6

Path-integral quantisation and Feynman rules: scalar and spinor fields

In this chapter we shall quantise scalar and spinor fields by path-integral quantisation, in analogy with the treatment of quantum mechanics in the last chapter. This will enable us to find the propagators for the scalar and spinor fields. We shall then introduce interactions, treat them perturbatively, and find the Feynman rules. After considering spinor fields in more detail, we conclude by calculating the pion–nucleon scattering cross section.

6.1 Generating functional for scalar fields[‡]

Suppose the scalar field $\phi(x)$ has a source, in the sense of §5.5, $J(x)$, then, analogously to expression (5.68), we may define the vacuum-to-vacuum transition amplitude in the presence of the source J as

$$Z[J] = \int \mathcal{D}\phi \exp\left\{\mathrm{i}\int \mathrm{d}^4x\left[\mathcal{L}(\phi) + J(x)\phi(x) + \frac{\mathrm{i}}{2}\varepsilon\phi^2\right]\right\}$$
$$\propto \langle 0, \infty | 0, -\infty\rangle^J. \tag{6.1}$$

Here we have made the substitution $\mathcal{D}q(t) \rightarrow \mathcal{D}\phi(x^\mu)$, and have put $\hbar = 1$. $\mathcal{L}$ is the Klein–Gordon Lagrangian (3.10). So, instead of dividing time up into segments, we divide space and time up, and Minkowski space is broken down into 4-dimensional cubes of volume δ^4 in each of which ϕ is taken to be constant

$$\phi \sim \phi(x_i, y_j, z_k, t_l).$$

Derivatives are approximated by, for example,

$$\left.\frac{\partial\phi}{\partial x}\right|_{i,j,k,l} \approx \frac{1}{\delta}[\phi(x_i + \delta, y_j, z_k, t_l) - \phi(x_i, y_j, z_k, t_l)].$$

Now let us replace the four indices (i, j, k, l) formally by one index n, and write

$$\mathcal{L}(\phi(x_i, y_j, z_k, t_l), \partial_\mu\phi(x_i, y_j, z_k, t_l)) = \mathcal{L}(\phi_n, \partial_\mu\phi_n) = \mathcal{L}_n.$$

‡ In this section and for much of this chapter I have drawn on the lectures of J. Wess, Karlsruhe University, 1974 (unpublished), and Popov (1983).

If i, j, k and l each take on N values, then n takes on N^4 values, so the action $S = \int \mathcal{L}\, d^4x$ becomes

$$S \approx \sum_{n=1}^{N^4} \delta^4 \mathcal{L}_n.$$

The vacuum-to-vacuum amplitude $Z[J]$ is then

$$Z[J] = \lim_{N\to\infty} \int \prod_{n=1}^{N^4} d\phi_n \exp\left\{ i \sum_{n=1}^{N^4} \delta^4 \left(\mathcal{L}_n + \phi_n J_n + \frac{i}{2}\varepsilon\phi_n^2 \right) \right\}. \tag{6.2}$$

Let us now calculate this for a free particle (field), for which

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2).$$

The corresponding vacuum-to-vacuum amplitude is (taking the limit $N \to \infty$)

$$Z_0[J] = \int \mathcal{D}\phi \exp\left(i \int \{ \tfrac{1}{2}[\partial_\mu\phi\partial^\mu\phi - (m^2 - i\varepsilon)\phi^2] + \phi J \}\, d^4x \right). \tag{6.3}$$

We use the identity

$$\int \partial_\mu\phi\partial^\mu\phi\, d^4x = \int \partial_\mu(\phi\partial^\mu\phi)\, d^4x - \int \phi\Box\phi\, d^4x,$$

and convert the first term on the right to a surface integral, using the 4-dimensional version of Gauss' theorem. This surface term vanishes if $\phi \to 0$ at infinity, so we have

$$\int \partial_\mu\phi\partial^\mu\phi\, d^4x = -\int \phi\Box\phi\, d^4x, \tag{6.4}$$

giving

■ $$Z_0[J] = \int \mathcal{D}\phi \exp\left\{ -i \int [\tfrac{1}{2}\phi(\Box + m^2 - i\varepsilon)\phi - \phi J]\, d^4x \right\}. \tag{6.5}$$

(Note that the field ϕ in this generating functional does *not* obey the Klein–Gordon equation (3.8).) To evaluate $Z_0[J]$, let us change ϕ to

$$\phi(x) \to \phi(x) + \phi_0(x). \tag{6.6}$$

Using the fact that

$$\int \phi_0(\Box + m^2 - i\varepsilon)\phi\, d^4x = \int \phi(\Box + m^2 - i\varepsilon)\phi_0\, d^4x,$$

which follows from an argument analogous to that leading to (6.4), we have, under (6.6),

$$\int [\tfrac{1}{2}\phi(\Box + m^2 - i\varepsilon)\phi - \phi J]\, d^4x \to \int [\tfrac{1}{2}\phi(\Box + m^2 - i\varepsilon)\phi + \phi(\Box + m^2 - i\varepsilon)\phi_0 + \tfrac{1}{2}\phi_0(\Box + m^2 - i\varepsilon)\phi_0 - \phi J - \phi_0 J]\, d^4x.$$

If ϕ_0 is now chosen to satisfy

$$(\Box + m^2 - i\varepsilon)\phi_0(x) = J(x) \tag{6.7}$$

then this becomes

$$\int [\tfrac{1}{2}\phi(\Box + m^2 - i\varepsilon)\phi - \tfrac{1}{2}\phi_0 J]\, d^4x. \tag{6.8}$$

Now the solution to (6.7) is

$$\phi_0(x) = -\int \Delta_F(x - y) J(y)\, d^4y \tag{6.9}$$

where $\Delta_F(x - y)$ is the so-called Feynman propagator, obeying

$$(\Box + m^2 - i\varepsilon)\Delta_F(x) = -\delta^4(x). \tag{6.10}$$

Substituting (6.9) into (6.8), we see that the exponent in (6.5) is $-i$ times

$$\tfrac{1}{2}\int \phi(\Box + m^2 - i\varepsilon)\phi\, d^4x + \tfrac{1}{2}\int J(x)\Delta_F(x - y)J(y)\, d^4x\, d^4y. \tag{6.11}$$

So $Z_0[J]$ now takes the form (where dx stands for d^4x, and similarly for y)

$$Z_0[J] = \exp\left[-\frac{i}{2}\int J(x)\Delta_F(x - y)J(y)\, dx\, dy\right] \times \int \mathcal{D}\phi \exp\left[-\frac{i}{2}\int \phi(\Box + m^2 - i\varepsilon)\phi\, dx\right]. \tag{6.12}$$

The superiority of this expression to (6.5) lies in the fact that here $Z_0[J]$ has separated into two factors, one depending on ϕ only, and the other on J only. In fact, the integral involving ϕ is actually a *number*, let us call it N, since the integral has been taken over all functions ϕ. Finally, then, we have

■ $$Z_0[J] = N \exp\left[-\frac{i}{2}\int J(x)\Delta_F(x - y)J(y)\, dx\, dy\right]. \tag{6.13}$$

Since we are only interested in normalised transition amplitudes, the value of N, in the applications we consider, is irrelevant.

The aim of this section, to derive equation (6.13) for the vacuum-to-vacuum transition amplitude, has now been achieved. In the next section we shall show how the same equation is derived using rather higher brow mathematical techniques of functional integration. Before finishing this section, however, we shall consider briefly the Feynman propagator $\Delta_F(x)$, defined by (6.10). It is easy to see that $\Delta_F(x)$ has a Fourier representation

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ikx}}{k^2 - m^2 + i\varepsilon}. \tag{6.14}$$

Note that the presence of the iε term, which was put in originally (see (6.1)) to ensure vacuum-to-vacuum boundary conditions, dictates the path of integration round the poles at $k_0 = \pm(\mathbf{k}^2 + m^2)^{1/2}$. In fact, the poles are at $k_0^2 = \mathbf{k}^2 + m^2 - \mathrm{i}\varepsilon$, and therefore at

$$k_0 = \pm(\mathbf{k}^2 + m^2)^{1/2} \mp \mathrm{i}\delta = \pm E \mp \mathrm{i}\delta. \tag{6.15}$$

This is shown in Fig. 6.1, where the integration path of k_0 is along its real axis, as shown. In the limit $\delta \rightarrow 0$, i.e. $\varepsilon \rightarrow 0$, which is implied in the expression (6.14), the poles reach the real axis, and in this case the integration path is as shown in Fig. 6.2.

There is another way of incorporating the vacuum-to-vacuum boundary conditions, which is to rotate the time axis, instead of through the small angle δ, shown in Fig. 5.7, through an angle of $\pi/2$, so that $t \rightarrow -\mathrm{i}\infty$. Defining

$$x_4 = \mathrm{i}t = \mathrm{i}x_0 \tag{6.16}$$

this limit is $x_4 \rightarrow \infty$. This space–time, with an imaginary time axis, is *Euclidean*, for the invariant interval is

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 = -\sum_{\mu=1}^{4}(\mathrm{d}x^\mu)^2.$$

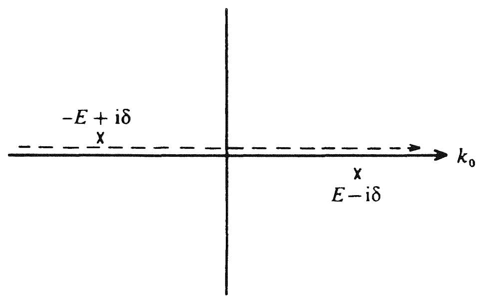


Fig. 6.1. Integration path along the real k_0 axis in the definition of $\Delta_F(x)$.

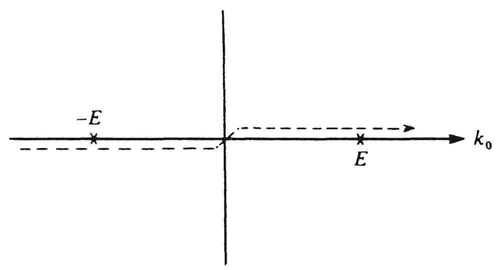


Fig. 6.2. New integration path on taking the limit $\varepsilon \rightarrow 0(\delta \rightarrow 0)$.

Defining, in addition,

$$k_4 = -\mathrm{i}k_0, \tag{6.17}$$

giving, in the Euclidean space,

$$k^2 = -(k_1^2 + k_2^2 + k_3^2 + k_4^2) = -k_E^2 \tag{6.18}$$

and

$$\mathrm{d}^4 k_E = \mathrm{d}^3 k\, \mathrm{d}k_4 = -\mathrm{i}\,\mathrm{d}^4 k,$$

the Feynman propagator is

$$\Delta_F(x) = \frac{-\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4 k_E \frac{\mathrm{e}^{-\mathrm{i}k_E x}}{k_E^2 + m^2}. \tag{6.19}$$

Here there is no problem of choosing a contour, for the poles are not on the real axis, but at $k_4 = \pm\mathrm{i}(\mathbf{k}^2 + m^2)^{1/2}$. Referring to equation (6.3) for $Z_0[J]$, noting that $\mathrm{d}^4x = -\mathrm{i}\,\mathrm{d}^4x_E$ and that $(\partial^\mu\phi)^2 = -(\partial_E^\mu\phi)^2$, the Euclidean transition amplitude is

$$Z_{0_E}[J] = \int \mathscr{D}\phi \exp\left(-\int\{\tfrac{1}{2}[(\partial_E^\mu\phi)^2 + m^2\phi^2] - \phi J\}\,\mathrm{d}^4x_E\right). \tag{6.20}$$

The exponent in the integrand is negative definite, so the integrand converges; the role of the ε term in (6.3) was, in effect, to make the integrand converge.

