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TOPOLOGICAL LANDAU-GINZBURG MODELS

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We derive a general expression for correlation functions of topological Landau-Ginzburg models on an arbitrary genus Riemann surface. The expressions we find for the correlation functions suggest that for $\hat{c}>1$ the perturbation of the theory by chiral primary fields of dimensions bigger than one is rather singular, though perturbation by relevant chiral primary fields seems sensible regardless of the value of $\hat{c}$.

The $N = 2$ superconformal Landau-Ginzburg (LG) models have been studied previously[1] in the context of classification of $N = 2$ superconformal theories and constructing string vacua. It was found there that a large class of $N = 2$ theories can be characterized by a superpotential $W(x_i)$ of an $N = 2$ supersymmetric LG theory, where x_i denote chiral superfields as $i = 1, \ldots, n$. Moreover, it was shown that a finite number of states of any $N = 2$ theory are topological in the sense that their naive operator product has no singularities, and they form a closed ring. This ring was named the chiral primary ring $\mathcal{R}$. For LG models this ring takes the particularly simple form

$$\mathcal{R} = \frac{C[x_i]}{dW}, \tag{1}$$

i.e., the free ring generated by x_i modulo setting the derivatives of W to zero (which corresponds to getting rid of the descendants). For W to give rise to a conformal theory, it was noted that W needs to be quasi-homogeneous, i.e., all the fields x_i have a definite charge q_i, once we associate charge 1 to W. For a review of various aspects and open questions in the context of $N=2$ superconformal theories see Ref. 2, also for some recent progress in their understanding see Refs. 3 and 4.

More recently, following general considerations in Ref. 5, it was found[6] that there is a twisted version of these conformal theories for which the chiral primary fields are the *only* physical excitations, thus deserving the name 'topological LG' models. It was shown in Ref. 7 that in general the twisted $N = 2$ models can serve as topological matter which could be coupled to topological gravity. It was conjectured by Witten[7] and generalized in Ref. 8 that topological gravity coupled to topological matter is equivalent to the matrix models[9] for an appropriate choice of topological matter, which itself is believed to be equivalent to ordinary matter coupled to $2d$ gravity. By now there is a lot of evidence for all these conjectures. More recently, in an interesting work Li[10] provided some evidence that the appropriate topological matter which gives rise to the n matrix model, is the n-th minimal twisted $N=2$ model. These models are among the $N=2$ models which do admit an LG description.[1] This conjecture of Li has been recently put on a firmer ground in Ref. 11 in which it was shown that the twisted minimal $N=2$ models indeed give the same correlation as the matrix model on the sphere. One of the key observations in Ref. 11 was the fact that the LG description of the minimal model is most suitable for computing the ring structures. In particular, the ring (1) of a superpotential which has been *deformed* from a quasi-homogeneous form, still gives the relevant ring of observables for the deformed topological theory.

In this paper we compute the correlation functions for an arbitrary topological LG theory on an arbitrary genus surface. The main new input is to use an explicit Lagrangian description of a topological LG theory.

There are two types of operators of interest in the topological LG[a]: those with dimension zero $\phi^{(0)}$, and those with dimension one (on left and right) $\phi^{(1)}$. The dimension zero fields are precisely the chiral primary fields before being twisted, and the dimension one fields are the first descendant of the chiral primary fields with respect to the generators of superconformal transformations $(G^{-}\overline{G}^{-})$. The U(1) charge conservation implies that on genus-g surface the non-vanishing amplitudes

$$\langle \phi_1^{(0)}\phi_2^{(0)}\cdots\phi_r^{(0)}\int\phi_{r+1}^{(1)}\cdots\int\phi_{r+n}^{(1)}\rangle_g$$

satisfy

$$\sum_{i=1}^{r} q_i + \sum_{i=r+1}^{r+n}(q_i-1)=\hat{c}(1-g)\,, \tag{2}$$

where $\hat{c}$ is the normalized central charge of the theory (which in turn is equal to the maximum U(1) charge in the untwisted superconformal ring). We see that for the topological LG theory we can have non-vanishing correlations on arbitrary genus consistent with U(1) charge conservation.

