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FUSION RULES AND MODULAR TRANSFORMATIONS IN 2D CONFORMAL FIELD THEORY

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We study conformal field theories with a finite number of primary fields with respect to some chiral algebra. It is shown that the fusion rules are completely determined by the behavior of the characters under the modular group. We illustrate with some examples that conversely the modular properties of the characters can be derived from the fusion rules. We propose how these results can be used to find restrictions on the values of the central charge and conformal dimensions.

1. Introduction

In conformal field theory local operators are characterized by their behavior under conformal transformations $z \to z'(z)$. In particular primary fields transform as conformal tensors under these analytic coordinate transformations [1]. Since the patching functions on Riemann surfaces are by definition analytic, CFT allows a very natural formulation on the moduli-space of Riemann surfaces [2]. This turns out to be an interesting and also very fruitful approach to CFT. The analytic and modular properties of the partition and correlation functions provide a powerful means of probing CFT and finding restrictions on the operator content [3]. This idea has lead to a successful classification of the modular invariant partition functions for the $c < 1$ discrete series [4].

The situation for $c \geqslant 1$ is completely different because of the existence of continuous marginal deformations, and an infinite number of primary fields w.r.t. the Virasoro algebra. However some CFT's with $c \geqslant 1$ have similar properties as the models of the discrete series, in the sense that their partition function can be decomposed into finitely many analytic building blocks, which can be interpreted as the characters of an extended algebra. These models all have the striking feature that the central charge and conformal dimensions of the primary fields are given by rational numbers, and are therefore often called rational CFT's [5]. RCFT's have many other special properties, which make them an interesting class of models to study.

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In this paper we consider the fusion rules of the primary fields and the modular properties of the characters of a RCFT. These are two at first sight separate aspects of these models, but we will show there is in fact an intimate connection. A first indication that a relation between fusion rules and modular transformations exists was found in the operator formulation of the gaussian model on Riemann surfaces [6, 7]. For this model we found that one could derive the modular behavior of the characters by studying the algebra of a set of operators in the Hilbert space. In analogy with this we will construct, using the primary fields and the fusion rules, a set of linear operators acting on the space of characters of a general RCFT. Then by considering the algebra of these operators and the role of the modular group we find that the matrix S, describing the behavior of the characters under the transformation $\tau \to -1/\tau$, diagonalizes the fusion rules. Furthermore, the matrix S contains sufficient information to determine the fusion rules completely.

An important ingredient of the derivation is the fact that the algebra of the linear operators is given by the fusion rules. We indeed find that the fusion rules are associative and have very similar properties as the product rules of the linear operators. Furthermore, we will check for many examples that the algebra of the operators and the fusion rules are equal, but we have not found a general proof of this fact.

Our results give constraints on the possible fusion rules of a RCFT, but also restrict the allowed representations of the modular group. By considering the behavior of the characters under the transformation $\tau \to \tau + 1$, we find for a given set of fusion rules powerful restrictions on the central charge and conformal dimensions.

In sect. 2 we describe the definition of the fusion rules of a RCFT. The relation with the modular properties of the characters is explained in sect. 3. This also contains a dimension formula for the number of generalized characters. In sect. 4 we discuss several examples, in particular the rational gaussian models, the SU(2) WZW models and the $c < 1$ unitary series. In sect. 5 it is shown that the fusion rules give restrictions on the values of the central charge and conformal dimensions. Finally, sect. 6 contains some concluding remarks.

2. The fusion rules

We start by reviewing some facts about the analytic structure of the partition and correlation functions of a CFT on Riemann surfaces. Consider some unnormalized correlator G of n primary fields on a genus g surface. This n-point function G depends on the positions $z = (z_1, \dots, z_n)$ of the fields and the moduli $\boldsymbol{m} = (m_1, \dots, m_{3g-3})$ of the Riemann surface. Here $(z, \boldsymbol{m})$ are a set of analytic coordinates on the moduli-space $\mathcal{M}_{g,n}$ of a Riemann surface with n punctures. The correlator $G(z, \boldsymbol{m}, \bar{z}, \bar{\boldsymbol{m}})$ has a very nice analytic structure, namely it can be decomposed into the sum of factorizable terms [1, 2].

$$G(z, m, \bar{z}, \bar{m}) = \sum_{\bar{I}, J} \bar{\mathscr{F}}_{\bar{I}}(\bar{z}, \bar{m}) h_{\bar{I}J} \mathscr{F}_J(z, m), \tag{2.1}$$

where $\bar{\mathscr{F}}_{\bar{I}}$ and $\mathscr{F}_J$ are the (anti-)analytic building blocks of the correlation function and $h_{\bar{I}J}$ is an hermitean metric. For the case $g = 1$ and $n = 0$ this equation gives the decomposition of the partition function in terms of characters. The correlator G and the blocks $\mathscr{F}_I$ depend on the representations of the primary fields at the punctures, but this is suppressed in the notation.

The $\mathscr{F}_I$ have in general non-trivial monodromy and modular properties, but the full correlation function G should of course be single-valued and modular invariant. This gives constraints on the metric $h_{\bar{I}J}$ and in particular for the genus one partition function one finds restrictions on the operator content of the CFT. In the remainder of this paper we will only consider one chiral half of CFT. Issues like modular invariance and crossing symmetry are not discussed, although the monodromy and the modular properties of the blocks $\mathscr{F}_I$ will play an important role.

