Toda equations for Hurwitz numbers

Andrei Okounkov[*]

Abstract

We consider ramified coverings of $\mathbb{P}^1$ with arbitrary ramification type over $0, \infty \in \mathbb{P}^1$ and simple ramifications elsewhere and prove that the generating function for the numbers of such coverings is a τ-function for the Toda lattice hierarchy of Ueno and Takasaki.

1 Introduction

1.1

In a recent paper [14], R. Pandharipande showed that the conjectural Toda equation for the Gromov-Witten potential $F_{\mathbb{P}^1}$ of the Riemann sphere $\mathbb{P}^1$ implies a Toda equation for the generating function of the Hurwitz numbers $H_{g,d}$. Here $H_{g,d}$ is the number of degree d connected coverings of $\mathbb{P}^1$ by a curve of genus g which have simple ramifications over some $2g + 2d - 2$ fixed points of $\mathbb{P}^1$. This Toda equation results in a simple recurrence relation for the numbers $H_{g,d}$.

The purpose of this paper is to prove that a more general generating function, where arbitrary ramifications are allowed over two given points of $\mathbb{P}^1$ (for example, over $0, \infty \in \mathbb{P}^1$) satisfies an entire hierarchy of PDE's, namely the Toda lattice hierarchy of Ueno and Takasaki [15]. This proves and generalizes the result of Pandharipande.

We will call these more general numbers of ramified coverings the double Hurwitz numbers.

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[*]Department of Mathematics, University of California at Berkeley, Evans Hall #3840, Berkeley, CA 94720-3840. E-mail: okounkov@math.berkeley.edu

1.2

Our approach is quite straightforward: we remark that the generating function for double Hurwitz numbers is almost by definition a certain matrix element in the infinite wedge space. It is a well known result of the Kyoto school that such matrix elements are τ-functions of integrable hierarchies.

These Toda equations should probably be not too surprising from the physical point of view. Physicists consider the enumeration of coverings of $\mathbb{P}^1$ as a sort of string theory on $\mathbb{P}^1$, see for example [8], and relate it to a number of other models where the occurrence of integrable hierarchies is common.

1.3

It is known, see [3] and also [7, 4], that ordinary Hurwitz numbers, that is, the numbers of coverings where the ramification over $\infty \in \mathbb{P}^1$ is arbitrary and all other ramifications are simple, can be expressed as certain Hodge integrals over moduli spaces of curves. It is an interesting problem to transfer the Toda equation to the Hodge integrals side. In particular, it would be nice to reproduce the KdV equations or Virasoro constraints, see for example [9, 5], for just the Gromov-Witten potential of a point, which corresponds to Hodge integrals with no λ-classes.

It is also interesting and challenging to find an expression for the double Hurwitz numbers in terms of the GW theory of $\mathbb{P}^1$. The conjectural Toda equation for the GW potential of $\mathbb{P}^1$, see [14], may be an indication of such a connection.

1.4

The Toda lattice equations is an infinite number of PDE's satisfied by the generating function for coverings, each resulting is some recurrence relations for the double Hurwitz numbers. These relations uniquely determine the double Hurwitz numbers.

In particular, the lowest equation of the Toda hierarchy, see Sections 2.8 and 2.9, describes what will happen if we add an extra sheet to our covering leaving the nontrivial part of the monodromy the same.

Many recurrent relations for Hurwitz numbers are known in the literature, see for example [7] and references therein, but as R. Pandharipande writes

in [14], the form of the Toda recurrence relations is "certainly the simplest recursion for the Hurwitz numbers known to the author" . It is absolutely beyond the scope of the present note to give any sort of survey on the vast literature on Hurwitz numbers and their applications.

1.5

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2 Coverings and Toda equations

2.1

Let $S(d)$ denote the symmetric group and fix some conjugacy classes

$$C_1, \ldots, C_s \subset S(d) \, .$$

Let $\mathrm{Cov}_d(C_1, \ldots, C_s)$ denote the weighted number of d-fold coverings of $\mathbb{P}^1$ ramified over s fixed points of $\mathbb{P}^1$ with monodromies in the conjugacy classes $C_1, \ldots, C_s$. The weight of any covering in Cov_d is the reciprocal of the order of its group of automorphisms. This automorphisms group is the same as the centralizer of all monodromies in $S(d)$. Note that Cov_d counts all coverings, that is, including the disconnected ones.