Let us focus on topological LG models not coupled to topological gravity (though these correlations are needed in the computation of their coupling to topological gravity). We can add operators of dimension 1 to the superpotential and thus obtain a new topological LG theory. So in order to compute arbitrary correlation functions it suffices to compute the correlation functions of operators of dimension zero in the presence of an arbitrary perturbation of W. The correlations involving the mixture of dimension zero fields and dimension one fields can be obtained by taking appropriate derivatives with respect to couplings in the superpotential. So we now relax the condition that W be quasi-homogeneous, and consider arbitrary functions W (finite perturbations of the superconformal W by chiral primary fields). With no loss of generality we assume that W is deformed in such a way that there are no degenerate critical points, i.e., for the points at which $dW=0$, i.e.,

$$\frac{dW}{dx_i}=0 \text{ for all } i$$

[a] There are, in addition, operators which are defined as integral over one-dimensional cycles over the surface, but those we will not consider here.

the Hessian H does not vanish

$$H = \det(\partial_i \partial_j W) \neq 0 .$$

We can set the couplings on the superpotentials to any values at the end of our computations.

It is convenient to use an explicit description of a topological LG model. One version was done in Ref. 12. However, we need a different form. In fact, we simply take the LG Lagrangian before twisting and then declare that the fields have different spins (as is meant by twisting[5]). We take the action

$$\begin{aligned} S &= \int d^2 z d^4 \theta X^i \overline{X}^{\bar{i}} + \int d^2 z d^2 \theta W(X) + c.c. \\ &= \int d^2 z (|\partial x^i|^2 + |\partial_i W|^2) + (\rho^i \bar{\partial} \varphi^{\bar{i}} + \bar{\rho}^i \partial \overline{\psi}^{\bar{i}}) \\ &\quad + \rho^i \partial_i \partial_j W \bar{\rho}^j + \overline{\psi}^{\bar{i}} \partial_{\bar{i}} \partial_{\bar{j}} \overline{W} \psi^{\bar{j}} . \end{aligned} \tag{3}$$

After twisting $\psi^{\bar{i}}$ and $\overline{\psi}^{\bar{i}}$ have dimension zero and ρ^i and $\bar{\rho}^i$ are (left/right) one forms. The BRST transformation which renders the action topological is given by the standard transformation $G^\dagger + \overline{G}^\dagger$,

$$\begin{gathered} \delta x^{\bar{i}} = \psi^{\bar{i}} + \overline{\psi}^{\bar{i}} , \quad \delta \overline{\psi}^{\bar{i}} = -\partial_i W , \quad \delta \psi^{\bar{i}} = \partial_i W , \\ \delta \rho^i = -\partial x^i , \quad \delta \bar{\rho}^i = -\bar{\partial} x^i , \\ \delta x^i = 0 . \end{gathered}$$

For general topological sigma models[5] the semiclassical configuration dominate the path-integral and in fact give an exact answer to the path-integral. The idea is to rescale the action $S \to \gamma S$ and note that the variation of the action is a BRST-commutator and thus does not change the correlations. Taking the limit $\gamma \to \infty$ restricts to the field configurations which give zero action and those, in the context of topological sigma models, are nothing but holomorphic instantons. The simplest kinds of instantons that are always present on an arbitrary manifold are the constant maps. The moduli for such instantons are precisely the manifold itself, and the correlation in these sectors boil down to computing intersection of cycles over the manifold.