6.2 Functional integration

We shall now generalise the usual formulae for Gaussian integration over a finite number of variables, to formulae for functional integrals, and then show how (6.12) (or (6.13)) follows from (6.5). To begin, we have, from (5A.1),

$$\int \mathrm{e}^{-ax^2/2}\,\mathrm{d}x = \left(\frac{2\pi}{a}\right)^{1/2} \tag{6.21}$$

(the limits $-\infty$ and ∞ are to be understood in this, and all subsequent, integrals). Now we take the product of n such integrals (with all $a_i > 0$)

$$\int \exp\left(-\tfrac{1}{2}\sum_n a_n x_n^2\right)\mathrm{d}x_1 \ldots \mathrm{d}x_n = \frac{(2\pi)^{n/2}}{\prod_{i=1}^{n} a_i^{1/2}}. \tag{6.22}$$

Let A be a diagonal matrix with elements $a_1, \ldots, a_n$, and let x be an n-vector $(x_1, \ldots, x_n)$. Then the exponent above is the inner product

$$\sum_n a_n x_n^2 = (x, Ax)$$

and the determinant of A is

$$\det A = a_1 a_2 \ldots a_n = \prod_{i=1}^{n} a_i.$$

Equation (6.22) then becomes

$$\int e^{-(x,Ax)/2}\, d^n x = (2\pi)^{n/2}(\det A)^{-1/2}. \tag{6.23}$$

Since this holds for any diagonal matrix, it also holds for any real symmetric, positive, non-singular matrix. Defining the measure

$$(dx) = d^n x (2\pi)^{-n/2},$$

equation (6.23) becomes

$$\int e^{-(x,Ax)/2}\,(dx) = (\det A)^{-1/2}. \tag{6.24}$$

This equation may be extended to quadratic forms

$$Q(x) = \tfrac{1}{2}(x, Ax) + (b, x) + c. \tag{6.25}$$

The minimum of Q lies at $\bar{x} = -A^{-1}b$ and

$$Q(x) = Q(\bar{x}) + \tfrac{1}{2}(x - \bar{x}, A(x - \bar{x}))$$

so we have

$$\int \exp\{-[\tfrac{1}{2}(x, Ax) + (b, x) + c]\}(dx) = \exp[\tfrac{1}{2}(b, A^{-1}b) - c](\det A)^{-1/2}. \tag{6.26}$$

This equation is analogous to (5A.3).

Let us make a small digression to the case of Hermitian matrices. Squaring (6.21) gives

$$\int e^{-a(x^2+y^2)/2}\, dx\, dy = \frac{2\pi}{a}.$$

Putting $z = x + iy$, $z^* = x - iy$, $dx\, dy = -\tfrac{1}{2}i\, dz^*\, dz$, we have

$$\int e^{-az^*z} \frac{dz^*}{(2\pi i)^{1/2}} \frac{dz}{(2\pi i)^{1/2}} = \frac{1}{a}. \tag{6.27}$$

The generalisation of this formula is that if A is a positive definite Hermitian matrix and we define the measure $(dz) = d^n z(2\pi i)^{-n/2}$, then

$$\int e^{-(z^*,Az)}\,(dz^*)(dz) = (\det A)^{-1}. \tag{6.28}$$

The formulae written down so far are rigorously true; they are simply the result of generalising the integration space from one dimension to a finite-dimensional vector space. We now assume that we can generalise the above formulae to an infinite-dimensional function space. This actually needs a careful mathematical justification, but, assuming that this can be made, then, if the generalisation is to the case of a single real variable $f(t)$, the inner product (f, f) is

$$(f, f) = \int [f(t)]^2 \, dt.$$

In our case, we are concerned with real functions of space–time, $\phi(x^\mu)$, so we have

$$(\phi, \phi) = \int [\phi(x)]^2 \, d^4x. \tag{6.29}$$

The generalisation of equation (6.24) is

$$\int \mathscr{D}\phi \exp\left[-\tfrac{1}{2}\int \phi(x) A \phi(x) \, dx\right] = (\det A)^{-1/2}; \tag{6.30}$$

A may be, in general, a differential operator. If ϕ is a complex field, we instead generalise (6.28) to give

$$\int \mathscr{D}\phi^* \mathscr{D}\phi \exp\left[-\int \phi^*(x) A \phi(x) \, dx\right] = (\det A)^{-1}. \tag{6.31}$$

We are now in a position to prove (6.12) from (6.5). The exponent of the integrand of (6.5) is a quadratic form, so we employ equation (6.26) (or, rather, its functional generalisation), with $A = i(\Box + m^2 - i\varepsilon)$, $b = -iJ$, $c = 0$, giving

$$Z_0[J] = \exp\left[\frac{i}{2}\int J(x)(\Box + m^2 - i\varepsilon)^{-1} J(y) \, dx \, dy\right][\det i(\Box + m^2 - i\varepsilon)]^{-1/2}.$$

The determinant term is defined by (6.30), and we have, from (6.10),

$$(\Box + m^2 - i\varepsilon)^{-1} = -\Delta_F(x - y), \tag{6.32}$$

so

$$Z_0[J] = \exp\left[-\frac{i}{2}\int J(x)\Delta_F(x - y)J(y) \, dx \, dy\right]$$
$$\times \int \mathscr{D}\phi \exp\left[-\frac{i}{2}\int \phi(\Box + m^2 - i\varepsilon)\phi \, dx\right]$$

which is equation (6.12). We recall that the last factor is just a number N, so $Z_0[J]$ can be written in the form (6.13).

6.3 Free particle Green's functions

We shall now show that the amplitude $Z_0[J]$ is the 'generating functional' for the free particle Green's functions, which will be defined as we proceed. We begin by expanding equation (6.13), to give

$$Z_0[J] = N\left\{1 - \frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2!}\left(\frac{\mathrm{i}}{2}\right)^2\left[\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]^2 + \frac{1}{3!}\left(\frac{-\mathrm{i}}{2}\right)^3\left[\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]^3 + \cdots\right\}. \quad (6.33)$$

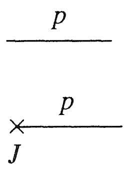
Introducing the Fourier transform of $J(x)$ by

$$J(x) = \int J(p)\,\mathrm{e}^{-\mathrm{i}px}\,\mathrm{d}^4p, \quad (6.34)$$

we have, using (6.14) and temporarily reverting to d^4x instead of $\mathrm{d}x$, etc.,

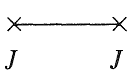
$$-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}^4x\,\mathrm{d}^4y = \frac{-\mathrm{i}}{2(2\pi)^4}\int\frac{J(p_1)\,\mathrm{e}^{-\mathrm{i}(p_1+k)x}\,\mathrm{e}^{-\mathrm{i}(p_2-k)y}\,J(p_2)}{k^2-m^2+\mathrm{i}\varepsilon} \times \mathrm{d}^4p_1\,\mathrm{d}^4p_2\,\mathrm{d}^4k\,\mathrm{d}^4x\,\mathrm{d}^4y$$
$$= -\frac{\mathrm{i}}{2}(2\pi)^4\int\frac{J(-k)J(k)}{k^2-m^2+\mathrm{i}\varepsilon}\,\mathrm{d}^4k \quad (6.35)$$

in which we have integrated over x and y to give two delta functions, and then over p_1 and p_2. We may represent this diagrammatically by using the following rules of correspondence (Feynman rules in momentum space).



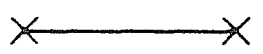
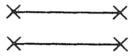
$$\frac{1}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon},$$
$$\mathrm{i}(2\pi)^4J(p). \quad (6.36)$$

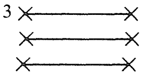
These Feynman rules may be compared with those in (5.45). Since the non-relativistic limit of $p^2 - m^2$ is $2m(T - \mathbf{p}^2/2m)$ (T = kinetic energy), the propagator above is clearly the (relativistic) propagator for *one particle*. The expression (6.35) then corresponds to the diagram

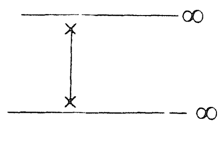
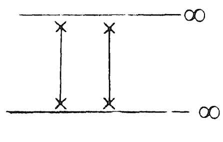
$\frac{1}{2}$ 

(6.37)

The vacuum-to-vacuum amplitude (6.33) may then be written in terms of Feynman diagrams (ignoring N)

$$Z_0 = 1 + \frac{1}{2}$$  $$+ \frac{1}{2!}\left(\frac{1}{2}\right)^2$$ 

$$+ \frac{1}{3!}\left(\frac{1}{2}\right)^3$$  $$+ \cdots$$

$$= 1 + \frac{1}{2}$$  $$+ \frac{1}{2!}\left(\frac{1}{2}\right)^2$$ 

$$+ \cdots \tag{6.38}$$

where, in the last line, we have resorted to a pictorial representation reminiscent of Fig. 5.6. We shall now show that the implied suggestion is correct; in other words, it is correct to interpret this series as the propagation of one particle between sources, the propagation of two particles between sources, and so on. We therefore have a *many-particle theory*, consistent with our original philosophy of using a field, which yielded particles on quantisation. Each term in the above series is a Green's function so $Z_0[J]$ is a *generating functional* for the Green's functions of the theory.

To understand how to interpret the power series expansion of a functional, let us first recall the formula for the power series expansion of a function, say $F(y_1, \ldots, y_k)$ of k variables $y_1, \ldots, y_k$. It is

$$F\{y\} = F(y_1, \ldots, y_k) = \sum_{n=0}^{\infty} \sum_{i_1=0}^{k} \cdots \sum_{i_n=0}^{k} \frac{1}{n!} T_n(i_1, \ldots, i_n) y_{i_1} \ldots y_{i_n}$$

where

$$T_n(i_1, \ldots, i_n) = \left.\frac{\partial^n F\{y\}}{\partial y_{i_1} \ldots \partial y_{i_n}}\right|_{y=0}. \tag{6.39}$$

Going over to the case of continuously many variables, $i \to x_i$, y_i $(i = 1, \ldots, k) \to y(x)$, $-\infty < x < \infty$ and $\sum_i \to \int \mathrm{d}x$, and we obtain the power series expansion of a functional

$$F[y] = \sum_{n=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \frac{1}{n!} T_n(x_1, \ldots, x_n) y(x_1) \ldots y(x_n)$$

where

$$T_n(x_1, \ldots, x_n) = \left.\frac{\delta}{\delta y(x_1)} \cdots \frac{\delta}{\delta y(x_n)} F[y]\right|_{y=0}. \tag{6.40}$$

$F[y]$ is called the *generating functional* of the functions $T_n(x_1, \ldots, x_n)$.