For a rational CFT we can give a more detailed description of the $\mathscr{F}_I$. Consider some RCFT with N primary operators ϕ_i $(i = 0, \dots, N-1)$, corresponding to irreducible representations $[\phi_i]$ of some chiral algebra. This algebra could be superconformal [8], affine [9, 10] and parafermionic [11], but there are many other possible algebras. Since we are only interested one chiral half of the RCFT, say the left, we don't bother to write an index for the representations of the right algebra. The label $i = 0$ is used for the representation [1] containing the identity, and a multiplet of primary fields in the same $[\phi_i]$ will be denoted collectively by ϕ_i. In this paper only algebras generated by operators with integer conformal spin are considered. We believe that this is not a restriction on the RCFT, but only on what we call its symmetry algebra.

An important property of the representations $[\phi_i]$ is the appearance of null states. In combination with the Ward identities of the chiral symmetry algebra they lead to a set of partial differential equations for the correlation functions of the primary fields [1]. These PDE's involve derivatives with respect to the coordinates z and also w.r.t. the moduli m [12]. The $\mathscr{F}_I$ are the analytic solutions to these PDE's, or to be precise they form a basis for the vector space of analytic solutions.

From the viewpoint of RCFT there are only a few natural choices for this basis. Because one can obtain any punctured Riemann surface by sewing 3-punctured spheres, a complete basis of analytic blocks $\mathscr{F}_I$ can be constructed by sewing three-point functions (cf. [13]). We can represent this sewing operation schematically by a g-loop Feynman diagram for a φ^3 theory, in which each propagator represents a sum over all states in one of the representation $[\phi_i]$. At the vertices the representations $[\phi_i]$, $[\phi_j]$ and $[\phi_k]$ of the three propagators have to be fused together using a three-point function $\langle \phi_i \phi_j \phi_k \rangle$. In order to decide whether a block $\mathscr{F}_I$ can indeed be constructed in this way, we need to know the fusion rules, telling us not only when but also in how many ways three representations can be joined. The basis of $\mathscr{F}_I$ one obtains for a given φ^3 diagram is unique up to phases. This implies that modular transformations which do not change the φ^3 diagram are necessarily represented by phases.

Before we give a precise definition of the fusion rules, we note that everything we discuss can be formulated in the language of vector bundles. Because the analytic blocks $\mathscr{F}_I$ have a non-trivial behavior under modular transformations, they should in fact not be considered as functions but as the holomorphic sections of a vector bundle $V_{g,n}$ over the moduli space $\mathscr{M}_{g,n}$ of the n-punctured surface. The bundle $V_{g,n}$ is uniquely characterized by the fact that its holomorphic sections satisfy the PDE, and is for a RCFT (by definition) finite dimensional [5]. It has several components which are distinguished by the representations of the fields at the punctures.

Now let us describe the definition of the fusion rules. Consider the components $V_{0,ijk}$ of the bundle $V_{0,3}$ corresponding to the sphere with the fields ϕ_i, ϕ_j and ϕ_k at the three punctures. Let the integers N_{ijk} be the dimensions of the bundles $V_{0,ijk}$. We define the fusion rules in terms of these N_{ijk} as the formal product rule

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k , \tag{2.2}$$

where the integers $N_{ij}{}^k$ are related to N_{ijk} by using the conjugation matrix $C_{ij} = N_{ij0}$ as a metric to raise the index k. These $N_{ij}{}^k$ can be interpreted as multiplicities counting the number of independent fusion paths from ϕ_i and ϕ_j to ϕ_k. In this respect the fusion rules are very analogous to the rules for decomposing tensor products of representations of groups. However, we want to stress that (2.2) has nothing to do with the decomposition of $[\phi_i] \otimes [\phi_j]$, because taking tensor products changes the central extension of the chiral algebra.

To determine the fusion rules in practice one has to analyse the three-point function $\langle \phi_i \phi_j \phi_k \rangle$, or equivalently the operator product expansion of two primary fields ϕ_i and ϕ_j. In a similar way as has been done for the Virasoro [1] and Kac–Moody [10] algebras one can use the PDE's to find restrictions on the three-point function and, at least in principle, determine which representations occur in the OPE of two primary fields. (Note that, since there are no moduli z and $\boldsymbol{m}$ associated with the 3-punctured sphere, the PDE must become algebraic.)

Let us return to the analytic blocks $\mathscr{F}_I$ for an arbitrary surface. For a given set of external primary fields the number of $\mathscr{F}_I$ ($= \dim V_{g,n}$) can be computed by using the φ^3 diagram and counting the number of ways the representations can be fused together. In this way one gets the following set of "Feynman rules". One writes for each vertex a factor N_{ijk}, and contracts the indices as indicated by the propagators. The result is equal to $\dim V_{g,n}$ and should be independent of the way the spheres are sewn together, i.e. of the choice for the φ^3 diagram. It is easy to see that it is sufficient to check this for the conformal blocks for the four-point function. This gives the following condition on the multiplicities N_{ijk}:

$$\sum_k N_{ij}{}^k N_{klm} = \sum_k N_{il}{}^k N_{kjm}. \tag{2.3}$$

An important consequence of this relation is that the fusion rules (2.2) are associative and can be interpreted as an algebra of linear operators ϕ_i. Furthermore, it implies that the matrices $(N_i)_j{}^k$ form a representation of this algebra. Because the matrices N_i are symmetric and mutually commuting, they can be simultaneously diagonalized. Their eigenvalues form the N one-dimensional representations of the fusion rules. These will play an important role in the next sections.