There is, however, a difference between the topological theory defined by (3) and topological sigma models, in that the action is *not* a BRST-commutator,[b] nevertheless it is still true that the $2d$ energy momentum tensor is a BRST-commutator[c]:

$$T_{zz} = -\delta(\partial x^{\bar{i}} \rho^i) , \; T_{z\bar{z}} = \delta(\partial_{\bar{i}} \overline{W} \psi^{\bar{i}}) , \; T_{\bar{z}\bar{z}} = -\delta(\bar{\partial} x^{\bar{i}} \bar{\rho}^i) .$$

So we cannot use the trick of rescaling the coefficient in front of the action by an arbitrary amount, as the action is not BRST-trivial. Instead, to compute the

[b] This is due to one of the F terms.

[c] Note that, since the trace of the energy momentum tensor is non-vanishing, this topological QFT is not a topological CFT whose properties have been discussed in detail in Ref. 11.

correlation function the trick in this case is to use the topological property and *rescale the worldsheet metric*

$$g \to \lambda^2 g.$$

The action changes according to

$$S = \int d^2 z(|\partial x^i|^2 + \lambda^2 |\partial_i W|^2) + (\rho^i \bar{\partial}\psi^{\bar{i}} + \bar{\rho}^i \partial \bar{\psi}^{\bar{i}}) + \rho^i \partial_i \partial_j W \bar{\rho}^j + \lambda^2 \bar{\psi}^{\bar{i}} \partial_{\bar{i}} \partial_{\bar{j}} \overline{W} \psi^{\bar{j}} . \tag{4}$$

The fact that λ is arbitrary allows us to take a large-λ limit and see from the bosonic piece of the action that the path-integral gets dominated by configurations where $dW \sim 0$ (in order to give a finite action) i.e., near configurations given by

$$x^i(z) = \text{const.} \quad \text{and} \quad \partial_j W(x^i) = 0 .$$

We are thus left with constant instantons which map the surface to the critical points of the superpotential. In particular, there is one instanton for each critical point of W, which in turn is equal to the number of chiral primary fields, as long as we restrict our attention to relevant perturbations of W. It is interesting to think about this in the following way: In contrast to the sigma model case where the moduli space of trivial instanton was isomorphic to a manifold, here the moduli space in question is a number of points, which suggests that the space itself is a number of points. This way of thinking is in accord with the way we think about the matrix model. According to Ref. 10 the n-matrix model is related to the minimal model with superpotential x^{n+1}. The deformation of this theory gives n isolated critical points. So the manifold in question consists of n-point, which could be identified with the number of matrices in the matrix model. It would be interesting to see if this analogy can be pushed further. In particular from this point of view it seems natural to expect that *it should be possible to describe theories with* $\hat{c} \geq 1$ *using a finite number of matrices*, as there seem to be one matrix corresponding to each critical point. In other words, LG theories with $\hat{c} \geq 1$ do have finitely many chiral primary fields and there should be a matrix model analog. In fact, probably more can be said: There is a (generalized) Dynkin diagram associated to every LG theory (for any $\hat{c}$) as is familiar from singularity theory. The number of nodes corresponds to the number of chiral primary fields, and the links between them represent the intersections of the mid-dimension homologies of the resolved singularity.[13] It is tempting to conjecture that there is a matrix model with the number of matrices equal to the number of nodes and with the interactions between them *dictated* by the links of the Dynkin diagram. This conjecture is motivated from the $\hat{c} \leq 1$ theories. Unfortunately at the present we do not know how to deal with a general matrix model even for all $\hat{c} < 1$ models, however, this suggests that we should try to develop techniques for dealing with them and their generalizations which would correspond to $\hat{c} \geq 1$.

Going back to our computation, we have seen that the path-integral is dominated by constant instantons at critical points of W. Near each of these instantons, the path-integral (for the non-zero modes) becomes simply the ratio of bosonic and fermionic determinants which in fact precisely cancel. The only subtlety is that the zero modes of both bosonic and fermionic modes should be treated separately. The constant modes of the bosonic field gives us

$$\int \prod d^2x^i \exp(-|\lambda\partial_i W|^2),$$

where, noting that λ is large, $\partial_i W$ can be replaced by its linear term and the above integral reduces to a simple Gaussian with the answer

$$Z_b = \lambda^{-2n}(H\overline{H})^{-1},$$

where H is the Hessian defined above evaluated at the critical point.