Now we return to $Z_0[J]$. It is convenient here to settle the question of its normalisation. $Z_0[J]$ is the vacuum-to-vacuum transition amplitude in the presence of the source, J, so it is sensible to normalise it to $Z_0[J = 0] = 1$. In that case, referring to (5.61), we may *define* $Z_0[J]$ by

$$Z_0[J] = \langle 0, \infty | 0, -\infty \rangle^J \tag{6.41}$$

which automatically obeys

$$Z_0[0] = 1. \tag{6.42}$$

Equations (6.5) and (6.13) must then be rewritten as

$$Z_0[J] = \frac{\int \mathcal{D}\phi \exp\left\{-\mathrm{i}\int [\frac{1}{2}\phi(\Box + m^2 - \mathrm{i}\varepsilon)\phi - \phi J]\,\mathrm{d}x\right\}}{\int \mathcal{D}\phi \exp\left[-\mathrm{i}\int \frac{1}{2}\phi(\Box + m^2 - \mathrm{i}\varepsilon)\phi\,\mathrm{d}x\right]} \tag{6.43}$$

and

$$Z_0[J] = \exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x - y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]. \tag{6.44}$$

These new definitions clearly obey (6.42). Then $Z_0[J]$, as defined by (6.44), is clearly the generating functional for

$$\tau(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n Z_0[J]}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J=0}. \tag{6.45}$$

At this point we refer back to equation (5.76), which was derived for its use here. The analogous equation in the case of fields, and using our new normalisation, is

$$\frac{\delta^n Z_0[J]}{\delta J(x_1) \ldots J(x_n)}\bigg|_{J=0} = \mathrm{i}^n \langle 0 | T(\phi(x_1) \ldots \phi(x_n)) | 0 \rangle \tag{6.46}$$

or

$$\langle 0 | T(\phi(x_1) \ldots \phi(x_n)) | 0 \rangle = \frac{1}{\mathrm{i}^n} \frac{\delta^n Z_0[J]}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J=0}. \tag{6.47}$$

Comparing (6.45) and (6.47) we have

$$\tau(x_1, \ldots, x_n) = \langle 0 | T(\phi(x_1) \ldots (x_n)) | 0 \rangle. \tag{6.48}$$

These quantities, the vacuum expectation values of time-ordered products of field operators, are called the *Green's functions* or n-point *functions* of the theory. They are closely related to the S matrix elements – we shall see this connection later, when we introduce interactions. We then have

$$Z_0[J] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n J(x_1) \ldots J(x_n) \tau(x_1, \ldots, x_n) \tag{6.49}$$

which expresses Z as the generating functional of the Green's functions τ. This equation corresponds to the pictorial equation (6.38).

Let us now calculate some n-point functions. We start with the 2-point function

$$\tau(x, y) = -\left.\frac{\delta^2 Z_0[J]}{\delta J(x) \delta J(y)}\right|_{J=0} \tag{6.50}$$

with $Z_0[J]$ given by (6.44). (Recall again that we are still concerned with a *free* field theory, since we started with the free Lagrangian, and thus with (6.3). The expressions we shall find below refer, therefore, to the free particle Green's functions. The corresponding Green's functions for interacting fields will differ from these, and will be found later.)

From (6.44) we have

$$\begin{aligned}
\frac{1}{\mathrm{i}} \frac{\delta Z_0[J]}{\delta J(x)} &= \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} \exp\left[-\frac{\mathrm{i}}{2} \int \mathrm{d}x_1 \, \mathrm{d}x_2 J(x_1) \Delta_\mathrm{F}(x_1 - x_2) J(x_2)\right] \\
&= -\int \Delta_\mathrm{F}(x - x_1) J(x_1) \, \mathrm{d}x_1 \exp\left[-\frac{\mathrm{i}}{2} \int \mathrm{d}x_1 \, \mathrm{d}x_2 J(x_1) \Delta_\mathrm{F}(x_1 - x_2) J(x_2)\right], \\
\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(y)} Z_0[J] &= \mathrm{i} \Delta_\mathrm{F}(x - y) \exp\left(-\frac{\mathrm{i}}{2} \int J \Delta_\mathrm{F} J\right) \\
&\quad + \int \Delta_\mathrm{F}(x - x_1) J(x_1) \, \mathrm{d}x_1 \int \Delta_\mathrm{F}(y - x_1) J(x_1) \, \mathrm{d}x_1 \\
&\quad \times \exp\left(-\frac{\mathrm{i}}{2} \int J \Delta_\mathrm{F} J\right).
\end{aligned} \tag{6.51}$$

Here we have abbreviated the notation in the exponent, for simplicity. Finally, putting $J = 0$,

$$\left.\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)} \frac{1}{\mathrm{i}} \frac{\delta}{\delta J(y)} Z_0[J]\right|_{J=0} = \mathrm{i} \Delta_\mathrm{F}(x - y),$$

or

$$\tau(x, y) = \mathrm{i} \Delta_\mathrm{F}(x - y). \tag{6.52}$$

What is the physical significance of this? From (6.48) the 2-point function is

$$\begin{aligned}
\tau(x, y) &= \langle 0 | T(\phi(x) \phi(y)) | 0 \rangle \\
&= \langle 0 | \theta(x_0 - y_0) \phi(x) \phi(y) + \theta(y_0 - x_0) \phi(y) \phi(x) | 0 \rangle.
\end{aligned} \tag{6.53}$$

From (4.14), we may decompose ϕ into its positive and negative frequency parts

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) \tag{6.54}$$

with

$$\phi^{(+)}(x) = \int \frac{d^3\mathbf{k}}{[(2\pi)^3 2\omega_k]^{1/2}} f_k(x) a(k)$$
$$\phi^{(-)}(x) = \int \frac{d^3\mathbf{k}}{[(2\pi)^3 2\omega_k]^{1/2}} f_k^*(x) a^\dagger(k)$$

and where $f_k(x)$ is given by (4.11). Because $a(k)$ and $a^\dagger(k)$ are annihilation and creation operators, the only terms to survive the vacuum expectation value in (6.53) are $\phi^{(+)}(x)\phi^{(-)}(x)$, so that

$$\tau(x, y) = \theta(x_0 - y_0)\langle 0|\phi^{(+)}(x)\phi^{(-)}(y)|0\rangle + \theta(y_0 - x_0)\langle 0|\phi^{(+)}(y)\phi^{(-)}(x)|0\rangle. \tag{6.55}$$

The first term is the amplitude for creating a particle at y, at time y_0, and destroying it at x, at time x_0 ($> y_0$). The second term is the amplitude for creating a particle at x, at time x_0, and destroying it at y, at time y_0 ($> x_0$). These are represented schematically in Fig. 6.3. We shall now verify that the sum of these terms is the Feynman propagator $i\Delta_F(x - y)$.

To prove this, let us first re-express $\Delta_F(x - y)$. From (6.14)

$$\begin{aligned}\Delta_F(x) &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 + i\varepsilon} \\ &= \int \frac{d^3\mathbf{k}\, dk_0}{(2\pi)^4} \frac{e^{-ikx}}{k_0^2 - (\mathbf{k}^2 + m^2) + i\varepsilon} \\ &= \int \frac{d^3\mathbf{k}\, dk_0}{(2\pi)^4} \frac{e^{-ikx}}{2\omega_k}\left(\frac{1}{k_0 - \omega_k + i\delta} - \frac{1}{k_0 + \omega_k - i\delta}\right)\end{aligned}$$

where $\omega_k^2 = \mathbf{k}^2 + m^2$. The integration path over k_0 is shown in Fig. 6.1. The exponential contains $e^{-ik_0x_0}$, so, when $x_0 > 0$, we may complete the integral in the lower half of the k_0 plane, and the contribution on the semicircle at infinity

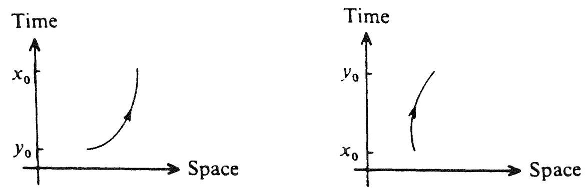


Fig. 6.3. Interpretation of equation (6.55). (See text.)

will vanish; the path encloses the pole at $k_0 = \omega_k - i\delta$. On the other hand, when $x_0 < 0$, we complete in the upper half plane and enclose the pole at $k_0 = -\omega_k + i\delta$. Applying Cauchy's theorem then gives

$$\Delta_F(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{2\omega_k} [\theta(x_0)(-i)\, e^{-i\omega_k x_0} - \theta(-x_0) i\, e^{i\omega_k x_0}].$$

In the second integral, we may change **k** to $-$**k** without affecting the value of the integral, so, finally,

$$\Delta_F(x-y) = -i\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} [\theta(x_0 - y_0)\, e^{-ik(x-y)} + \theta(y_0 - x_0)\, e^{ik(x-y)}]. \tag{6.56}$$

This is the form we want for $\Delta_F(x-y)$. Now we substitute (6.54) into (6.55), giving

$$\begin{aligned}\tau(x, y) &= \int \frac{d^3k\, d^3k'}{(2\pi)^3 (2\omega_k 2\omega_{k'})^{1/2}} [\theta(x_0 - y_0)\langle 0|a(k) f_k(x) a^\dagger(k') f^*_{k'}(y)|0\rangle \\ &\quad + \theta(y_0 - x_0)\langle 0|a(k) f_k(y) a^\dagger(k') f^*_{k'}(x)|0\rangle] \\ &= \int \frac{d^3k\, d^3k'}{(2\pi)^6 2\omega_k 2\omega_{k'}} [\theta(x_0 - y_0)\, e^{-i(kx - k'y)}\, \langle 0|a(k) a^\dagger(k')|0\rangle \\ &\quad + \theta(y_0 - x_0)\, e^{-i(ky - k'x)}\, \langle 0|a(k) a^\dagger(k')|0\rangle]\end{aligned}$$

where we have used (4.11) for $f_k(x)$. Now we use the commutation relation (4.16*a*), to find

$$\begin{aligned}\tau(x, y) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} [\theta(x_0 - y_0)\, e^{-ik(x-y)} + \theta(y_0 - x_0)\, e^{ik(x-y)}] \\ &= i\Delta_F(x - y)\end{aligned} \tag{6.57}$$

from (6.56). We have now proved (6.52) again, and have in the process found an interpretation for the 2-point function in terms of the creation, propagation and destruction of our particle between two points.