The fusion rules (2.2) do not seem to give any information about the central charge and the conformal dimensions h_i of the primary fields ϕ_i. It is even possible that different CFT's have the same fusion rules. However, we will show in this paper that a given set of fusion rules can only occur for a countable number of c- and h-values. This fact will be a consequence of the relation between the fusion rules and modular transformations, which we are about to discuss. In the following we will mainly restrict our attention to the genus one case, although many of the presented ideas can be generalized to higher genus.

3. Relation with modular transformations

Let us consider the characters χ_i of the chiral symmetry algebra. To describe their definition we choose two oriented cycles $\boldsymbol{a}$ and $\boldsymbol{b}$ on the torus. The $\boldsymbol{b}$-cycle indicates the direction of the time evolution, which is generated by L_0, while the cycle $\boldsymbol{a}$ connects points of equal time. The character χ_i is defined as the trace of the evolution operator over the representation $[\phi_i]$:

$$\chi_i = \mathrm{tr}_{[\phi_i]}\left(q^{L_0+\varepsilon}\right), \tag{3.1}$$

where $q = e^{2\pi i\tau}$ and $\varepsilon = -\frac{1}{24}c$, and τ is the modular parameter of the torus.

The moduli space $\mathscr{M}_1$ of genus one surfaces is obtained from the upper half of the complex plane by dividing out the action of the modular group generated by T: $\tau \to \tau + 1$ and S: $\tau \to -1/\tau$. For chiral algebras generated by operators with integer conformal spin the characters χ_i transform in a finite dimensional representation of the modular group and form a basis of the holomorphic sections of the vector bundle V_1 over $\mathscr{M}_1$ [2]. For other algebras one has to specify boundary conditions and, consequently, has to consider coverings of the moduli space. For example, the characters of the superconformal algebra are naturally defined on the spin-covering of $\mathscr{M}_1$.

The behavior under T: $\tau \to \tau + 1$ depends only on the central charge c and the conformal dimension h_i of the primary field ϕ_i. It follows directly from its definition (3.1) that χ_i transforms as:

$$T: \qquad \chi_i \to e^{2\pi i(h_i + \varepsilon)} \chi_i . \tag{3.2}$$

The behavior of χ_i under S: $\tau \to -1/\tau$ gives additional information about the CFT. On the characters this modular transformation is represented by a unitary matrix which is also denoted by S:

$$S: \qquad \chi_i \to \sum_j S_i^{\,j} \chi_j . \tag{3.3}$$

Using the fact that under $\tau \to -1/\tau$ the cycles $\boldsymbol{a}$ and $\boldsymbol{b}$ are mapped onto $-\boldsymbol{b}$ resp. $\boldsymbol{a}$ one sees that S^2 inverts the time direction and consequently transforms χ_i into the character χ^i of the conjugate representation. So in general the matrix S will satisfy $S^2 = C$, and only when all representations are self-conjugate does one have $S^2 = 1$.

We now like to show that there is a connection between the unitary matrix S and the fusion rules of the primary fields. The basic idea is to use the primary fields to manipulate the characters and then compare the situation before and after the modular transformation. First consider the character χ_0 of the representation [1] of the identity. We can obtain the other characters χ_i from χ_0 in the following way. We insert the identity operator inside the trace (3.1) and rewrite it as the OPE of the primary field ϕ_i and its conjugate field. Next we move ϕ_i along the $\boldsymbol{b}$-cycle and then, after it has gone round once, we let the two fields annihilate again. As a result the trace is no longer over the representation [1] but over $[\phi_i]$. To see this one can for example map the torus on an annular region of the complex plane $|q|^{1/2} \leqslant z < |q|^{-1/2}$ and pinch the $\boldsymbol{a}$-cycle i.e. send $q \to 0$. In this picture the field ϕ_i is moved from ∞ to the origin and thus changes the representation. We conclude that this operation, which will be denoted by $\phi_i(\boldsymbol{b})$, indeed transforms χ_0 into χ_i:

$$\phi_i(\boldsymbol{b}) \chi_0 = \chi_i . \tag{3.4}$$

What happens when we apply the same procedure to some other character χ_j? Using the same picture we see that in this case, when we move ϕ_i to the origin, we have to take the operator product with the primary field at the origin, which is ϕ_j. Thus again the representation $[\phi_j]$ over which the trace is taken will change, namely precisely according to the fusion rules. So we find that under the operation $\phi_i(\boldsymbol{b})$ a character χ_j will in general transform into a linear combination of several characters:

$$\phi_i(\boldsymbol{b}) \chi_j = \sum_k A_{ij}^{\ \ k} \chi_k \tag{3.5}$$

where $A_{ij}^{\ k}$ are some coefficients which are only non-zero if the three-point function $\langle\phi_i\phi_j\phi^k\rangle$ is non-vanishing, i.e. if $N_{ij}^{\ k}\neq 0$.