The fermionic zero modes are also easy to deal with. Though here we have to recall that the zero modes for $\rho^i(\overline{\rho}^i)$ correspond to the holomorphic (anti-holomorphic) one-form and we have g of them, and the zero modes for $\psi^{\bar{i}}(\overline{\psi}^{\bar{i}})$ are just the constant functions and that is only one. Therefore we find

$$Z_f = \left|\int \prod d\rho^i d\overline{\rho}^i \exp(-\rho^i\partial_i\partial_j W\overline{\rho}^j)\right|^g \times \int \prod d\overline{\psi}^{\bar{i}} d\psi^{\bar{i}} \exp(-\lambda^2\overline{\psi}^{\bar{i}}\partial_{\bar{i}}\partial_{\bar{j}}\overline{W}\psi^{\bar{j}}) = \lambda^{2n}H^g\overline{H}.$$

Putting the bosonic and fermionic zero mode contributions together we finally obtain (the λ independent answer)

$$Z = Z_b Z_f = H^{g-1}.$$

The appearance of the power $(g-1)$ can be intuitively understood as the reflection of the fact that there is a U(1) charge violation by $g-1$ units, and that is equivalent to inserting $g-1$ copies of spectral flow in the genus-g amplitude. Moreover, the operator H in the superconformal theory is the operator which corresponds to the spectral flow[1] and this seems to hold true even in the perturbed theory.

So just to review, we have found that we can evaluate any correlation by restricting our path-integral to constant maps corresponding to the critical points of W, and the contribution of the path-integral near each of these configurations is H^{g-1}. Therefore, the correlations from a $2d$ theory reduce to a zero-dimensional theory, summed over critical points of W and we obtain for arbitrary correlations

$$\langle F_1(x_i)F_2(x_i)\dots F_N(x_i)\rangle_g = \sum_{dW=0} F_1F_2\dots F_N H^{g-1}, \tag{5}$$

where F_l are polynomials in the superfields x_i and on the right-hand side of (5) are to be evaluated at the critical points. There is a simple way to recast Eq. (5) to make contact with the operator formulation of topological theories. Let α label the set of critical points of W, i.e., where $dW = 0$. For each α we consider the ket $|\alpha\rangle$. This Hilbert space is in fact isomorphic to the physical Hilbert space of our theory, as there is a one-to-one correspondence between these minima and the chiral primary fields. In other words, instead of realizing the space of chiral primary fields as identified with monomials in x^i we can take them to be this new Hilbert space.

In fact the appearance of this Hilbert space is clear from the fact that our path-integrals could be restricted to this class. This Hilbert space gives a description of the physical states in a BRST equivalent way to the description involving the fields x^i. But the advantage of choosing the basis of critical points as the Hilbert space is that the fields x^i are diagonal in this basis. But it is clear that we can rewrite (5) in a basis independent way as

$$\langle F_1(x^i)\dots F_N(x^i)\rangle_g = \mathrm{Tr}\ F_1(x^i)\dots F_N(x^i)H^{g-1}\,. \tag{6}$$

Written in this way it makes contact with the general form one expects for the correlation functions of a topological matter theory derived in Ref. 7 where H is to be identified as the operator one gets from a genus-one surface with two punctures (this is also familiar from the study of Verlinde algebra[14]). Moreover, this guarantees that we are not missing a genus-dependent normalization from our path-integral derivation (modulo the possibility of rescaling the coupling constant).