We have found an expression for the 2-point function. What is the 1-point function? It is clear that

$$\tau(x) = \langle 0|T\phi(x)|0\rangle = \langle 0|\phi(x)|0\rangle = \frac{1}{i}\frac{\delta Z_0[J]}{\delta J(x)}\bigg|_{J=0} = 0. \tag{6.58}$$

Let us now find the 3-point function.

$$\tau(x_1, x_2, x_3) = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_3)} Z_0[J]\bigg|_{J=0}.$$

We saw above (equation (6.51)) that

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_2)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_3)}Z_0[J] = \mathrm{i}\Delta_{\mathrm{F}}(x_2 - x_3)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)$$
$$+ \int\Delta_{\mathrm{F}}(x_2 - x)J(x)\,\mathrm{d}x\int\Delta_{\mathrm{F}}(x_3 - x)J(x)\,\mathrm{d}x\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right).$$

Further differentiation then gives

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_1)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_2)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_3)}Z_0[J]$$
$$= -\mathrm{i}\Delta_{\mathrm{F}}(x_2 - x_3)\int\Delta_{\mathrm{F}}(x_1 - x)J(x)\,\mathrm{d}x\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)$$
$$-\mathrm{i}\Delta_{\mathrm{F}}(x_2 - x_1)\int\Delta_{\mathrm{F}}(x_3 - x)J(x)\,\mathrm{d}x\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)$$
$$-\mathrm{i}\Delta_{\mathrm{F}}(x_3 - x_1)\int\Delta_{\mathrm{F}}(x_2 - x)J(x)\,\mathrm{d}x\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)$$
$$-\int\Delta_{\mathrm{F}}(x_2 - x)J(x)\,\mathrm{d}x\int\Delta_{\mathrm{F}}(x_3 - x)J(x)\,\mathrm{d}x$$
$$\times\int\Delta_{\mathrm{F}}(x_1 - x)J(x)\,\mathrm{d}x\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right). \qquad (6.59)$$

Putting $J = 0$ reduces this expression to zero, so

$$\tau(x_1, x_2, x_3) = \langle|T(\phi(x_1)\phi(x_2)\phi(x_3))|0\rangle = 0. \qquad (6.60)$$

To find the 4-point function we carry on in the same way, by differentiating (6.59) once more and putting $J = 0$. We have

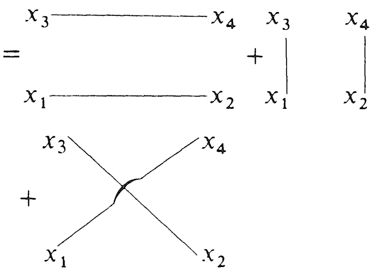
$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_4)}Z_0[J] = -\Delta_{\mathrm{F}}(x_2 - x_3)\Delta_{\mathrm{F}}(x_1 - x_4)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)$$
$$-\Delta_{\mathrm{F}}(x_2 - x_1)\Delta_{\mathrm{F}}(x_3 - x_4)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)$$
$$-\Delta_{\mathrm{F}}(x_3 - x_1)\Delta_{\mathrm{F}}(x_2 - x_4)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)$$
$$+\ (\text{terms which vanish when } J = 0),$$

so the 4-point function is

$$\tau(x_1, x_2, x_3, x_4) = \langle 0|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|0\rangle$$
$$= -[\Delta_{\mathrm{F}}(x_1 - x_2)\Delta_{\mathrm{F}}(x_3 - x_4)$$
$$+ \Delta_{\mathrm{F}}(x_1 - x_3)\Delta_{\mathrm{F}}(x_2 - x_4)$$
$$+ \Delta_{\mathrm{F}}(x_1 - x_4)\Delta_{\mathrm{F}}(x_2 - x_3)]. \qquad (6.61)$$

This is simply the sum of products of 2-point functions, and may be written

$\tau(x_1, x_2, x_3, x_4) =$



(6.62)

Going to higher orders, it is clear that if n is odd, the n-point function always vanishes,

$$\tau(x_1, x_2, \ldots, x_{2n+1}) = 0, \tag{6.63}$$

and it turns out that, if n is even, the n-point function is a sum of products of 2-point functions,

$$\tau(x_1, x_2, \ldots, x_{2n}) = \sum_{\text{perms}} \tau(x_{p_1}, x_{p_2}) \ldots \tau(x_{p_{2n-1}}, x_{p_{2n}}), \tag{6.64}$$

where

$$\tau(x, y) = \mathrm{i}\Delta_{\mathrm{F}}(x - y).$$

This important result, when derived using the 'canonical' method, employing the operator commutator relations, is known as *Wick's theorem*.

In this section we have derived the Green's functions in a scalar free field theory. The interesting case, however, is the one for which interactions are present; how do we calculate the Green's functions then? When we know this, we are one step nearer calculating a scattering amplitude for a real physical process!

6.4 Generating functionals for interacting fields

The Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 = \mathscr{L}_0 + \mathscr{L}_{\text{int}} \tag{6.65}$$

describes a scalar field which interacts with itself, through the term in ϕ^4. We shall first show how to find the Green's functions for a general interaction $\mathscr{L}_{\text{int}}$, and then in the next section apply the formulae to the case of ϕ^4 theory. The normalised generating functional is

$$Z[J] = \frac{\int \mathcal{D}\phi \exp\left(\mathrm{i}S + \mathrm{i}\int J\phi \,\mathrm{d}x\right)}{\int \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}S}} \tag{6.66}$$

with $S = \int \mathcal{L}\,\mathrm{d}x$. It is clear that, when $\mathcal{L}_{\text{int}} = 0$, this becomes the expression (6.43), which we were able to show is the same as (6.44). Equation (6.44) is in a form suitable for functional differentiation with respect to J, and therefore for finding the Green's functions. We want to find the expression which corresponds to (6.44) in the case of interacting fields. We proceed by finding the differential equation satisfied by $Z[J]$, and then solving it in terms of $Z_0[J]$.

First note that, from (6.44),

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)} Z_0[J] = -\int \Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}y \exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\,\mathrm{d}x\,\mathrm{d}y\right),$$

so, since Δ_{F} is minus the inverse of $\Box + m^2$,

$$(\Box + m^2)\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)} Z_0[J] = J(x) Z_0[J]. \tag{6.67}$$

This is the differential equation satisfied by $Z_0[J]$.

Now we have, from (6.66),

$$\frac{1}{\mathrm{i}}\frac{\delta Z[J]}{\delta J(x)} = \frac{\int \exp\left(\mathrm{i}S + \mathrm{i}\int J\phi\,\mathrm{d}x\right)\phi(x)\mathcal{D}\phi}{\int \mathrm{e}^{\mathrm{i}S}\mathcal{D}\phi}. \tag{6.68}$$

We define the functional

$$\hat{Z}[\phi] = \frac{\mathrm{e}^{\mathrm{i}S}}{\int \mathrm{e}^{\mathrm{i}S}\mathcal{D}\phi}. \tag{6.69}$$

Then

$$Z[J] = \int \hat{Z}[\phi] \exp\left[\mathrm{i}\int J(x)\phi(x)\,\mathrm{d}x\right]\mathcal{D}\phi. \tag{6.70}$$

This is the functional analogue of the Fourier transform. Now we take the functional derivative of $\hat{Z}[\phi]$, noting that

$$S = \int(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + \mathcal{L}_{\text{int}})\,\mathrm{d}^4x = -\int[\tfrac{1}{2}\phi(\Box + m^2)\phi - \mathcal{L}_{\text{int}}]\,\mathrm{d}^4x. \tag{6.71}$$

We obtain

$$\begin{aligned}
\mathrm{i}\frac{\delta \hat{Z}[\phi]}{\delta\phi(x)} &= \mathrm{i}\frac{\delta}{\delta\phi}\left\{\exp\left[-\mathrm{i}\int[\tfrac{1}{2}\phi(\Box + m^2)\phi - \mathscr{L}_{\text{int}}]\,\mathrm{d}^4x\right]\right\}\left[\int \mathrm{e}^{\mathrm{i}S}\,\mathscr{D}\phi\right]^{-1} \\
&= (\Box + m^2)\phi(x)\hat{Z}[\phi] - \frac{\partial\mathscr{L}_{\text{int}}}{\partial\phi}\hat{Z}[\phi] \\
&= (\Box + m^2)\phi(x)\hat{Z}[\phi] - \mathscr{L}'_{\text{int}}(\phi)\hat{Z}[\phi] \qquad (6.72)
\end{aligned}$$

where the prime on $\mathscr{L}'_{\text{int}}$ means differentiation with respect to the argument. Now we multiply both sides of (6.72) by $\exp[\mathrm{i}\int J(x)\phi(x)\,\mathrm{d}x]$ and integrate over ϕ. The right-hand side gives

$$\frac{\int(\Box + m^2)\phi(x)\exp\left(\mathrm{i}S + \mathrm{i}\int J\phi\,\mathrm{d}x\right)\mathscr{D}\phi}{\int \mathrm{e}^{\mathrm{i}S}\,\mathscr{D}\phi} - \frac{\int\mathscr{L}'_{\text{int}}(\phi)\exp\left(\mathrm{i}S + \mathrm{i}\int J\phi\,\mathrm{d}x\right)\mathscr{D}\phi}{\int \mathrm{e}^{\mathrm{i}S}\,\mathscr{D}\phi}$$

$$= (\Box + m^2)\frac{1}{\mathrm{i}}\frac{\delta Z[J]}{\delta J(x)} - \mathscr{L}'_{\text{int}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right]Z[J] \qquad (6.73)$$

where (6.68) has been used, and the argument of $\mathscr{L}'_{\text{int}}$ has been changed from ϕ to $(1/\mathrm{i})(\delta/\delta J)$, since it operates on $Z[J]$. The left-hand side of (6.72) gives

$$\begin{aligned}
\mathrm{i}\int\frac{\delta\hat{Z}[\phi]}{\delta\phi}\exp\left(\mathrm{i}\int J\phi\,\mathrm{d}x\right)\mathscr{D}\phi &= \mathrm{i}\exp\left(\mathrm{i}\int J\phi\,\mathrm{d}x\right)\hat{Z}[\phi]\bigg|_{\phi\to\infty} \\
&\quad + \int J(x)\hat{Z}[\phi]\exp\left(\mathrm{i}\int J\phi\,\mathrm{d}x\right)\mathscr{D}\phi \\
&= J(x)Z[J] \qquad (6.74)
\end{aligned}$$

from (6.70). Equating (6.73) and (6.74) gives

$$(\Box + m^2)\frac{1}{\mathrm{i}}\frac{\delta Z[J]}{\delta J(x)} - \mathscr{L}'_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)Z[J] = J(x)Z[J]. \qquad (6.75)$$

We must solve this equation for $Z[J]$. In the free field case, $\mathscr{L}_{\text{int}} = 0$ and the equation reduces to (6.67), for $Z_0[J]$. We shall now show that the solution to (6.75) is

$$Z[J] = N\exp\left[\mathrm{i}\int\mathscr{L}_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\mathrm{d}x\right]Z_0[J] \qquad (6.76)$$

where N is a normalising factor. The proof is in two stages.

Proof

(*a*) We first prove the identity

$$\exp\left[-\mathrm{i}\int \mathscr{L}_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)\mathrm{d}y\right] J(x) \exp\left[+\mathrm{i}\int \mathscr{L}_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)\mathrm{d}y\right]$$
$$= J(x) - \mathscr{L}'_{\text{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right). \quad (6.77)$$

This follows by observing that the functional analogue of

$$\left[x_i, \frac{1}{\mathrm{i}}\frac{\partial}{\partial x_j}\right] = \mathrm{i}\delta_{ij}$$

is

$$\left[J(x), \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right] = \mathrm{i}\delta(x-y).$$

Repeated application of this equation gives

$$\begin{aligned}\left[J(x), \left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)^n\right] &= \mathrm{i}\delta(x-y)\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)^{n-1} \\ &\quad + \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\left[J(x), \left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)^{n-1}\right] \\ &\quad \vdots \\ &= \mathrm{i}\delta(x-y)\,n\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)^{n-1}. \end{aligned} \quad (6.78)$$

By expanding the function

$$F(\phi) = F(0) + \phi F'(0) + \frac{\phi^2}{2!}F''(0) + \ldots = \sum_{n=0}^{\infty}\frac{\phi^n}{n!}F^{(n)}(0)$$

and making the replacement $\phi \to (1/\mathrm{i})(\delta/\delta J)$, it then follows from (6.78) that

$$\left[J(x), \int F\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)\mathrm{d}y\right] = \mathrm{i}F'\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right). \quad (6.79)$$

Now we use the Hausdorff formula

$$\mathrm{e}^A B \mathrm{e}^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots \quad (6.80)$$

where A and B are operators, and put $A = -\mathrm{i}\int \mathscr{L}_{\text{int}}((1/\mathrm{i})(\delta/\delta J(y)))\,\mathrm{d}y$ and $B = J(x)$. Since, in this case, A commutes with $[A, B]$ (from (6.79)), only the first two terms on the right-hand side of (6.80) appear, and (6.77) is proved.