For this weighted number there is the following formula, see for example [2], which goes back to Burnside

$$\mathrm{Cov}_d(C_1, \ldots, C_s) = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{s} f_{C_i}(\lambda) \, , \qquad (1)$$

where the sum is over all partitions λ of d, $\dim \lambda$ is the dimension of the corresponding representation of the symmetric group $S(d)$, and

$$f_{C_i}(\lambda) = |C_i| \frac{\chi^\lambda(C_i)}{\dim \lambda} \, . \qquad (2)$$

Here $|C_i|$ is the cardinality of the conjugacy class C_i and $\chi^\lambda(C_i)$ is the value of the irreducible character χ^λ on any element of C_i. The expression (2) is in fact a polynomial in λ in a certain natural sense, see for example [13].

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2.2

Recall that the Schur functions s_λ are certain distinguished symmetric functions associated to any partition λ. In terms of the power-sum symmetric functions $p_1, p_2, \ldots$ they have the following expression, see [11],

$$s_\lambda(p_1, p_2, \ldots) = \frac{1}{d!} \sum_{|\mu|=d} \chi^\lambda(C_\mu) \, |C_\mu| \, p_\mu \, ,$$

where the summation is over all partitions μ of $d = |\lambda|$, C_μ is the conjugacy class corresponding to the partition μ, and $p_\mu = \prod p_{\mu_i}$. We will treat the p_k's as independent variables.

Now let $P = (p_1, p_2, \ldots)$ and $P' = (p'_1, p'_2, \ldots)$ be two sets of variables. We have the following equality:

$$\sum_{d=0}^\infty q^d \sum_{|\mu|=|\nu|=d} p_\mu \, p'_\nu \, \mathrm{Cov}_d(C_\mu, C_\nu, C_1, \ldots, C_s) =$$

$$\sum_\lambda q^{|\lambda|} s_\lambda(P) \, s_\lambda(P') \prod_{i=1}^s f_{C_i}(\lambda) \, , \quad (3)$$

where q is a new variable which keeps track of the degree d of the covering. The variables q is redundant since for every term its degree d in q is equal to both its degree in P and in P', where

$$\deg p_k = \deg p'_k = k \, ,$$

but it is convenient to keep it.

2.3

Let $C_{(2)}$ be the conjugacy class of a transposition. Introduce the following generating function

$$\tau(P, P', \beta, q) = \sum_{d,b,\mu,\nu} q^d \, \beta^b \, p_\mu \, p'_\nu \, \mathrm{Cov}_d(C_\mu, C_\nu, \underbrace{C_{(2)}, \ldots, C_{(2)}}_{b \text{ times}})/b! \, .$$

This is the generating function for degree d possibly disconnected coverings of $\mathbb{P}^1$ such that the ramification over $0, \infty \in \mathbb{P}^1$ can be fixed arbitrarily and the

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other b ramifications are simple. It is a general property of the exponential generating functions, see e.g. Chapter 3 in [6], that

$$\log \tau(P, P', \beta, q) = H(P, P', \beta, q) \,,$$

where $H(P, P', \beta, q)$ is the generating function for connected coverings

$$H(P, P', \beta, q) = \sum_{d,b,\mu,\nu} q^d \, \beta^b \, p_\mu \, p'_\nu \, \mathrm{Hur}_{d,b}(\mu, \nu)/b! \,,$$

Here $\mathrm{Hur}_{d,b}(\mu, \nu)$ is the weighted number of *connected* degree d coverings of $\mathbb{P}^1$ with monodromy around $0, \infty \in \mathbb{P}^1$ being μ and ν, respectively, and b additional simple ramifications. The genus of such a covering is

$$g = (b + 2 - \ell(\mu) - \ell(\nu))/2 \,,$$

where $\ell(\mu)$ is the number of parts of μ. The weight of each covering in $\mathrm{Hur}_{d,b}(\mu, \nu)$ is the reciprocal of the order of its automorphism group. We call the numbers $\mathrm{Hur}_{d,b}(\mu, \nu)$ the *double Hurwitz numbers*.

We also point out that if when one of ramifications μ and ν is trivial and the covering is connected of degree $d > 2$ then the automorphism group is trivial and so $H = \log \tau$ counts any such covering with weight 1.

