A finite dimensional proof of the Verlinde formula

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Motivation: Jacobian variety and theta functions

- Let C be a smooth projective curve of genus g,

$$J_C = \{\text{vector bundles } E \text{ of } \mathrm{rk}(E) = 1,\, \deg(E) = 0 \text{ on } C\}$$

- Fix a $\kappa_0 \in \mathrm{Pic}^{g-1}(C)$, we have theta divisor

$$\theta_{\kappa_0} = \{\, E \in J_C \,|\, \mathrm{H}^0(C, E \otimes \kappa_0) \neq 0 \,\} \subset J_C$$

- Let $\mathcal{E}$ be a universal line bundle on $C \times J_C \xrightarrow{\pi} J_C$, then

$$\Theta_{J_C} := \mathcal{O}(\theta_{\kappa_0})^k = \det R\pi(\mathcal{E} \otimes \kappa_0)^{-k}$$

- $\mathrm{H}^0(J_C, \Theta_{J_C})$ is the so called space of **theta functions of order** k

$$\dim \mathrm{H}^0(J_C, \Theta_{J_C}) = k^g$$

Motivation: Moduli spaces and generalized theta functions

- A. Weil observed: $J_C \cong \mathrm{Hom}(\pi_1(C), U(1))$, suggested to study

$$\mathrm{Hom}(\pi_1(C), U(r))$$

- (Mumford, Narasimhan-Seshadri): On $\mathrm{Hom}(\pi_1(C), U(r))$, there is a natural **structure of projective variety** $\mathcal{U}_C$, which is the **moduli spaces** of semi-stable vector bundles of rank r and degree 0.

$$\theta_{\kappa_0} = \{\, E \in \mathcal{U}_C \,|\, \mathrm{H}^0(C, E \otimes \kappa_0) \neq 0 \,\} \subset \mathcal{U}_C$$

$$\Theta_{\mathcal{U}_C} := \mathcal{O}(\theta_{\kappa_0})^k = \det R\pi(\mathcal{E} \otimes \kappa_0)^{-k}.$$

- A formula was predicted by **Conformal Field Theory**, when $r = 2$,

$$\dim \mathrm{H}^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = \left(\frac{k}{2}\right)^g \left(\frac{k+2}{2}\right)^{g-1} \sum_{i=0}^{k} \frac{1}{(\sin \frac{(i+1)\pi}{k+2})^{2g-2}}$$

The moduli spaces: $\mathcal{U}_{C,\omega} = \mathcal{U}_C(r, d, \omega)$

- C: projective curve of genus $g \geq 0$ with at most one node
- $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in I})$: a finite set $I \subset C$ of smooth points,

$$\vec{n}(x) := (n_1(x), n_2(x), \cdots, n_{l_x+1}(x))$$

$$\vec{a}(x) := (a_1(x), a_2(x), \cdots, a_{l_x+1}(x))$$

  and an integer $k > 0$ such that

$$0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) \leq k.$$

- $\mathcal{U}_{C,\omega}$: moduli space of semistable parabolic sheaves of rank r and degree d on C with parabolic structures determined by ω

The moduli spaces: Parabolic sheaves

- A torsion free sheaf E has a parabolic structure of type $\vec{n}(x)$ and weights $\vec{a}(x)$ at a smooth point $x \in C$, we mean a choice of

$$E_x = Q_{l_x+1}(E)_x \twoheadrightarrow \cdots\cdots \twoheadrightarrow Q_1(E)_x \twoheadrightarrow Q_0(E)_x = 0$$

  of fibre E_x with $n_i(x) = \dim(ker\{Q_i(E)_x \twoheadrightarrow Q_{i-1}(E)_x\})$ and a sequence of integers

$$0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k.$$

- For any $F \subset E$, let $Q_i(E)_x^F \subset Q_i(E)_x$ be the image of F,

$$n_i^F(x) = \dim(ker\{Q_i(E)_x^F \twoheadrightarrow Q_{i-1}(E)_x^F\})$$

$$\mathrm{par}\chi(F) := \chi(F) + \frac{1}{k} \sum_{x \in I} \sum_{i=1}^{l_x+1} a_i(x) n_i^F(x).$$