(*b*) We must now show that (6.76) is the solution of (6.75). From (6.76) and (6.77)

$$J(x)Z[J] = NJ(x)\exp\left[\mathrm{i}\int \mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)\mathrm{d}y\right]Z_0[J]$$

$$= N\exp\left[\mathrm{i}\int \mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)\mathrm{d}y\right]\left[J(x) - \mathcal{L}'_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]Z_0[J].$$

The first of these terms is transformed using (6.67) and, in the second, the order of $\mathrm{e}^{\mathrm{i}\int \mathcal{L}_{\mathrm{int}}}$ and $\mathcal{L}'_{\mathrm{int}}$ may be interchanged, giving

$$J(x)Z[J] = N\exp\left[\mathrm{i}\int \mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)\mathrm{d}y\right](\Box + m^2)\frac{1}{\mathrm{i}}\frac{\delta Z_0}{\delta J(x)}$$

$$- N\mathcal{L}'_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\exp\left[\mathrm{i}\int \mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(y)}\right)\mathrm{d}y\right]Z_0[J]$$

$$= (\Box + m^2)\frac{1}{\mathrm{i}}\frac{\delta Z[J]}{\delta J(x)} - \mathcal{L}'_{\mathrm{int}}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right]Z[J],$$

where (6.76) was used. This is equation (6.75). QED.

We are now in a position to calculate the Green's functions in the interacting field case, which we proceed to do, as usual in quantum theory, by perturbation theory.

6.5 ϕ^4 theory

Generating functional

As we saw in (6.65), the interaction Lagrangian in ϕ^4 theory is

$$\mathcal{L}_{\mathrm{int}} = -\frac{g}{4!}\phi^4. \tag{6.81}$$

The normalised generating functional $Z[J]$ is

$$Z[J] = \frac{\exp\left[\mathrm{i}\int \mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)\mathrm{d}z\right]\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]}{\left\{\exp\left[\mathrm{i}\int \mathcal{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)\mathrm{d}z\right]\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]\right\}\Big|_{J=0}}. \tag{6.82}$$

The only way of treating $\exp(\mathrm{i}\int \mathcal{L}_{\mathrm{int}})$ is as a power series in the coupling constant g, i.e. by perturbation theory. Substituting (6.81) into (6.82) and expanding in powers of g, the numerator of $Z[J]$ is

$$\left[1 - \frac{\mathrm{i}g}{4!}\int\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)^4\mathrm{d}z + O(g^2)\right]\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right].$$

To order g^0, we just have the free particle generating functional $Z_0[J]$. To order g, we proceed as follows:

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]$$
$$=-\int\Delta_F(z-x)J(x)\,\mathrm{d}x\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right];$$

$$\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)^2\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]$$
$$=\left\{\mathrm{i}\Delta_F(0)+\left[\int\Delta_F(z-x)J(x)\,\mathrm{d}x\right]^2\right\}\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right];$$

$$\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)^3\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]$$
$$=\left\{3[-\mathrm{i}\Delta_F(0)]\int\Delta_F(z-x)J(x)\,\mathrm{d}x-\left[\int\Delta_F(z-x)J(x)\,\mathrm{d}x\right]^3\right\}$$
$$\times\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right];$$

$$\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)^4\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]$$
$$=\left\{-3[\Delta_F(0)]^2+6\mathrm{i}\Delta_F(0)\left[\int\Delta_F(z-x)J(x)\,\mathrm{d}x\right]^2+\left[\int\Delta_F(z-x)J(x)\,\mathrm{d}x\right]^4\right\}$$
$$\times\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right].\tag{6.83}$$

We may write this expression diagramatically. Let

$$x \,\text{———}\, y\rightarrow\Delta_F(x-y)\tag{6.84}$$

represent the free particle propagator. $\Delta_F(0)=\Delta_F(x-x)$ is then represented by a closed loop

$$\bigcirc\rightarrow\Delta_F(0).\tag{6.85}$$

Equation (6.83) may then be written

$$\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(z)}\right)^4\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right)=\{-3\,\bigcirc\!\!\bigcirc+6\mathrm{i}\,{\times}\!\!-\!\!\bigcirc\!\!-\!\!{\times}+\times\!\!\!\!\times\}\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right).\tag{6.86}$$

The meeting of four lines at a point in the three diagrams in (6.86) is clearly a consequence of the fact that $\mathcal{L}_{\text{int}}$ contains ϕ^4. Moreover, the coefficients 3, 6 and 1 follow from rather simple considerations of symmetry. The first term on the right-hand side of (6.86), for example, results from joining up the two pairs of lines in the third term, in all possible ways; and there are three ways to do this. The second term is got by joining any two lines of the third term, and there are six ways to do this. These numerical coefficients are known as *symmetry factors*. The first term, with two closed loops, is known as a *vacuum graph*. because it has no external lines. The term with one closed loop has two external lines (i.e. two Js) and the last term four external lines (four Js). It is now an easy matter to write down the denominator of (6.82). Putting $J = 0$ eliminates the second and third terms in (6.86), so we have

$$\left[\exp\left(\mathrm{i}\int \mathcal{L}_{\text{int}}\right)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)\right]\Bigg|_{J=0} = 1 - \frac{\mathrm{i}g}{4!}\int(-3\ \bigcirc\!\!\bigcirc\ )\,\mathrm{d}z. \quad (6.87)$$

The complete generating functional, given by equation (6.82), is, to order g,

$$Z[J] = \frac{\left[1 - \frac{\mathrm{i}g}{4!}\int\left(-3\ \bigcirc\!\!\bigcirc + 6\mathrm{i}\ {\times\!\!-\!\!\bigcirc\!\!-\!\!\times} + \bowtie\right)\mathrm{d}z\right]\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right)}{1 - \frac{\mathrm{i}g}{4!}\int(-3\ \bigcirc\!\!\bigcirc\ )\,\mathrm{d}z}$$

$$= \left[1 - \frac{\mathrm{i}g}{4!}\int\left(6\mathrm{i}\ {\times\!\!-\!\!\bigcirc\!\!-\!\!\times} + \bowtie\right)\mathrm{d}z\right]\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right), \quad (6.88)$$

where the denominator has been expanded by the binomial theorem. The interesting thing that has happened is that the *vacuum diagram has disappeared* in $Z[J]$. It turns out that this is true to all orders in perturbation theory, and is a general property of *normalised* generating functionals.

2-point function

The 2-point function is defined by

$$\tau(x_1, x_2) = -\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_1)}\bigg|_{J=0}. \quad (6.89)$$

By looking at (6.88), it is seen that the first term in Z will give $\mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2)$ in τ; this is the free particle propagator. The term in $\bowtie$ in (6.88) contains four Js and so gives no contribution to the 2-point function. The term in ${\times\!\!-\!\!\bigcirc\!\!-\!\!\times}$ is

$$\frac{g}{4}\Delta_{\mathrm{F}}(0)\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\Delta_{\mathrm{F}}(z - x)J(x)\Delta_{\mathrm{F}}(z - y)J(y)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_{\mathrm{F}}J\right).$$

On differentiation we get

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_1)}\Big(\quad\Big) = -\frac{\mathrm{i}g}{4}\Delta_\mathrm{F}(0)2\int \mathrm{d}y\, \mathrm{d}z \Delta_\mathrm{F}(z-x_1)\Delta_\mathrm{F}(z-y)J(y) \times \exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_\mathrm{F}J\right) + \cdots,$$

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_2)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x_1)}\Big(\quad\Big) = -\frac{g}{2}\Delta_\mathrm{F}(0)\int \mathrm{d}z \Delta_\mathrm{F}(z-x_1)\Delta_\mathrm{F}(z-x_2) \times \exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_\mathrm{F}J\right) + \ldots,$$

where the omitted terms vanish when $J = 0$. We then have

$$\tau(x_1, x_2) = \mathrm{i}\Delta_\mathrm{F}(x_1 - x_2) - \frac{g}{2}\Delta_\mathrm{F}(0)\int \mathrm{d}z \Delta_\mathrm{F}(z-x_1)\Delta_\mathrm{F}(z-x_2) + O(g^2) \tag{6.90}$$

$= \mathrm{i}$ $- \frac{g}{2}$  $+ O(g^2)$. (6.91)

To order g, this represents the effect of interaction on the free-particle propagation. The free-particle propagator is, from (6.14),

$$\Delta_\mathrm{F}(x-y) = \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}}{k^2 - m^2 + \mathrm{i}\varepsilon}\mathrm{d}^4k,$$

and its Fourier transform contains a pole at $k^2 = m^2$. This identifies the mass of the particle as m. We shall now show that the effect of the interaction is to change the value of the physical mass away from m. Indeed, the second term in (6.90) is

$$\begin{aligned}
-\tfrac{1}{2}g\Delta_\mathrm{F}(0)\int \Delta_\mathrm{F}(x_1 - z)\Delta_\mathrm{F}(x_2 - z)\,\mathrm{d}z
&= -\frac{g}{2}\frac{\Delta_\mathrm{F}(0)}{(2\pi)^8}\int \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x_1-z)}}{p^2 - m^2 + \mathrm{i}\varepsilon}\cdot\frac{\mathrm{e}^{-\mathrm{i}q\cdot(x_2-z)}}{q^2 - m^2 + \mathrm{i}\varepsilon}\mathrm{d}^4p\,\mathrm{d}^4q\,\mathrm{d}^4z \\
&= -\frac{g}{2}\frac{\Delta_\mathrm{F}(0)}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x_1-x_2)}}{(p^2 - m^2 + \mathrm{i}\varepsilon)^2}\delta^4(p+q)\,\mathrm{d}^4p\,\mathrm{d}^4q \\
&= -\frac{g}{2}\frac{\Delta_\mathrm{F}(0)}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x_1-x_2)}}{(p^2 - m^2 + \mathrm{i}\varepsilon)^2}\mathrm{d}^4p,
\end{aligned}$$

giving, for the 2-point function (6.90),

$$\tau(x_1, x_2) = \frac{\mathrm{i}}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x_1-x_2)}}{p^2 - m^2 + \mathrm{i}\varepsilon}\left[1 + \frac{\frac{1}{2}\mathrm{i}g\Delta_\mathrm{F}(0)}{p^2 - m^2 + \mathrm{i}\varepsilon}\right]\mathrm{d}^4p. \tag{6.92}$$

Formally, the term in square brackets above can be written as

$$\left[1 - \frac{\frac{1}{2}\mathrm{i}g\Delta_{\mathrm{F}}(0)}{p^2 - m^2 + \mathrm{i}\varepsilon}\right]^{-1},$$

so we have

$$\tau(x_1, x_2) = \frac{\mathrm{i}}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x_1 - x_2)}}{p^2 - m^2 - \frac{1}{2}\mathrm{i}g\Delta_{\mathrm{F}}(0) + \mathrm{i}\varepsilon}\,\mathrm{d}^4 p. \tag{6.93}$$

The Fourier transform of $\tau(x_1, x_2)$ will now possess a pole at p^2 equal to

$$m^2 + \tfrac{1}{2}\mathrm{i}g\Delta_{\mathrm{F}}(0) \equiv m^2 + \delta m^2 = m_{\mathrm{r}}^2 \tag{6.94}$$

where

$$\delta m^2 = \tfrac{1}{2}\mathrm{i}g\Delta_{\mathrm{F}}(0); \tag{6.95}$$

m_{r} is now identified as the *physical mass*, or *renormalised mass*. The change in (mass)^2, δm^2, is a quadratically divergent quantity, since $\Delta_{\mathrm{F}}(0)$ contains four powers of p in the numerator ($\mathrm{d}^4 p$) and two in the denominator. It happens, then, that the renormalisation of the mass is by an infinite quantity, but that is a distinct circumstance; the essence of renormalisation is that the physical quantity (mass, in this case) is not the same as the parameter in the Lagrangian, if an interaction is present. More will be said about renormalisation in Chapter 9.