One can show that the operators $\phi_i(\boldsymbol{b})$ mutually commute and as a consequence the coefficients $A_{ij}^{\ k}$ have the following properties.

$$A_{ij}^{\ k}=A_{ji}^{\ k}, \tag{3.6}$$

$$\sum_k A_{ij}^{\ k}A_{kl}^{\ m}=\sum_k A_{il}^{\ k}A_{kj}^{\ m}. \tag{3.7}$$

Using (3.7) one finds that the operators $\phi_i(\boldsymbol{b})$ satisfy the following associative algebra:

$$\phi_i(\boldsymbol{b})\phi_j(\boldsymbol{b})=\sum_k A_{ij}^{\ k}\phi_k(\boldsymbol{b}). \tag{3.8}$$

This algebra is very reminiscent of the fusion rules (2.2), and eventually we would like to show that both algebras (2.2) and (3.8) are indeed the same.

The manipulation described above can be performed for any cycle on the torus. In particular it is interesting to consider what happens when we transport ϕ_i along the $\boldsymbol{a}$- instead of the $\boldsymbol{b}$-cycle. One can easily convince oneself that in this case, because the $\boldsymbol{a}$-cycle consists of points with equal time, the representation $[\phi_j]$ does not change. In other words, the characters χ_j are eigenstates of the operators $\phi_i(\boldsymbol{a})$.

$$\phi_i(\boldsymbol{a})\chi_j=\lambda_i^{(j)}\chi_j. \tag{3.9}$$

These eigenvalues $\lambda_i^{(n)}$ are real or complex numbers, not necessarily phases. The operators $\phi_i(\boldsymbol{a})$ will satisfy the same algebra (3.8) as $\phi_i(\boldsymbol{b})$. This gives the following multiplication rule for the eigenvalues $\lambda_i^{(n)}$

$$\lambda_i^{(n)}\lambda_j^{(n)}=\sum_k A_{ij}^{\ k}\lambda_k^{(n)}. \tag{3.10}$$

Our description of the operators $\phi_i(\boldsymbol{a})$ and $\phi_i(\boldsymbol{b})$ has been quite intuitive. It is possible to analyse the different manipulations more carefully. In fact, one finds that in (3.4) there is in general a factor different from one in front χ_0. But, because the $\phi_i(\boldsymbol{b})$ act as linear operators, we can normalize them so that (3.4) holds. In this way we can express the coefficients $A_{ij}^{\ k}$ in terms of the monodromy properties of the conformal blocks of the four-point function. The resulting expression is however rather unpleasant and doesn't seem to be very useful.

Fortunately, there is another way to get information about the coefficients, namely by considering the role of the modular transformation S. This transformation interchanges the cycles $\boldsymbol{a}$ and $\boldsymbol{b}$ and hence also the operators $\phi_i(\boldsymbol{a})$ and $\phi_i(\boldsymbol{b})$. In particular the transformed characters (3.3) become eigenstates of $\phi_i(\boldsymbol{b})$. Thus we arrive at the important conclusion that the unitary matrix $S_i^{\ j}$ diagonalizes the coefficients $A_{ij}^{\ \ k}$. This gives, by combining (3.3), (3.5) and (3.9) the following identity:

$$A_{ij}^{\ \ k} = \sum_n S_j^n \lambda_i^{(n)} S_n^{\dagger k}. \tag{3.11}$$

Note that the r.h.s. indeed satisfies the property (3.7). By using in addition to (3.11) the fact that $A_{i0}^{\ \ k} = \delta_i^k$ we can also express the eigenvalues $\lambda_i^{(n)}$ in terms of the entries of the matrix S_i^n

$$\lambda_i^{(n)} = S_i^n / S_0^n. \tag{3.12}$$

Together with (3.11) this allows us to compute the coefficients A_{ijk} if we know the matrix S. In this way we have calculated the coefficients $A_{ij}^{\ \ k}$ for many rational CFT's for which the modular properties are known, and for all of them we have indeed found that:

$$A_{ijk} = N_{ijk}. \tag{3.13}$$

We conjecture that this is true for every RCFT, but a proof of this fact requires a better understanding of the operators $\phi_i(\boldsymbol{b})$. We have shown that the coefficients A_{ijk} have the same properties (3.6) and (3.7) as the multiplicities N_{ijk}. Furthermore, we know that $A_{ijk} = 0$ if $N_{ijk} = 0$ and that $A_{ij0} = N_{ij0}$, but all this is not sufficient to prove our conjecture. Note that (3.13) implies the algebra (3.8) of these operators has precisely the same form as the fusion rules (2.2). In particular, it allows us to rephrase (3.11) in a more notation-independent statement:

The modular transformation S: $\tau \to -1/\tau$ diagonalizes the fusion rules!

This is, in words, the main result of this paper. In an attempt to convince the reader, that our conjecture is correct we will in the next section discuss several examples.