The moduli spaces: Semi-stability

- E is called **semistable** (resp., **stable**) for $\frac{\vec{a}}{k}$ if for any nontrivial subsheaf $F \subset E$ such that E/F is torsion free, one has

$$\mathrm{par}\chi(F) \leq \frac{\mathrm{par}\chi(E)}{r} \cdot r(F) \text{ (resp., } <).$$

- There exists a seminormal projective variety

$$\mathcal{U}_{C,\,\omega} = \mathcal{U}_C(r, d, \omega)$$

  which is the coarse moduli space of s-equivalence classes of semistable parabolic sheaves E of rank r and $\deg(E) = d$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$.

- If C is smooth, then it is normal, with only rational singularities.

Generalized theta functions on $\mathcal{U}_{C,\omega}$

- There is an algebraic family of ample line bundles $\Theta_{\mathcal{U}_{C,\omega}}$ on $\mathcal{U}_{C,\omega}$ (the so called Theta line bundles) when

$$\ell := \frac{k\chi - \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) r_i(x)}{r}$$

is an integer, where

$$d_i(x) = a_{i+1}(x) - a_i(x)$$

$$r_i(x) = n_1(x) + \cdots + n_i(x).$$

- $\mathrm{H}^0(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}})$: The space of generalized theta functions. An explicit formula of

$$D_g(r,d,\omega) = \dim\, \mathrm{H}^0(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}})$$

was predicted by **Conformal Field Theory**.

Verlinde formula: $D_g(r,d,\omega)=?$

$$D_g(r,d,\omega)=(-1)^{d(r-1)}\left(\frac{k}{r}\right)^g\left(r(r+k)^{r-1}\right)^{g-1}$$

$$\sum_{\vec{v}}\frac{\exp\left(2\pi i\left(\frac{d}{r}-\frac{|\omega|}{r(r+k)}\right)\sum_{i=1}^r v_i\right)S_\omega\left(\exp 2\pi i\frac{\vec{v}}{r+k}\right)}{\prod_{i<j}\left(2\sin\pi\frac{v_i-v_j}{r+k}\right)^{2(g-1)}}$$

where $\vec{v}=(v_1,v_2,\ldots,v_r)$ runs through the integers

$$0=v_r<v_{r-1}<\cdots<v_2<v_1<r+k.$$

- For given $\omega=(k,\{\vec{n}(x),\vec{a}(x)\}_{x\in I})$, let $\lambda_i=k-a_i(x)$

$$\lambda_x=(\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1(x)},\overbrace{\lambda_2,\ldots,\lambda_2}^{n_2(x)},\ldots,\overbrace{\lambda_{l_x+1},\ldots,\lambda_{l_x+1}}^{n_{l_x+1}(x)})$$

- Let $S_{\lambda_x}(z_1,...,z_r)$ be Schur polynomial, $|\lambda_x|=\sum\lambda_i n_i(x)$,

$$S_\omega(z_1,...,z_r)=\prod_{x\in I}S_{\lambda_x}(z_1,...,z_r),\quad |\omega|=\sum_{x\in I}|\lambda_x|.$$

- Let Λ be a finite set with an involution $\lambda \mapsto \lambda^*$, a **RCFT** is a functor:
$$(C, \vec{p}; \vec{\lambda}) \mapsto V_C(\vec{p}; \vec{\lambda})$$
where $\vec{p} = (p_1, \ldots, p_n)$, $p_i \in C$, $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, satisfies axioms:

- **A0:** $V_{\mathbb{P}^1}(\emptyset) = \mathbb{C}$, **A1:** $V_C(\vec{p}; \vec{\lambda}) \cong V_C(\vec{p}; \vec{\lambda}^*)$