4-point function

Let us proceed, finally, to the 4-point function, given by (equation (6.45))

$$\tau(x_1, x_2, x_3, x_4) = \left.\frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}, \tag{6.96}$$

where $Z[J]$ is given by (6.88). The first (order g^0) term in τ is the same as that found in (6.61)

$$-[\Delta_{\mathrm{F}}(x_1 - x_2)\Delta_{\mathrm{F}}(x_3 - x_4) + \Delta_{\mathrm{F}}(x_1 - x_3)\Delta_{\mathrm{F}}(x_2 - x_4) + \Delta_{\mathrm{F}}(x_1 - x_4)\Delta_{\mathrm{F}}(x_2 - x_3)] = -(= + \,||\, + \times) = -3(=). \tag{6.97}$$

This is the free particle 4-point function and will not contribute to the scattering. The next term in $Z[J]$, of order g, gives, as may be checked,

$$\left.\frac{g}{4}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\left\{ -\!\!\bigcirc\!\!- \exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_{\mathrm{F}}(x - y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]\right\}\right|_{J=}$$

$$= \frac{g}{4}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}$$

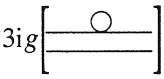
$$\times \left\{\Delta_F(0)\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\Delta_F(x-z)\Delta_F(y-z)J(y)J(x)\exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right)\right\}\Bigg|_{J=0}$$

$$= \frac{-\mathrm{i}g}{2}\Delta_F(0)\int \mathrm{d}z[\Delta_F(z-x_1)\Delta_F(z-x_2)\Delta_F(x_3-x_4)$$

$$+ \Delta_F(z-x_1)\Delta_F(z-x_3)\Delta_F(x_2-x_4) + \Delta_F(z-x_1)\Delta_F(z-x_4)\Delta_F(x_2-x_3)$$

$$+ \Delta_F(z-x_2)\Delta_F(z-x_3)\Delta_F(x_1-x_4) + \Delta_F(z-x_2)\Delta_F(z-x_4)\Delta_F(x_1-x_3)$$

$$+ \Delta_F(z-x_3)\Delta_F(z-x_4)\Delta_F(x_1-x_2)]$$

$= -3\mathrm{i}g$  . (6.98)

The diagram above does service for the six equivalent terms in the expression above it. Each of these terms contributes twice, so the 'symmetry factor' of the diagram is 12.

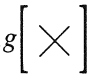
The final term in $Z[J]$, of order g, gives

$\frac{-\mathrm{i}g}{4!}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big\{$  $\exp\left[-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)\,\mathrm{d}x\,\mathrm{d}y\right]\Big\}\Big|_{J=0}$

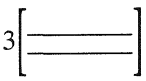
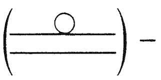
$$= \frac{-\mathrm{i}g}{4!}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}$$

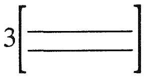
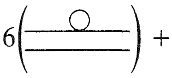
$$\times \left\{\left[\int \Delta_F(z-x)J(x)\,\mathrm{d}x\right]^4 \exp\left(-\frac{\mathrm{i}}{2}\int J\Delta_F J\right)\right\}\Bigg|_{J=0}$$

$$= -\mathrm{i}g\int \Delta_F(x_1-z)\Delta_F(x_2-z)\Delta_F(x_3-z)\Delta_F(x_4-z)\,\mathrm{d}z$$

$= -\mathrm{i}g$  . (6.99)

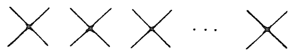
The complete 4-point function, to order g, is then

$\tau(x_1, x_2, x_3, x_4) = -3$  $- 3\mathrm{i}g$  $- \mathrm{i}g$ 

$= -3$  $\frac{-\mathrm{i}g}{4!}\Big[12 \times 6$  $+ 24$  $\Big]$.

(6.100)

The first term, of order g^0, does not contribute to the scattering. The numerical coefficients above are easily derived by simple combinatorics, and this suggests a rather direct way to write down all the diagrams of a given order. Let us consider, for example, all diagrams contributing to the 4-point function of order g, in $g\phi^4/4!$ theory. We deduce them as follows. First of all, we are considering a $g\phi^4$ theory, so to order g^n we have the n vertices

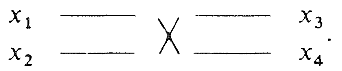


(6.101)

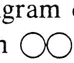
and corresponding to the 4-point function we draw four external points

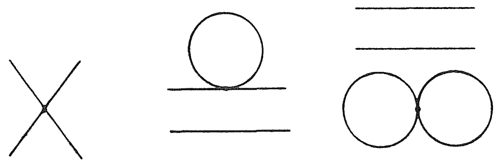
(6.102)

The 4-point function in $g\phi^4$ theory to order g is then constructed from the following *prediagram*



(6.103)

(This is called a prediagram to distinguish it from a real Feynman diagram.) We now join up all the lines. There are three topologically distinct types of Feynman diagram which result, drawn in Fig. 6.4. The multiplicities are calculated as follows. To get diagram (a) (in Fig. 6.4) join x_1 up to one of the legs of the vertex in (6.103). There are four ways to do this. Now join x_2 up to one of the remaining three legs – there are three ways. There are $4! = 24$ ways to complete the diagram (a), which is the coefficient in equation (6.100). Next, to make diagram (b) join x_1 directly to one of the other external points x_2, x_3, x_4. There are three ways to do this. Choose one leg of the vertex and join it up to one of the two remaining external points. There are 4×2 ways to do this. Join one of the three remaining legs of the vertex to the one remaining point. There are three ways to do this. Join the remaining two legs together. The total multiplicity is $3 \times 4 \times 2 \times 3 = 12 \times 6$, as in (6.100). The reader will easily convince himself that the multiplicity of diagram (c) is $3 \times 3 = 9$. The reason this diagram does not appear in (6.100) is that it is a multiple of the vacuum diagram  (see above) and $Z[J]$, being properly normalised, produces no vacuum diagrams.



(a) (b) (c)

Fig. 6.4. First-order terms in the 4-point function.

In summary, the Feynman rules for ϕ^4 scalar field theory, in co-ordinate space, are

line	x———y	$\Delta_F(x - y)$	
vertex	$\times\, z$ integration over z	$-ig$	(6.104)
symmetry factor	·	$S/4!$	

In calculations of realistic processes, involving, for example, electrons and photons, the particles are not identical and there is then no symmetry factor to contend with. We postpone, until we consider these real scattering processes, the derivation of the S matrix from the Green's function.

It is however, convenient to mention here that, of the two order g diagrams in (6.100), the first one, ═O═, only contributes to the trivial (diagonal) part of the S matrix, so is not interesting. It describes the two particles moving independently, and the effect of the interaction is to modify the propagator of one of them. This graph is called *disconnected*. The other order g graph, $\times$, is *connected*, since every line in it is connected to every other line. Only connected Feynman diagrams contribute to $S - 1$, i.e. to the non-trivial part of the S matrix.

6.6 Generating functional for connected diagrams

Now it turns out that there is a generating functional W, which generates only connected Feynman diagrams or connected Green's functions. It is related to Z by

$$Z[J] = e^{iW[J]} \tag{6.105}$$

or

$$W[J] = -i \ln Z[J]. \tag{6.106}$$

We shall now show, by considering the 2-point and 4-point functions in ϕ^4 theory, that $W[J]$ generates no disconnected graphs. We have, firstly,

$$\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)} = \frac{i}{Z^2}\frac{\delta Z}{\delta J(x_1)}\frac{\delta Z}{\delta J(x_2)} - \frac{i}{Z}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}. \tag{6.107}$$

When $J = 0$, we have

$$\left.\frac{\delta Z[J]}{\delta J(x)}\right|_{J=0} = 0, \quad Z[0] = 1 \tag{6.108}$$

so

$$\left.\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} = -i\left.\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} = i\tau(x_1, x_2) \tag{6.109}$$

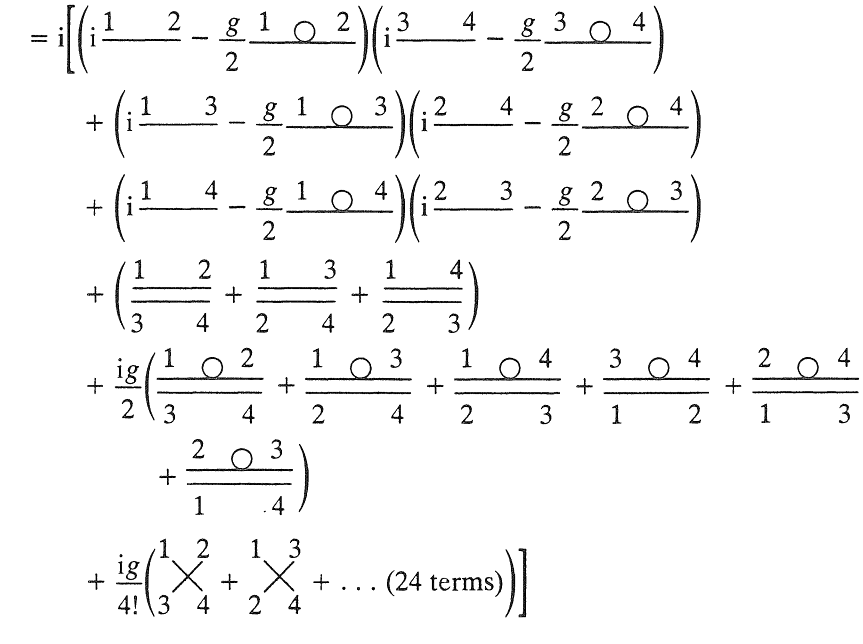
showing that W generates the propagator, to any order in g. This is as we expected, since the propagator has no disconnected part. To find the 4-point function, we differentiate (6.107) twice more and set $J = 0$, to give

$$\left.\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} = \mathrm{i}\left[\frac{1}{Z^2}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\frac{\delta^2 Z}{\delta J(x_3)\delta J(x_4)} + \frac{1}{Z^2}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_3)}\frac{\delta^2 Z}{\delta J(x_2)\delta J(x_4)} + \frac{1}{Z^2}\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_4)}\frac{\delta^2 Z}{\delta J(x_2)\delta J(x_3)} - \left.\frac{1}{Z}\frac{\delta^4 Z}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right]\right|_{J=0}$$

$$= \mathrm{i}[\tau(x_1, x_2)\tau(x_3, x_4) + \tau(x_1, x_3)\tau(x_2, x_4) + \tau(x_1, x_4)\tau(x_2, x_3) - \tau(x_1, x_2, x_3, x_4)]. \tag{6.110}$$

We have to show that this contains no disconnected diagrams. The most convenient way to do this is to substitute the diagrams themselves into (6.110). Working, as always to order g, and substituting equations (6.91) and (6.100) into (6.110) gives (in a notation which is meant to be self-explanatory; 1 stands for x_1, etc.)

$$\left.\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}$$



$$= \frac{-g}{4!}\left(\underset{3\;\;4}{\overset{1\;\;2}{\times}} + \underset{2\;\;4}{\overset{1\;\;3}{\times}} + \ldots \text{(24 terms)} \right) = -g \times . \qquad (6.111)$$

We see, indeed, that only the connected diagram survives*.