But first we like to mention one consequence concerning the number of generalized characters (= $\dim V_g$). For those RCFTs for which (3.13) holds (as we will see this includes many of the known models) we can use the result (3.11)–(3.12) to give a very simple expression for the dimension of V_g, in terms of the entries S_{n0} of the matrix S. Following the recipe of the preceding section we represent the surface by a g-loop φ^3 diagram, write a factor N_{ijk} for each vertex and contract all indices. By choosing a convenient φ^3 diagram we can write the result as a trace

$$\dim V_g = \mathrm{tr}\left(\sum_{i=0}^{N-1} N_i^2 \right)^{g-1}, \tag{3.14}$$

which can be evaluated in terms of the eigenvalues $\lambda_i^{(n)}$ of the matrices $(N_i)_j^{\ k}$. Finally one uses (3.12) and $S^2 = C$ to obtain the following expression for the dimension of the vectorbundle V_g

$$\dim V_g = \sum_{n=0}^{N-1} |S_{n0}|^{-2(g-1)}. \tag{3.15}$$

It is an amusing exercise to check that the r.h.s. gives integer dimensions for the RCFT's for which the unitary matrix S is known. For example, for the Ising model one recovers the familiar result: $\dim V_g^{\text{Ising}} = 2^{g-1}(2^g+1) =$ the number of even spin structures on a genus g Riemann surface [2].

4. Some examples

In this section we will illustrate the presented ideas with some concrete examples and in particular we will check our conjecture (3.13). We explicitly work out the $c=1$ gaussian model, the SU(2) WZW models and the unitary series and we briefly mention some other examples.

The simplest class of RCFT's are the rational gaussian models. They can be described by a free scalar field φ which is compactified on a circle with a rational value for the (radius)^2. They have as symmetry algebra the U(1) current algebra extended with some chiral vertex-operator with conformal spin $\frac{1}{2}N$ and momentum (= U(1) charge) $\sqrt{N}$ (N is an even integer). There are N primary fields $[\phi_p]$ being the vertex-operators with momentum $p/\sqrt{N}$ with $p \in \mathbb{Z}_N$. The fusion rules follow directly from momentum-conservation:

$$\phi_p \times \phi_{p'} = \phi_{p+p'}, \qquad p, p' \in \mathbb{Z}_N. \tag{4.1}$$

For these models the operators $\phi_p(c)$ can be defined in a more explicit way using the operator formulation on Riemann surfaces. They can be expressed in terms of the loop-momentum operators introduced in [6]; this is discussed in [7]. Their action on the characters can be calculated by inserting the operator

$$\phi_p(c) = \exp\left(\frac{p}{\sqrt{N}} \oint_c \partial\varphi\right) \tag{4.2}$$

into the trace (3.1). (Note that this is not the zero mode of a vertex operator.) This operator measures the momentum flux through the cycle c but at the same time increases the flux through the cycles intersecting c. This is reflected in the relation:

$$\phi_p(a)\phi_{p'}(b) = e^{2\pi i p p'/N} \phi_{p'}(b)\phi_p(a). \tag{4.3}$$

The operators $\phi_p(a)$ and $\phi_p(b)$ act on the characters χ_p precisely in the way we expect when we apply the results of the preceding section to this case.

$$\phi_p(\boldsymbol{a})\chi_{p'} = e^{2\pi i p p'/N}\chi_{p'},$$
$$\phi_p(\boldsymbol{b})\chi_{p'} = \chi_{p+p'}. \tag{4.4}$$

The operator $\phi_p(\boldsymbol{b})$ shifts the momentum and according to (3.11) has to be diagonalized by modular transformation S. This implies that S should act as a Fourier transformation on the characters χ_p,

$$S: \qquad \chi_p \to \frac{1}{\sqrt{N}} \sum_{p' \in Z_N} e^{2\pi i p p'/N}\chi_{p'}. \tag{4.5}$$

Note that S^2 maps χ_p onto χ_{-p} and so only for $N = 2$ the modular transformation S has order two. The genus one (and higher genus) characters can be expressed in terms of theta functions with known modular properties agreeing with (4.5). For the gaussian models this method can be generalized to arbitrary genus. The analysis for these models is rather easy in comparison with other CFT's because according to (4.1) the OPE of two primary field contains only one representation. In a general CFT this is usually not the case, as we can see in the next (less trivial) example: the $SU(2)_k$ WZW models for arbitrary k.

The field content of these models is organized by the Kac–Moody algebra $SU(2)_k$. Gepner and Witten [10] showed that $SU(2)_k$ has $k+1$ integrable representations $[\phi_l]$, namely the ones with SU(2) isospin $\frac{1}{2}l \leqslant \frac{1}{2}k$. By making use of the null states in these representations they found the following fusion rules

$$\phi_l \times \phi_{l'} = \sum_{j=|l-l'|}^{\min(l+l',2k-l-l')} \phi_j, \tag{4.6}$$

where $j - |l-l'|$ is an even integer.

Using the Weyl–Kac character formula one can express the affine $SU(2)_k$ characters χ_l in theta functions. However, as we will now show, one can derive the behavior of the characters under S: $\tau \to -1/\tau$ without needing these explicit expressions. The first step is to determine the numbers $\lambda_l^{(n)}$ which, assuming (3.13) is correct, have to satisfy an algebra of the same form as the fusion rules (4.6). To find these numbers we only need the finite Weyl-character formula for the group SU(2). The Weyl character $\mathrm{ch}_l(\theta)$ of the isospin $\frac{1}{2}l$ representation of SU(2), not to be confused with the affine $SU(2)_k$ character χ_l, is defined by

$$\mathrm{ch}_l(\theta) = \sum_{m=-l}^{l} e^{im\theta} = \frac{\sin(l+1)\theta}{\sin\theta} \tag{4.7}$$

with $m+l=$ even. The product of two Weyl characters can be decomposed into a sum of Weyl characters

$$\mathrm{ch}_l(\theta)\mathrm{ch}_{l'}(\theta)=\sum_{j=|l-l'|}^{l+l'}\mathrm{ch}_j(\theta), \tag{4.8}$$

which is nothing but the well-known Clebsch–Gordan formula.