Let us discuss some simple properties of (5). First we note that if we compute any correlations involving fields proportional to $\partial_i W$ it vanishes, as the sum is over the critical points of W. This indeed is consistent with the statement that the observable states in the theory are given by the ring (1). Also note that the partition function at genus-one, with no operators inserted, is simply equal to the number of critical points. Let us ask how continuously (5) varies as we change W. In particular suppose we start from a quasihomogeneous W, which corresponds to a (twisted) superconformal theory. If we add operators of charge less than one, the degenerate critical point at the origin of field space split to a number of critical points near the origin. So all the correlations (5) change continuously under such perturbations. However, the same cannot be said about perturbations with fields of charge bigger than one. Such operators necessarily exist among the chiral primary fields for theories with $\hat{c} > 1$ (e.g., there is always a field with[1] $q = \hat{c}$). These operators before perturbation have dimension bigger than one and cannot be added to the action in a renormalizable way. In the twisted theory one might have thought that their dimension is one and can be added with no penalty. However, adding such terms gives rise to *new* critical points coming from infinity in the field configuration space. Therefore the correlations, (5), do not change continuously under such perturbations. For example, consider a theory with W given by

$$W = \frac{x^n}{n} + \frac{y^n}{n} - \varepsilon\frac{(xy)^\alpha}{\alpha}\,,$$

where ε is a small perturbation parameter and we take $\alpha > n/2$ so that the perturbation is by a chiral primary field with dimension bigger than one. For $n > 4$ the unperturbed theory has $\hat{c} > 1$ (the general formula is[1] $\hat{c} = \Sigma(1 - 2q_i)$ which gives $\hat{c} = 2 - (4/n)$). We now see that we get additional critical points from infinity (the reader can easily verify that $x = y = \varepsilon^{-1/(2\alpha-n)}$ is a new critical point). Therefore (5) will not change continuously under such perturbations (for instance, the genus-one partition function which counts the number of critical points of W jumps). Put differently, the dimension of the chiral ring defined by (1) changes. This seems to indicate that only fields with charge less than or equal to one can be added to the superpotential. It would be interesting to see if this is a topological reflection of the $d=1$ barrier in the usual formulation of $2d$ gravity. At any rate, in the following we will restrict our attention to perturbations which can be obtained by adding operators of charge less than (or equal to) one.

On the face of it (5) looks quite complicated. In particular one cannot expect to be able to find all the critical points of W explicitly. However, from the formulation of it given by (6) it is clear that the answer is quite simple. This is because we can compute the trace in any basis, and in particular if we compute it in the basis of the monomials, where each monomial is represented by a matrix in this space (using $dW=0$), we have a simple trace to compute. There is another way to do this, which is even simpler. Let us start with the case of one variable. In that case (5) can be rewritten as

$$\begin{aligned}&\langle F_1(x)\dots F_N(x)\rangle_g \\ &= \sum_{dW=0}(F_1\dots F_N(\partial^2W)^g)/\partial^2 W = \mathrm{res}\left(\frac{F_1\dots F_N(\partial^2 W)^g}{\partial W}\right)\end{aligned} \tag{7}$$

(where the residue involves taking a contour at large radius). This is the same result obtained in Ref. 11 for the topological CFT based on the LG model. This reformulation of (5) has the advantage of making it explicitly possible to compute (by expanding the polynomials at large x and taking the contour around infinity). There is a similar trick which works for the case of many fields. We can write (5) using a residue for n variables[15] namely[d]

$$\langle F(x)\rangle_g = \int \frac{dx_1 dx_2\dots dx_n}{\partial_1 W \partial_2 W\dots\partial_n W}\, F(x)H^{g-1}(x). \tag{8}$$