- **A2:** Let $(C, \vec{p}; \vec{\lambda}) = (C', \vec{p}'; \vec{\lambda}') \sqcup (C'', \vec{p}''; \vec{\lambda}'')$. Then
$$V_C(\vec{p}; \vec{\lambda}) = V_{C'}(\vec{p}'; \vec{\lambda}') \otimes V_{C''}(\vec{p}''; \vec{\lambda}'')$$

- **A3:** For a family $\{C_t, \vec{p_t}; \vec{\lambda}\}_{t \in \triangle}$, there are canonical isomorphisms
$$V_{C_t}(\vec{p_t}; \vec{\lambda}) \cong V_{C_0}(\vec{p_0}; \vec{\lambda})$$

- **A4:** If C_0 has a node x, $\pi^{-1}(x) = \{x_1, x_2\}$, $\pi : \widetilde{C_0} \to C_0$. Then
$$V_{C_t}(\vec{p_t}; \vec{\lambda}) \cong \bigoplus_{\nu} V_{\widetilde{C_0}}(\vec{p_0}, x_1, x_2; \vec{\lambda}, \nu, \nu^*)$$

The fusion rules

- $\dim V_C(\overrightarrow{p};\overrightarrow{\lambda})$ depends only on $g = g(C)$ and

$$(\lambda_1, ..., \lambda_n) := \lambda_1 + \cdots + \lambda_n.$$

- $\mathbb{N}^{(\Lambda)} := \{x = \lambda_1 + \cdots + \lambda_n \,|\, n \geq 0,\ \lambda_i \in \Lambda\}$,

$$N_g : \mathbb{N}^{(\Lambda)} \to \mathbb{N}, \qquad N_g(x) := \dim_{\mathbb{C}} V_C(\overrightarrow{p};\overrightarrow{\lambda}).$$

- $N_g(x) = \sum_{\lambda\in\Lambda} N_{g-1}(x + \lambda + \lambda^*)$
- $N_0(0) = 1$
- $N_0(x) = N_0(x^*)$ ($\forall\ x \in \mathbb{N}^{(\Lambda)}$)
- $N_0(x + y) = \sum_{\lambda\in\Lambda} N_0(x + \lambda)N_0(y + \lambda^*)$ ($\forall\ x,\ y \in \mathbb{N}^{(\Lambda)}$).

The fusion ring $\mathcal{F}$ and Verlinde formula

- Let $\mathcal{F} = \mathbb{Z}^{(\Lambda)}$ be the free abelian group generated by Λ, define

$$\lambda \cdot \mu = \sum_{\nu \in \Lambda} N_0(\lambda + \mu + \nu^*) \cdot \nu.$$

- $\mathcal{F}$ is called the **fusion ring** associated to the **RCFT**,

- Let $\Sigma = \{\, \chi : \mathcal{F} \to \mathbb{C} \,\}$ be the set of characters of $\mathcal{F}$. Then

$$\dim V_C(\overrightarrow{p}; \overrightarrow{\lambda}) = \sum_{\chi \in \Sigma} \chi(\lambda_1) \cdots \chi(\lambda_n) \left(\sum_{\lambda \in \Lambda} |\chi(\lambda)|^2 \right)^{g-1}$$

Tsuchiya-Ueno-Yamada (1989): WZW model

- **Wess-Zumino-Witten (WZW) model** is associated to a simple complex Lie algebra $\mathfrak{g}$ and integer $k > 0$.

- Given a simple Lie algebra $\mathfrak{g}$ and integer $k > 0$, let P_k be the set of dominant weight of level $\leq k$, $V_{\vec{\lambda}} := V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ $(\lambda_i \in P_k)$ and
$$V_C(\vec{p}; \vec{\lambda}) := \mathrm{Hom}_{\mathfrak{g}\otimes A_C}(\mathcal{H}_k, V_{\vec{\lambda}}), \quad A_C = \mathcal{O}_C(C - \{q\})$$
where $\mathcal{H}_k$ is the basic representation of level k of affine Lie algebra $\widehat{\mathfrak{g}}$, and $\mathfrak{g} \otimes A_C \hookrightarrow \mathfrak{g} \otimes \mathbb{C}((z)) \subset \widehat{\mathfrak{g}}$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$.