This multiplication rule is almost what we want to have for the numbers λ_l, only in (4.6) the sum is over a subset and is truncated below $k+1$. This can be achieved by choosing the variable θ such that $\mathrm{ch}_{k+1}(\theta)=0$, which has precisely $k+1$ independent solutions, $\theta=(n+1)\pi/(k+2)$, $n=0,\dots,k$ corresponding to the $k+1$ integrable representations $[\phi_n]$. So we get:

$$\lambda_l^{(n)}=\mathrm{ch}_l\left(\frac{n+1}{k+2}\pi\right). \tag{4.9}$$

By repeatedly multiplying the equation $\mathrm{ch}_{k+1}(\theta)=0$ by $\mathrm{ch}_{l=1}(\theta)$ and using (4.8) one finds that $\mathrm{ch}_{k+1+l}(\theta)=-\mathrm{ch}_{k+1-l}(\theta)$ for $\theta=(n+1)\pi/(k+2)$. Using this relation it is easily verified that the eigenvalues $\lambda_l^{(n)}$ have indeed the correct multiplication rule (4.6).

Finally, to find the behavior of the affine characters χ_l under $\tau\to-1/\tau$ we use (3.12), which determines the unitary matrix S_{ln} up to a sign. We obtain the following expression

$$S_{ln}=\left(\frac{2}{k+2}\right)^{1/2}\sin\frac{(l+1)(n+1)}{k+2}\pi \tag{4.10}$$

which is in agreement with the modular behavior of the explicit expressions for the $\mathrm{SU}(2)_k$ characters [10].

We briefly discuss the WZW models for an arbitrary Lie group G. Also for these theories the modular properties of the characters are known. Using the expressions for the unitary matrix S given in [10] we can compute the numbers $\lambda_i^{(n)}$ and find that also in this case they are equal to the finite Weyl characters of G, evaluated for some special elements of the dual Cartan subalgebra. The Weyl characters have similar multiplication rules as (4.8) with integer multiplicities, which supports our conjecture (3.13).

Now let us consider the unitary series of the Virasoro algebra. For central charge $c=1-6/m(m+1)$ $(m=3,4,\dots)$, the Virasoro algebra has $\frac{1}{2}m(m-1)$ unitary representations $[\phi_{pq}]$, where $p=1,\dots,m-1$, $q=1,\dots,m$ and $p+q=$ even [14].

The corresponding primary fields ϕ_{pq} have conformal dimension:

$$h_{pq}=\frac{(p(m+1)-qm)^2-1}{4m(m+1)}. \tag{4.11}$$

We will compute the eigenvalues $\lambda_{pq}^{(p'q')}$ starting from the modular properties of the characters, and verify whether their multiplication rules agree with the known fusion rules.

The representation theory of these models is related to that of the SU(2) WZW models by the GKO coset construction [15]. By considering the branching of the representations of $SU(2)_1\times SU(2)_{m-2}$ w.r.t. the diagonal embedding of $SU(2)_{m-1}$ one finds the unitary representations ϕ_{pq} of the Virasoro-algebra

$$[\phi_0]_1\otimes[\phi_{p-1}]_{m-2}=\sum_{\substack{q=1\\ p+q=\text{even}}}^{m}[\phi_{pq}]\otimes[\phi_{q-1}]_{m-1},$$

$$[\phi_1]_1\otimes[\phi_{m-p-1}]_{m-2}=\sum_{\substack{q=1\\ p+q=\text{even}}}^{m}[\phi_{pq}]\otimes[\phi_{m-q}]_{m-1}. \tag{4.12}$$

For the characters one gets of course the same decomposition [15]. Then by considering the modular behavior of both sides one can express the matrix S for the Virasoro-characters χ_{pq} in terms of the matrix S (4.10) for the $SU(2)_k$ characters [4]. The eigenvalues λ can be obtained using (3.12) and are equal to the product of two SU(2) Weyl characters

$$\lambda_{pq}^{(p'q')}=\mathrm{ch}_{p-1}\left(\frac{p'}{m}\pi\right)\mathrm{ch}_{q-1}\left(\frac{q'}{m+1}\pi\right). \tag{4.13}$$

Finally applying (4.8) gives the fusion rules for the primary fields ϕ_{pq} of the unitary $c<1$ models:

$$\phi_{pq}\times\phi_{p'q'}=\sum_{r=|p-p'|+1}^{\min(p+p'-1,2m-p-p'-1)}\ \sum_{s=|q-q'|+1}^{\min(q+q'-1,2m-q-q'+1)}\phi_{rs}. \tag{4.14}$$

This is indeed the truncated fusion rules for minimal models [1,4].

Recently it has become clear that the $c<1$ unitary series are just special cases of a much larger class of discrete series, which are obtained by generalizing the GKO construction to other G/H cosets [16]. This is reflected in the modular properties of the characters and consequently also in the fusion rules. We expect that, similarly as for the discrete series, the eigenvalues λ can again be expressed in the Weyl characters of G and H, and hence have a multiplication rule with integer multiplicities.