Finally, let us briefly consider the n-point function. It is

$$\tau(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J=0}. \qquad (6.112)$$

We then define the *irreducible n-point function* $\phi(x_1, \ldots, x_n)$ by

$$\phi(x_1, \ldots, x_n) = \frac{1}{\mathrm{i}^n} \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J=0}. \qquad (6.113)$$

To justify the name, note that we have, from (6.100) and (6.111),

$$\left.\begin{aligned} \tau(x_1, \ldots, x_4) &= -\mathrm{i}g \times - 3\mathrm{i}g \, \overset{\circ}{=} - 3 \, = \\ \mathrm{i}\phi(x_1, \ldots, x_4) &= -\mathrm{i}g \times \end{aligned}\right\} \qquad (6.114)$$

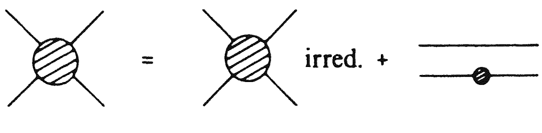
and from (6.110)

$$\begin{aligned} \mathrm{i}\phi(x_1, \ldots, x_4) = \tau(x_1, \ldots, x_4) &- \tau(x_1, x_2)\tau(x_3, x_4) \\ &- \tau(x_1, x_3)\tau(x_2, x_4) - \tau(x_1, x_4)\tau(x_2, x_3). \end{aligned}$$

We have, however, $\tau(x_1, x_2) = \mathrm{i}\phi(x_1, x_2)$, so

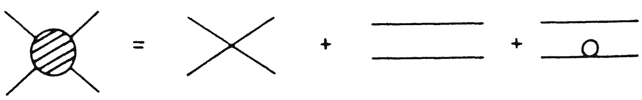
$$\tau(x_1, \ldots, x_4) = \mathrm{i}\phi(x_1, \ldots, x_4) - \sum_p \phi(x_{i_1}, x_{i_2})\phi(x_{i_3}, x_{i_4}) \qquad (6.115)$$

where $\sum_p$ stands for the sum over all possible partitions of the indices $(1 \ldots 4)$ into classes $(i_1, i_2)(i_3, i_4)$. Equation (6.115) is now the same as (6.114). The 4-point function is decomposed into an 'irreducible' (or 'connected') part $\times$ and reducible parts $=$ and $\overset{\circ}{=}$, in which, in general, a subset of the incident particles scatters into a subset of the final particles, quite independently of the rest. For the 4-point function we have,



(6.116)

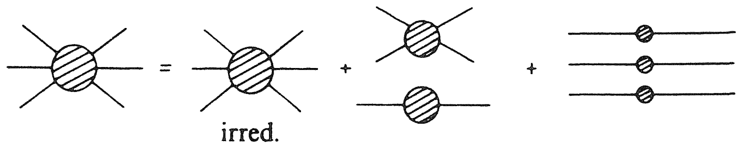
which gives, to first order in g,



(6.117)

* For a more general proof that $W[J]$ generates only connected diagrams, see J. Zinn-Justin, (1997), section 6.1.1. I am grateful to Dr B. Sheikholeslami-Sabzevari for bringing this reference to my attention.

This generalises to n-point functions, for example



(6.118)

6.7 Fermions and functional methods

We saw in §4.3 that there is a connection between spin and statistics, so that fermion fields obey the anticommutation relations

$$\{\psi(x), \psi(y)\}|_{x^0=y^0} = 0.$$

(Actually, the restriction $x^0 = y^0$ is unnecessary; the fields anticommute at all times.) In the canonical approach to field theory, $\psi(x)$ are regarded as operators, so we deal with a set of anticommuting operators. In the functional approach, however, the generating functional for the Green's functions is written as a functional integral over the fields, which are regarded as *classical* functions: c-numbers. To extend functional methods to Fermi fields, therefore, demands that, in the functional integral, they are regarded as *anticommuting c-numbers*. To most physicists this notion is strange, if not contradictory, but in the mathematical literature it goes back to 1855, where it appears in a paper by Hermann Grassmann on linear algebra. The generators C_i of an n-dimensional *Grassmann algebra* obey

$$\{C_i, C_j\} \equiv C_iC_j + C_jC_i = 0 \tag{6.119}$$

where $i, j = 1, 2, \ldots, n$. In particular,

$$C_i^2 = 0. \tag{6.120}$$

The expansion of a function contains only a finite number of terms; for example, for two variables

$$\begin{aligned} f(C_1, C_2) &= a_0 + a_1C_1 + a_2C_2 + a_3C_1C_2 \\ &= a_0 + a_1C_1 + a_2C_2 - a_3C_2C_1 \end{aligned}$$

where $a_0, \ldots, a_3$ are ordinary c-numbers. There are no terms in $C_1^2C_2$, etc. because of (6.120). We may define differentiation as *left* differentiation, so that, for example, operating on the function above

$$\left.\begin{aligned} \frac{\partial f}{\partial C_1} &= \frac{\partial^{\mathrm{L}} f}{\partial C_1} = a_1 + a_3C_2, \\ \frac{\partial f}{\partial C_2} &= \frac{\partial^{\mathrm{L}} f}{\partial C_2} = a_2 - a_3C_1. \end{aligned}\right\} \tag{6.121a}$$

Of course, it would also be possible to define it as *right* differentiation

$$\frac{\partial^{R} f}{\partial C_1} = a_1 - a_3 C_2, \tag{6.121b}$$

but I shall usually choose differentiation to mean left differentiation. Note that

$$\begin{aligned} C_1 \frac{\partial f}{\partial C_1} &= a_1 C_1 + a_3 C_1 C_2 \\ C_1 f &= a_0 C_1 + a_2 C_1 C_2 \\ \frac{\partial}{\partial C_1}(C_1 f) &= a_0 + a_2 C_2, \end{aligned}$$

hence

$$\left(C_1 \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_1} C_1\right) f = f$$

or

$$C_1 \frac{\partial}{\partial C_1} + \frac{\partial}{\partial C_1} C_1 = 1$$

as an operator identity. In general we have

$$\left\{C_i, \frac{\partial}{\partial C_j}\right\} = \delta_{ij}, \quad \left\{\frac{\partial}{\partial C_i}, \frac{\partial}{\partial C_j}\right\} = 0; \tag{6.122}$$

the second identity is one which the reader will have no problem in verifying.

We now need to define integration with respect to Grassmann variables. We clearly need the infinitesimals $\mathrm{d}C_i$ to be Grassmann quantities, so that

$$\left.\begin{aligned} \{C_i, \mathrm{d}C_j\} &= 0, \\ \{\mathrm{d}C_i, \mathrm{d}C_j\} &= 0. \end{aligned}\right\} \tag{6.123}$$

Multiple integrals are to be interpreted as iterated, so for example

$$\int \mathrm{d}C_1 \, \mathrm{d}C_2 f(C_1, C_2) \equiv \int \mathrm{d}C_1 \left[\int \mathrm{d}C_2 f(C_1, C_2)\right].$$

What are $\int \mathrm{d}C_1$ and $\int C_1 \, \mathrm{d}C_1$? We have

$$\begin{aligned} \left(\int \mathrm{d}C_1\right)^2 &= \int \mathrm{d}C_1 \int \mathrm{d}C_2 \\ &= \int \mathrm{d}C_1 \, \mathrm{d}C_2 \\ &= -\int \mathrm{d}C_2 \, \mathrm{d}C_1 \\ &= -\left(\int \mathrm{d}C_1\right)^2, \end{aligned}$$

hence

$$\int \mathrm{d}C_1 = \int \mathrm{d}C_2 = 0.$$

Since there is no other scale to Grassmann variables, we are free to define

$$\int \mathrm{d}C_1 C_1 = 1, \text{ etc.}$$

In the n-dimensional case these last two equations become

$$\int \mathrm{d}C_i = 0, \quad \int \mathrm{d}C_i C_i = 1. \tag{6.124}$$

(there is no summation over i in the second equation.) Referring to the function $f(C_1, C_2)$ above, we then have

$$\begin{aligned}\int \mathrm{d}C_1 f &= \int \mathrm{d}C_1[a_0 + a_1C_1 + a_2C_2 + a_3C_1C_2] \\ &= a_0\int \mathrm{d}C_1 + a_1\int \mathrm{d}C_1C_1 - a_2C_2\int \mathrm{d}C_1 + a_3C_2\int \mathrm{d}C_1C_1 \\ &= a_1 + a_3C_2.\end{aligned}$$

From (6.121a) we see that *differentiation and integration give the same result!*

Now let η and $\bar{\eta}$ be independent (complex) Grassmann quantities, so that

$$\int \mathrm{d}\eta = \int \mathrm{d}\bar{\eta} = 0, \quad \int \mathrm{d}\eta\eta = \int \mathrm{d}\bar{\eta}\bar{\eta} = 1.$$

Because $\eta^2 = \bar{\eta}^2 = 0$, we have

$$\mathrm{e}^{-\bar{\eta}\eta} = 1 - \bar{\eta}\eta$$

and hence

$$\begin{aligned}\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\mathrm{e}^{-\bar{\eta}\eta} &= \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta - \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\bar{\eta}\eta \\ &= 0 + \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\eta\bar{\eta} \\ &= 1.\end{aligned}$$

We now look for the generalisation of this formula to higher dimensions: let us consider the 2-dimensional case,

$$\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \quad \bar{\eta} = \begin{pmatrix}\bar{\eta}_1\\ \bar{\eta}_2\end{pmatrix}.$$

The exponent $\bar{\eta}\eta$ (which should properly be written $\bar{\eta}^{\mathrm{T}}\eta$, where T stands for transpose) is

$$\bar{\eta}\eta = \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2$$

so

$$\begin{aligned}(\bar{\eta}\eta)^2 &= (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)(\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)\\ &= \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 + \bar{\eta}_2\eta_2\bar{\eta}_1\eta_1\\ &= 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\end{aligned}$$

and higher powers of $\bar{\eta}\eta$ are zero, so we have

$$e^{-\bar{\eta}\eta} = 1 - (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) + \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2.$$

Applying the integration rules above, and *defining* $d\bar{\eta}\, d\eta = d\bar{\eta}_1\, d\eta_1\, d\bar{\eta}_2\, d\eta_2$, we then see that

$$\begin{aligned}\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} &= \int d\bar{\eta}_1\, d\bar{\eta}_2\, d\eta_1\, d\eta_2 \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\\ &= 1,\end{aligned} \tag{6.125}$$

as in the 1-dimensional case.