2.4

Denote by f_2 the polynomial (2) corresponding to the class $C_{(2)}$ of a transposition. From (3) we have

$$\tau(P, P', \beta, q) = \sum_\lambda q^{|\lambda|} \, e^{\beta \, f_2(\lambda)} \, s_\lambda(P) \, s_\lambda(P') \,. \tag{4}$$

Explicitly, the polynomial f_2 is given by

$$f_2 = \frac{1}{2} \sum_i \left[\left(\lambda_i - i + \tfrac{1}{2} \right)^2 - \left(-i + \tfrac{1}{2} \right)^2 \right] \,. \tag{5}$$

Remark that the upper limit of summation can be taken to be $+\infty$ in the above sum since $\lambda_i = 0$ for all sufficiently large i. Using some standard combinatorics, see for example Section 5 of [1], one can rewrite the formula (5) as follows

$$f_2 = \sum_{k \in \mathfrak{S}(\lambda)_+} \frac{k^2}{2} - \sum_{k \in \mathfrak{S}(\lambda)_-} \frac{k^2}{2} \tag{6}$$

where
$$\mathfrak{S}(\lambda) = \{\lambda_i - i + \tfrac{1}{2}\} \subset \mathbb{Z} + \tfrac{1}{2}$$
and, by definition, for any $S \subset \mathbb{Z} + \tfrac{1}{2}$ we set
$$S_+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2}\right), \quad S_- = \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2}\right) \setminus S.$$

2.5

The sum (4) admits a straightforward interpretation as a certain matrix element in the infinite wedge space. It will be convenient for us to use the notation of [12], see especially the Appendix to [12] for a summary of the infinite wedge basics. A good general reference on the infinite wedge space is Chapter 14 of [9].

Note that the power-sum variables P and P' are related to the variables t and t' used in [12] by

$$t_k = \frac{p_k}{k}, \quad t'_k = \frac{p'_k}{k}, \quad k = 1, 2, \dots,$$

We will use the following abbreviations

$$\Gamma_+ = \Gamma_+\left(p_1, \frac{p_2}{2}, \frac{p_3}{3}, \dots\right), \quad \Gamma_- = \Gamma_-\left(p'_1, \frac{p'_2}{2}, \frac{p'_3}{3}, \dots\right).$$

The formula (4) becomes

$$\tau(P, P', \beta, q) = \left(\Gamma_+ \, q^H e^{\beta F_2} \, \Gamma_- \, v_\emptyset, v_\emptyset\right), \tag{7}$$

where $v_\emptyset$ is the vacuum vector of the infinite wedge space, H is the energy operator, and F_2 is the following operator

$$F_2 = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{k^2}{2} \, :\psi_k \, \psi_k^*:,$$

where the colons denote the normal ordering. It is clear from (6) that the operator F_2 acts as follows

$$F_2 \, v_\lambda = f_2(\lambda) \, v_\lambda,$$

where

$$v_\lambda = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots, \quad \{s_1, s_2, \dots\} = \mathfrak{S}(\lambda),$$

is the vector corresponding to the partition λ.

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2.6

Any operator of the form $\xi = \psi_i \psi_j^*$ satisfies the relation

$$[\xi \otimes 1 + 1 \otimes \xi, \Omega] = 0, \quad \Omega = \sum \psi_k \otimes \psi_k^*,$$

Since the operators q^H and $e^{\beta F_2}$ are exponentials of sums of such operators ξ, we have

$$\left[q^H e^{\beta F_2} \otimes q^H e^{\beta F_2}, \Omega\right] = 0.$$

It follows that the sequence

$$\tau_n = \left(\Gamma_+ q^H e^{\beta F_2} \Gamma_- v_n, v_n\right), \quad n \in \mathbb{Z}, \tag{8}$$

where

$$v_n = \underline{n - \tfrac{1}{2}} \wedge \underline{n - \tfrac{3}{2}} \wedge \underline{n - \tfrac{5}{2}} \wedge \dots,$$

is the vacuum vector in the charge n subspace, is a sequence of τ-functions for the Toda lattice hierarchy. Our old $\tau(P, P', \beta, q)$ is the τ_0 term of this sequence.

2.7

We will now show that all terms of the sequence (8) can be expressed in terms of $\tau(P, P', \beta, q)$. We have $v_n = R^n v_\emptyset$, where R is the translation operator, and also $[R, \Gamma_\pm] = 0$. We compute

$$R^{-n} F_2 R^n = \sum_{k>0} \frac{k^2}{2} \psi_{k-n} \psi_{k-n}^* - \sum_{k<0} \frac{k^2}{2} \psi_{k-n}^* \psi_{k-n}$$

$$= \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{(k+n)^2}{2} :\psi_k \psi_k^*: + \sum_{k=1/2}^{n-1/2} \frac{k^2}{2}$$