The only class of solvable CFT's we have not yet considered are the orbifold models [17]. One obtains a rational orbifold model by starting with some RCFT and dividing out a discrete symmetry of the model. The generators of this discrete group can act as inner or as outer automorphisms on the representations ϕ_i of the original model. So to give a systematic analysis for orbifold models one has to distinguish several cases. Work in this direction is in progress [18], and seems to indicate that also for orbifold models (3.13) is correct.

5. Restrictions on the central charge and conformal dimensions

Encouraged by the previous examples we will now assume that for every RCFT $A_{ijk} = N_{ijk}$ and study what the applications and implications are. In particular we discuss how this fact might help in a possible classification of rational CFT's. One starts by considering the possible fusion rules for a RCFT with N primary fields. There are some conditions on the allowed fusion rules: each representation must have precisely one conjugate representation and, secondly, they have to be associative.

Next one uses the relation with the modular transformations to find restrictions on the central charge c and the conformal dimensions h_i of the primary fields. Given a set of fusion rules we can, by reversing the argumentation of sect. 3, try to find the modular behavior of the characters. First we determine the eigenvalues $\lambda_i^{(n)}$ and subsequently the unitary matrices S diagonalizing the fusion rules. Unfortunately S is not completely determined by (3.12), because in general we don't know the precise correspondence between the eigenvalues $\lambda_i^{(n)}$ and the characters χ_n. This is partially resolved by the requirement that, since S describes the modular behavior under $\tau \to -1/\tau$, its square has to map each representation onto its conjugate, i.e. $S^2 = C$. Some of the fusion rules one started with are eliminated by this condition, but for the remaining cases there are possibly several matrices S with the right properties.

The relations we gave for the matrix S leaves its sign still undetermined. There is however a way to fix this sign, which is based on the following observation*. Because the transformation $\tau \to -1/\tau$ has a fixed point $\tau = i$, we know that

$$\sum_k S_j{}^k \chi_k(i) = \chi_j(i). \tag{5.1}$$

From the definition (3.1) of the characters one easily sees that $\chi_j(i)$ is real and positive for all j. So the matrix S should have an eigenvector with eigenvalue 1 and

*This observation is due to C. Vafa (private communication).

with positive real components. Since there are no other eigenvectors with the same property, this indeed fixes the sign of S.

The final step is to consider the behavior of the characters under $\tau \to \tau + 1$ as given in (3.2) and impose the condition that they form a representation of the modular group, i.e. that $(ST)^3 = 1$. By counting the number of equations one sees that the eigenvalues $e^{2\pi i(h_i + \epsilon)}$ of T are in fact overdetermined. This means that, if we want to have solutions to $(ST)^3 = 1$, the matrix S must have extra symmetries. For those cases one finds in general only a countable set of allowed c- and h-values, but sometimes some of these quantities remain unrestricted.

Let us make some speculative remarks on why we may expect to find rational values for c and h_i. The numbers $\lambda_i^{(n)}$ are for all known examples given by a polynomial with integer coefficients in some primitive root of unity ζ. Because the unitary matrix S is related to these numbers one can expect that the conditions $(ST)^3 = 1$ restricts the eigenvalues $e^{2\pi i(h_i + \epsilon)}$ of T to certain powers of ζ. This clearly gives rational c- and h-values.

We illustrate the above program with the simplest cases. First of all, if there are no other primary operators than the identity, the representation must be one-dimensional. For this case $S = 1$ and one finds for the central charge:

$$c = 0 \pmod 8. \tag{5.2}$$

Examples are the $k = 1$ WZW models on the famous groups E_8 and SO(32).

In the next case we have in addition to the identity one primary field ϕ, with conformal dimension h. Since ϕ is necessarily self-conjugate the possible fusion rules are:

$$\phi \times \phi = 1 + n\phi. \tag{5.3}$$

These fusion rules are associative for all values of n. The eigenvalues λ of ϕ are the solutions of the quadratic equation $\lambda^2 = 1 + n\lambda$. The matrices S and T are of the form:

$$S = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{5.4}$$

$$T = e^{2\pi i\epsilon} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i h} \end{pmatrix} \tag{5.5}$$

with $\sin\theta \geqslant 0$. The requirement that S diagonalizes the fusion rules (5.3) yields the relation $\tan\theta = \lambda$. The condition $(ST)^3 = 1$ is easily worked out, and gives for c and h the following equations:

$$12h - c = 2 \pmod 8,$$

$$\cos 2\pi h = -\tfrac{1}{2}n\lambda. \tag{5.6}$$

We find that the value of the central charge is only determined up to multiples of 8. This can be understood as follows. If one takes the tensor product of a RCFT with the E_8 model the central charge c is increased with 8, but the conformal dimension h and the fusion rules are not changed.

Now let us give some examples of RCFT's with two primary fields. For the case $n = 0$ we have found as examples the $k = 1$ WZW models on the group manifolds of SU(2) and E_7, which have $(c, h) = (1, \frac{1}{4})$ resp. $(7, \frac{3}{4})$. Note that $SU(2)_{k=1}$ is equivalent to one of the rational gaussian models discussed in sect. 4, namely for $N = 2$ in (4.1). In fact one can show with this method that any CFT with fusion rules of the gaussian type (4.1) for arbitrary N must have an integer c-value. This suggests that such models can always be represented by free scalar fields, compactified on some torus.