(This residue has been investigated previously in the context of $N=2$ superconformal LG models in Ref. 4.) Let us define a generalized residue by

$$\mathrm{res}_W(F(x)) = \langle F(x)\rangle_{g=0}\,.$$

It is clear that the higher genus correlation functions can be computed using this residue by multiplying F with a factor H^g. Now we will show a number of properties hold for this residue which allows us to effectively compute it. The following properties hold,

$$\begin{aligned}&\text{(i)}\ \ \mathrm{res}_W(aF+bG) = a\,\mathrm{res}_W(F) + b\,\mathrm{res}_W(G)\,,\\ &\text{(ii)}\ \ \mathrm{res}_W(F) = \mathrm{res}_W(F+G\partial_i W)\,,\\ &\text{(iii)}\ \ \mathrm{res}_W(H) = \mu\,,\\ &\text{(iv)}\ \ \mathrm{res}_W(F) = 0\ \text{ if }\ Q_F < \hat{c}\,.\end{aligned} \tag{9}$$

[d] This can be defined precisely using the Dolbeaut isomorphism discussed in Ref. 15 as an integral of a $2n-1$ form in the boundary of a ball of a large radius.

Property (i) is obvious, property (ii) follows from formula (5). Property (iii) where H denotes the Hessian and μ is the criticality index of (the unperturbed) W which is the same as the number of chiral primary fields, again follows in a straightforward manner from (5). The only property in need of explanation is property (iv), which states that if F is a field with charge Q (well-defined in the unperturbed theory) less than the maximum allowed, i.e., $\hat{c}$, then its residue vanishes. This property is clear in the unperturbed theory as it violates the selection rule (2). We are stating that it is true even *after* perturbing W. The proof of that is simple using the representation of (5) given by (8). We simply note that we can deform all the contours[e] by rescaling the x_i according to

$$x_i \to \lambda^{q_i} x_i \,,$$

where q_i denotes the charge of x_i in the unperturbed theory. Using the quasi-homogeneity property of the unperturbed W and recalling that we are allowing perturbations with monomials of charge less than one, allows us to see that in the large-λ limit we end up with a W which in the leading order is unperturbed and the integral gets a prefactor

$$\lambda^{Q-\Sigma(1-2q_i)} = \lambda^{Q-\hat{c}} \to 0$$

and so we obtain the desired result.

Now we show that the properties (9) are in fact sufficient for the computation of the residue. First we recall an important fact about the $N = 2$ unperturbed chiral ring: Any field F with charge bigger than $\hat{c}$ vanishes by adding terms proportional to $\partial_i W$. After perturbation this is no longer true, but since we have perturbed by terms with charge less than one, the fact that it vanishes before perturbation means that by the addition of the same terms proportional to $\partial_i W$ we see that this field is equivalent to a field of lower charge. In this way we can inductively find a representative of any field which involves only states with charge less than or equal to $\hat{c}$. The state with charge $\hat{c}$ is equivalent to a multiple of H up to addition of lower charge states. This follows from the fact that in the unperturbed theory there is a *unique* state with charge $\hat{c}$.[1] So we see that for any F we have

$$F = \alpha H + G^i \partial_i W + (Q < \hat{c} \text{ fields})$$

then according to (9) we obtain

$$\mathrm{res}_W (F) = \alpha\mu \,.$$

This concludes our derivation of a relatively simple method for the computation of the correlations (5). Note that the final result applies also to the case where the W has degenerate critical points.

One can generalize the considerations of this paper in many directions. One direction is to consider twisted LG orbifolds.[16] It seems that the considerations of this paper do generalize to these cases and, when $\hat{c}$ is integer, this will give a realization of topological sigma models in a simple way.

[e] This can be made precise along the lines indicated in Ref. 15.

Probably the most urgent question is how to couple topological LG to topological gravity. In this direction, one would need to understand the exact relation between the model proposed here and the topological conformal theory related to it. Also, it would be interesting to see in exactly what way are the twisted and untwisted $N = 2$ supergravity theories related. This would be interesting to uncover also in connection with the critical $N = 2$ strings.[17] It would be interesting to see if techniques similar to the ones developed here for understanding correlations of topological LG theories can be combined with the techniques developed in Ref. 18 to elucidate the structure of $2d$ gravity theories and in particular prove the conjectures made in Ref. 19.

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