$$= F_2 + n H + \frac{n^2}{2} C + \frac{n(4n^2 - 1)}{24},$$

where H and C are the energy and charge operators. Similarly,

$$R^{-n} H R^n = H + nC + \frac{n^2}{2}.$$

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Since the charge operator C commutes with $\Gamma_{\pm}$ and $Cv_{\emptyset} = 0$, we have

$$
\begin{aligned}
\tau_n &= \left(\Gamma_+ \, q^H e^{\beta F_2} \, \Gamma_- \, v_n, v_n \right) \\
&= \left(\Gamma_+ \, R^{-n} q^H e^{\beta F_2} R^n \, \Gamma_- \, v_{\emptyset}, v_{\emptyset} \right) \\
&= q^{n^2/2} e^{n(4n^2-1)\beta/24} \left(\Gamma_+ \, (e^{n\beta}q)^H \, e^{\beta F_2} \, \Gamma_- \, v_{\emptyset}, v_{\emptyset} \right) \\
&= q^{n^2/2} e^{n(4n^2-1)\beta/24} \, \tau(P, P', \beta, e^{n\beta}q) \,.
\end{aligned}
$$

Substituting this expression into the Hirota bilinear equations for the Toda lattice hierarchy, we obtain the following

Theorem. *The function $\tau(P, P', \beta, q)$ satisfies the following Hirota-type bilinear equations:*

$$
q^{m+1} e^{m(m+1)\beta/2} [z^{-1-m}] \, e^{-2\sum n\, s_n/z^n} \, \diamondsuit \, \heartsuit = [z^{m+1}] \, e^{2\sum n\, s'_n/z^n} \, \clubsuit \, \spadesuit \,, \qquad (9)
$$

where $m \in \mathbb{Z}$, z is a formal variable, $[z^{m+1}]$ denotes the coefficient of z^{m+1}, $S = (s_1, s_2, \dots)$ and $S' = (s'_1, \dots)$ are two arbitrary sequences,

$$
\begin{aligned}
\diamondsuit &= \tau(P + S, P' + S' + \vec{z}, \beta, e^{(m+1)\beta}q), & \clubsuit &= \tau(P + S - \vec{z}, P' + S', \beta, e^{m\beta}q), \\
\heartsuit &= \tau(P - S, P' - S' - \vec{z}, \beta, e^{-\beta}q), & \spadesuit &= \tau(P - S + \vec{z}, P' - S', \beta, q),
\end{aligned}
$$

and $\vec{z} = (z, z^2, z^3, \dots)$.

Remark that, as pointed out earlier,

$$
\begin{aligned}
\tau(P, P', \beta, q) &= \tau(q\, p_1, q^2\, p_2, q^3\, p_3, \dots, P', \beta, 1) \\
&= \tau(P, q\, p'_1, q^2\, p'_2, q^3\, p'_3, \dots, \beta, 1) \,.
\end{aligned}
$$

2.8

In particular, setting $m = 0$ and taking the coefficient of s_1 gives

$$
\tau \, \frac{\partial^2 \tau}{\partial p_1 \partial p'_1} - \frac{\partial \tau}{\partial p_1} \frac{\partial \tau}{\partial p'_1} = q \, \tau(\dots, e^{\beta}q) \, \tau(\dots, e^{-\beta}q) \,.
$$

This is equivalent to

$$
\frac{\partial^2}{\partial p_1 \partial p'_1} \log \tau = q \, \frac{\tau(\dots, e^{\beta}q) \, \tau(\dots, e^{-\beta}q)}{\tau^2} \,. \qquad (10)
$$

8

Recall that
$$\log \tau = H$$
is the generating function for the connected coverings.

The combinatorial meaning of the operator $\dfrac{\partial^2}{\partial p_1 \partial p_1'}$ is that it peels off a sheet of the covering which is unramified over the special points $0, \infty \in \mathbb{P}^1$. The equation (10) gives thus a recursive procedure for the computation of the double Hurwitz numbers with a fixed nontrivial part of the monodromy around $0, \infty$.

2.9

In the particular case when there is no ramification over $0, \infty$, that is, when,

$$p_2 = p_3 = \cdots = p_2' = p_3' = \cdots = 0$$

the function τ depends on the variables p_1, p_1', and q only via their product $q p_1 p_1'$. If we set
$$u = \log q \, p_1 \, p_1'$$
then the equation (10) becomes

$$e^{-u} \frac{\partial^2}{\partial u^2} \log \tau = \frac{\tau(u + \beta, \beta)\,\tau(u - \beta, \beta)}{\tau^2},$$

which is equivalent to the Toda equation derived in [14].

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