bar l} Y^{\bar l}, \quad \text{here } R_{i\bar j}{}^{\bar q}{}_{\bar l} := -\partial_i \overline{\Gamma^q_{jl}}$$

$$[\nabla_i, \nabla_{\bar j}] a_p = -R_{i\bar j p}{}^q a_q.$$

$$[\nabla_i, \nabla_{\bar j}] b_{\bar q} = R_{i\bar j}{}^{\bar l}{}_{\bar q} b_{\bar l}$$

here $[\nabla_i, \nabla_{\bar j}] = \nabla_i \nabla_{\bar j} - \nabla_{\bar j}\nabla_i$.

$[\nabla_i, \nabla_j] = 0.$

Lem. (holomorphic normal coordinate system)

For Kähler mfd (M^n, g) and fixed pt $x \in M$, $\exists$ a holo. coord. chart (U, z) centered at x s.t. at x

$$g_{i\bar j}(x) = \delta_{i\bar j} \quad \text{and} \quad \partial_k g_{i\bar j}(x) = 0, \quad \forall i, j, k \in \{1, 2, \dots, n\}.$$

Notations: $\omega = \sqrt{-1}\, g_{i\bar j}\, dz^i \wedge d\bar z^j$ —— Kähler metric

$\beta = \sqrt{-1}\, \beta_{i\bar j}\, dz^i \wedge d\bar z^j$ ~ a real $(1,1)$-form

$f \in C^\infty(M)$

Define $\operatorname{tr}_\omega \beta := g^{i\bar j} \beta_{i\bar j}$,

$$\operatorname{tr}_\omega(\sqrt{-1}\,\partial f \wedge \bar\partial f) = g^{i\bar j}\partial_i f \partial_{\bar j} f = |\partial f|^2 \left(= \tfrac{1}{2} |\nabla_{\mathbb{R}} f|^2\right)$$

$$\operatorname{tr}_\omega(\sqrt{-1}\,\partial\bar\partial f) = g^{i\bar j}\partial_i \partial_{\bar j} f = \Delta f = \Delta_{\bar\partial} f$$

$$\Delta_d = (d+d^*)^2, \quad \Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

$\Rightarrow \frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar\partial}$ for Kähler metric.

$$\Delta_{\bar\partial} := \frac{1}{2} g^{i\bar j}(\nabla_i \nabla_{\bar j} + \nabla_{\bar j}\nabla_i)$$

$$\Delta f := g^{i\bar j}\partial_i \partial_{\bar j} f$$

$$\Rightarrow \begin{cases} n\omega^{n-1}\wedge\beta = g^{i\bar j}\beta_{i\bar j}\,\omega^n = (\mathrm{tr}_\omega \beta)\cdot\omega^n \\ n\omega^{n-1}\wedge(\sqrt{-1}\,\partial f\wedge\bar\partial f) = |\partial f|_g^2\,\omega^n \end{cases}$$

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