Now let us change variables, putting

$$\eta = M\alpha, \quad \bar{\eta} = N\bar{\alpha} \tag{6.126}$$

where M and N are 2×2 matrices, and α and $\bar{\alpha}$ are the new independent Grassmann quantities. We have

$$\begin{aligned}\eta_1\eta_2 &= (M_{11}\alpha_1 + M_{12}\alpha_2)(M_{21}\alpha_1 + M_{22}\alpha_2)\\ &= (M_{11}M_{22} - M_{12}M_{21})\alpha_1\alpha_2\\ &= (\det M)\alpha_1\alpha_2.\end{aligned}$$

However, in order to preserve the integration rules

$$\int d\eta_1\, d\eta_2 \eta_1\eta_2 = \int d\alpha_1\, d\alpha_2 \alpha_1\alpha_2$$

we must require

$$d\eta_1\, d\eta_2 = (\det M)^{-1}\, d\alpha_1\, d\alpha_2 \tag{6.127}$$

in contrast to the normal rule for a change of variable. Substituting (6.126) into (6.125), and remembering (6.127), we now have

$$(\det MN)^{-1} \int d\bar{\alpha}\, d\alpha\, e^{-\bar{\alpha}N^{\mathrm{T}}M\alpha} = 1.$$

But since $\det MN = \det M^{\mathrm{T}}N$, this gives, putting $M^{\mathrm{T}}N = A$,

$$\int d\bar{\alpha}\, d\alpha\, e^{-\bar{\alpha}A\alpha} = \det A. \tag{6.128}$$

This formula, or rather its generalisation to the infinite-dimensional case, will be used in the next chapter, in finding the Feynman rules for gauge fields.

To describe Fermi fields, we now make the transition to an *infinite-dimensional Grassmann algebra*, whose generators may be denoted $C(x)$. They obey the relations

$$\left.\begin{aligned} &\{C(x), C(y)\} = 0, \\ &\frac{\partial^{L,R} C(x)}{\partial C(y)} = \delta(x - y), \\ &\int dC(x) = 0; \int C(x)\, dC(x) = 1. \end{aligned}\right\} \tag{6.129}$$

Integrals like (6.128) then become functional integrals over complex Grassmann variables. As in the case of scalar fields, we shall treat these formulae with the confidence that, one day, a rigorous mathematical justification for them will be found. In this spirit, we write down an expression for the generating functional for free Dirac fields, by analogy with our treatment of scalar fields – see equation (6.1). Since the Lagrangian for the Dirac field is

$$\mathscr{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi,$$

the normalised generating functional for free Dirac fields is

$$Z_0[\eta, \bar{\eta}] = \frac{1}{N}\int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp\left\{i\int [\bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)]\, dx\right\} \tag{6.130}$$

where the integral over x is 4-dimensional, and

$$N = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp\left[i\int \bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x)\, dx\right]. \tag{6.131}$$

Here $\bar{\eta}(x)$ represents the source term for $\psi(x)$, and $\eta(x)$ the source for $\bar{\psi}(x)$. It is now our aim to express this in a form analogous to equation (6.13), so that we can perform functional differentiation, and calculate Green's functions and S-matrix elements. To simplify the appearance of the formulae, we define the operator S^{-1}

$$S^{-1} = i\gamma^\mu \partial_\mu - m. \tag{6.132}$$

Then

$$Z_0[\eta, \bar{\eta}] = \frac{1}{N}\int \mathscr{D}\bar{\psi}\mathscr{D}\psi \exp\left[i\int (\bar{\psi}S^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta)\, dx\right]. \tag{6.133}$$

Putting

$$Q(\psi, \bar{\psi}) = \bar{\psi}S^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta,$$

we now find the value of ψ which minimises Q. It is

$$\psi_m = -S\eta, \quad \bar{\psi}_m = -\bar{\eta}S$$

(where we have assumed that S^{-1} possesses an inverse; it will be shown below that it does), and the minimum value of Q is

$$Q_m = Q(\psi_m, \bar{\psi}_m) = -\bar{\eta}S\eta.$$

We then have

$$Q = Q_m + (\bar{\psi} - \bar{\psi}_m)S^{-1}(\psi - \psi_m)$$

and

$$\begin{aligned} Z_0 &= \frac{1}{N}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ \mathrm{i}\int [Q_m + (\bar{\psi} - \bar{\psi}_m)S^{-1}(\psi - \psi_m)]\,\mathrm{d}x \right\} \\ &= \frac{1}{N}\exp\left[-\mathrm{i}\int \bar{\eta}(x)S\eta(y)\,\mathrm{d}x\,\mathrm{d}y \right] \det(-\mathrm{i}S^{-1}). \end{aligned}$$

In the last step, $\mathrm{e}^{\mathrm{i}Q_m}$ has been placed outside the integral, since Q_m does not depend on ψ or $\bar{\psi}$, and equation (6.128) has been used, duly extended to the functional case. Moreover, it is clear that $N = \det(-\mathrm{i}S^{-1})$, so, finally,

$$Z_0[\eta, \bar{\eta}] = \exp\left[-\mathrm{i}\int \bar{\eta}(x)S(x-y)\eta(y)\,\mathrm{d}x\,\mathrm{d}y \right]. \tag{6.134}$$

It is easy to show that S exists. It is given by

$$S(x) = (\mathrm{i}\gamma\cdot\partial + m)\Delta_F(x) \tag{6.135}$$

where $\Delta_F(x)$ is the Feynman propagator. With equation (6.132) we have

$$\begin{aligned} S^{-1}S &= (\mathrm{i}\gamma\cdot\partial - m)(\mathrm{i}\gamma\cdot\partial + m)\Delta_F(x) \\ &= (-\Box - m^2)\Delta_F(x) \\ &= \delta^4(x). \end{aligned}$$

We may now find the free propagator for the Dirac field. By analogy with equation (6.50), it is defined by

$$\begin{aligned} \tau(x, y) &= -\left.\frac{\delta^2 Z_0[\eta, \bar{\eta}]}{\delta\eta(x)\delta\bar{\eta}(y)}\right|_{\eta=\bar{\eta}=0} \\ &= -\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\bar{\eta}(y)}\exp\left.\left\{ -\mathrm{i}\int \bar{\eta}(x)S(x-y)\eta(y)\,\mathrm{d}x\,\mathrm{d}y \right\}\right|_{\eta=\bar{\eta}=0} \\ &= \mathrm{i}S(x-y). \end{aligned} \tag{6.136}$$

Let us summarise our formulae for the free propagators of scalar and spinor fields. For scalar fields, with Lagrangian (up to a total divergence)

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 = -\tfrac{1}{2}\phi(\Box + m^2)\phi$$

we found the 2-point function

$$\tau(x, y) = \mathrm{i}\Delta_F(x - y) \tag{6.52}$$

where the Feynman propagator Δ_F obeys

$$(\Box + m^2)\Delta_F(x - y) = -\delta^4(x - y). \tag{6.10}$$

For spinor fields, the Lagrangian is (see (6.132))

$$\mathscr{L}_0 = \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = \bar{\psi}S^{-1}\psi$$

and the 2-point function is i times the propagator

$$\tau(x, y) = \mathrm{i}S(x - y).$$

In each case, it is seen that *the propagator is the inverse of the operator appearing in the quadratic term in the Lagrangian*. (The factor $\frac{1}{2}$ in the scalar Lagrangian is immaterial, and appears because ϕ is real; for complex ϕ, it is absent.) It is possible to take this as a *definition* of a propagator, and this is what we shall do when we consider gauge fields.

Finally, it will be convenient here to point out a further consequence of fields obeying Fermi statistics. It follows from the relation obeyed by the differential operators for the Grassmann fields. By a generalisation of (6.123), we have

$$\frac{\delta^2}{\delta\eta(x)\delta\eta(y)} = -\frac{\delta^2}{\delta\eta(y)\delta\eta(x)} \tag{6.137}$$

where η is a fermion source, and the operation of differentiation refers to either left or right differentiation. For left differentiation we have

$$\frac{\delta}{\delta\eta(x_1)}[\eta(x)\eta(y)] = \delta^4(x_1 - x)\eta(y) - \delta^4(x_1 - y)\eta(x).$$

What we want to show is that these rules result in a factor of -1 for each closed fermion loop in a Feynman diagram. For example, a spinor field coupled to a scalar field will give a correction to the free scalar propagator shown in Fig. 6.5, which contains a closed fermion loop and two interaction vertices. The appropriate 2-point function will be derived from the generating functional with interactions included. By a generalisation of (6.76), this will take the form

$$Z[\eta, \bar{\eta}] = \exp\left[\mathrm{i}\int \mathscr{L}_{\mathrm{int}}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\eta}, \frac{1}{\mathrm{i}}\frac{\delta}{\delta\bar{\eta}}\right)\mathrm{d}x\right]Z_0[\eta, \bar{\eta}] \tag{6.138}$$

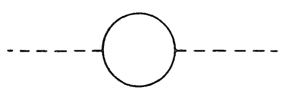


Fig. 6.5. Modification to the scalar field propagator by a closed fermion loop.

with Z_0 given by (6.134). The third term in its expansion is

$$-\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'\bar{\eta}(x)S(x-y)\eta(y)\bar{\eta}(x')S(x'-y')\eta(y').$$

The loop in Fig. 6.5 will contribute a term of the form

$$\frac{\delta^2}{\delta\bar{\eta}_i(z)\delta\eta_j(z)}\frac{\delta^2}{\delta\bar{\eta}_k(z')\delta\eta_l(z')}Z[\eta,\bar{\eta}]$$

(i, j, k and l are spinor indices). On substituting (6.138) for $Z[\eta, \bar{\eta}]$ and applying (6.137), this term is seen to be

$$+\ S_{il}(z-z')S_{kj}(z'-z).$$

The overall sign would be $-$ if the fields obeyed Bose statistics, hence the -1 factor for the fermion loop.

6.8 The *S* matrix and reduction formula

We have seen how to calculate the Green's functions for an interacting theory, but we now want to calculate quantities which we measure directly in experiments. The commonest types of process which concern the elementary particle physicist are firstly scattering processes, in which a *cross section* for a particular reaction is measured, and secondly a decay of one particle into two or more, in which a partial lifetime is measured. The calculation of both these quantities, the cross section and the lifetime, is carried out by first calculating the quantum mechanical *amplitude* that the process takes place. Once we have the amplitude, the rest of the calculation is fairly straightforward. In this section we show how to calculate the amplitude, which we call the scattering amplitude, and show how a particular scattering amplitude is related by a simple formula to a corresponding Green's function. We apply this to the case of the pion–nucleon interaction, and in the next section show how to obtain the scattering cross section. Let us consider, quite generally, a process in which an initial configuration of particles α ends up as a final configuration β. We denote the scattering amplitude for this $S_{\beta\alpha}$, and call it the ($\beta\alpha$) 'matrix element' of the *scattering matrix* or *S matrix*. S is the collection of all $S_{\beta\alpha}$. The states α and β are defined asymptotically, at times $t\to-\infty$ and $t\to\infty$ respectively, so we define

$$S_{\beta\alpha} = \langle\beta, t\to\infty|\alpha, t\to-\infty\rangle. \tag{6.139}$$

In the absence of interactions with long-range forces, these asymptotic states consist of *free particles*, which is a great simplification. Long-range interactions, like electromagnetism, bring complications which we prefer to leave aside. An alternative notation is to define the 'in' and 'out' states

$$|\alpha\rangle_{\text{in}} = |\alpha, t\to-\infty\rangle, \quad |\beta\rangle_{\text{out}} = |\beta, t\to\infty\rangle.$$