The condition (5.6) gives for the case $n = 1$ as allowed values for the conformal dimension of ϕ: $h = \pm\frac{1}{5}, \pm\frac{2}{5}$ (mod 1). Again there are some $k = 1$ WZW models which fit the description, in this case on the non-simply laced groups G_2 and F_4. For these models $(c, h) = (\frac{14}{5}, \frac{2}{5})$ resp. $(\frac{26}{5}, \frac{3}{5})$. Also one of the non-unitary minimal models has all the right properties*; it has $(c, h) = (-\frac{22}{5}, -\frac{1}{5})$ and describes the Lee–Yang singularity. We like to mention that the representation of the modular group for both cases $n = 0$ and $n = 1$ is finite. The transformations S and T act on the ratio of the two characters $z = \chi_1/\chi_0$ as fractional linear Sl(2, C) transformations, which by identifying the complex plane with the two-sphere generate precisely the symmetry groups of the octa- resp. icosahedron [4].

For multiplicities $n \geqslant 2$ the relations (5.6) don't seem to give very nice c- and h-values, and we don't know of any RCFT corresponding to one of these cases. We expect that for some reason multiplicities $n \geqslant 2$ are not allowed for RCFT's with two primary fields.

Finally we briefly discuss some CFT's with two non-trivial primary fields ϕ_1 and ϕ_2. We only consider fusion rules with multiplicities $N_{ijk} \leqslant 1$. The two fields ϕ_1 and ϕ_2 can be either conjugate to each other or to themselves. In the first case we find that the fusion rules are necessarily of the gaussian type (4.1) with $N = 3$. Examples are the $k = 1$ SU(3) and E_6 WZW models.

The Ising model is a well-known example of the second possibility. It is the first model in the discrete series and corresponds to the case $m = 3$ in (4.14). In a more conventional notation the two non-trivial primary fields of the Ising model are a spin field σ and a majorana fermion ψ with $h = \frac{1}{16}$ resp. $\frac{1}{2}$. The fusion rules are

$$\begin{aligned} \sigma \times \sigma &= 1 + \psi\,, \\ \psi \times \psi &= 1\,, \\ \sigma \times \psi &= \sigma\,. \end{aligned} \tag{5.7}$$

*I thank J.-B. Zuber for pointing this out to me.

These are in fact the same fusion rules as those for the $SU(2)_{k=2}$ (4.6) and $SO(2m+1)_{k=1}$ WZW models. All these models can be described by an odd number of real fermions with correlated spin structures. The matrix S is also the same for all models. The condition $(ST)^3 = 1$ gives in this case only two equations for the central charge c and the conformal dimensions h_1 and h_2: $h_1 = c/8$ and $h_2 = \frac{1}{2}$ (again mod 1). For all examples the central charge is half-integer, but to prove that these are the only allowed values one probably has to consider the modular behavior of the genus two (or higher) characters.

The constraints of associativity combined with $S^2 = 1$ allows only one other set of fusion rules for a CFT with three primary fields and $N_{ijk} \leqslant 1$. They are given by:

$$\begin{aligned} \phi_1 \times \phi_1 &= 1 + \phi_2 , \\ \phi_2 \times \phi_2 &= 1 + \phi_1 + \phi_2 , \\ \phi_1 \times \phi_2 &= \phi_1 + \phi_2 . \end{aligned} \tag{5.8}$$

Also in this case the eigenvalues of ϕ_1 and ϕ_2 are elements of a cyclotomic field. We find: $\lambda_1 = -\zeta - \zeta^{-1}$ and $\lambda_2 = 1 + \zeta^2 + \zeta^{-2}$ with $\zeta^7 = 1$. The relation $(ST)^3 = 1$ permits the existence of a CFT with these fusion rules for c equal to any multiple of $\frac{4}{7}$, except multiples of 4, but we have only found an example for $c = -\frac{68}{7}$, namely one of the non-unitary minimal models.

6. Conclusion

The connection we have found between the fusion rules and the modular behavior of a RCFT can be summarized by the statement that the transformation S: $\tau \to -1/\tau$ diagonalizes the fusion rules. We have given an intuitive derivation of this result, but in particular a proof of the fact that the coefficients A_{ijk} are equal to the multiplicities N_{ijk} is still lacking. It is very likely that we have to use some extra ingredients, for example the fact that the characters have a q-expansion with integer coefficients, or some other special property of the characters.

Another interesting question is whether the method of sect. 3 can be generalized to surfaces with higher genus or with punctures. In particular one would like to know if the fusion rules contain sufficient information about the RCFT to determine the monodromy of the conformal blocks, or the modular behavior of the generalized characters. If such a generalization to other topologies exists, it would undoubtedly give new restrictions on the central charge, the conformal dimensions and the fusion rules.

In this paper we have only considered RCFT's. The reason for this is more of a practical nature than a matter of principle. Most of the quantities we worked with can also be defined for CFT's with an infinite number of primary fields. There is no reason why it should not be possible to generalize the relation between fusion rules and modular transformations to those CFT's. From the viewpoint of string theory it is particularly interesting to see what this implies for models with a space-time interpretation.

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