- Tsuchiya-Ueno-Yamada (1989): Functor: $(C, \vec{p}; \vec{\lambda}) \mapsto V_C(\vec{p}; \vec{\lambda})$ satisfies the **axioms A0 to A4**.

- **The characters of its fusion ring** $R_k(\mathfrak{g})$ are determined by **Beauville** (for $\mathfrak{sl}_r$) and **Faltings** (for all the classical algebras and G_2).

Moduli stack of G-bundles and double quotient

- Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\dot{C} = C \setminus \{q\}$, $D = \mathrm{Spec}(\hat{\mathcal{O}}_q)$, $\dot{D} = D \cap \dot{C}$. Then there is a bijective map of sets

$$G(\mathcal{O}(\dot{C}))\backslash G(\mathcal{O}(\dot{D}))/G(\hat{\mathcal{O}}_q) \xrightarrow{\bar{\phi}} \mathrm{Bund}_G$$

  $X := G(\mathcal{O}(\dot{D}))/G(\hat{\mathcal{O}}_q)$ is called affine Grassmannian, which is a inductive limit of generalized Schubert varieties $\{\, X_w \mid w \in \widetilde{\mathcal{W}}/\mathcal{W} \,\}$.

- There is an algebraic G-bundle $\mathcal{P} \to C \times X$, which defines the morphism of stacks:

$$X = G(\mathcal{O}(\dot{D}))/G(\hat{\mathcal{O}}_q) \xrightarrow{\phi} \mathrm{Bund}_G$$

- $\mathrm{H}^0(\mathrm{Bun}_G, \Theta_{\mathrm{Bun}_G}) = \mathrm{H}^0(X, \phi^*\Theta_{\mathrm{Bun}_G})^{\Gamma}, \quad \Gamma := G(\mathcal{O}(\dot{C}))$.

WZW model and generalized theta functions

- **Beauville- Laszlo** (1994): For $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$, we have

$$V_C(\emptyset) \cong \mathrm{H}^0(\mathrm{Bun}_{\mathrm{SL(r)}}, \Theta_{\mathrm{Bun}_{\mathrm{SL(r)}}})$$

- **Faltings** (1994): It is true for arbitrary simple Lie algebra $\mathfrak{g}$!

$$V_C(\emptyset) \cong \mathrm{H}^0(\mathrm{Bun}_G, \Theta_{\mathrm{Bun}_G})$$

Bun_G is the moduli stack of G-bundles on C.

- **Pauly** (1996): For $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$, when $g \geq 2$, we have

$$V_C(\overrightarrow{p}; \overrightarrow{\lambda}) \cong \mathrm{H}^0(\mathcal{U}_{C,\,\omega}^{\mathcal{O}_C}, \Theta_{\mathcal{U}_{C,\,\omega}^{\mathcal{O}_C}})$$

where $\mathcal{U}_{C,\,\omega}^{\mathcal{O}_C}$ is the moduli spaces of semi-stable parabolic bundles on C with trivial determinant.

Faltings: A proof for the Verlinde formula

- The referee has asked me to remark that there are several interpretations of the term "Verlinde formula".
- One is equality of dimensions for spaces of global sections and their analogues in conformal field theory, and this is proved here.
- Another version is an explicit formula for these numbers. It follows from the previous and from **certain facts** about integrable representations of Kac-Moody Lie algebras.
- According to experts **these facts** (Conjecture 5.1) are "true" and "known", but I have not found any written proof.
- D. Kazhdan has suggested to use the methods from [GK] (Gelfand-Kazhdan: Examples of tensor categories, Invent. Math. 109 (1992), but I have not understood his argument.

Faltings's conjecture 5.1

- Let $T \subset G$ be maximal torus, $\mathfrak{g} = \mathrm{Lie}(G)$. $\forall\, \gamma \in T$,

$$\chi_\gamma : R_k(\mathfrak{g}) \to \mathbb{C}, \quad E \mapsto \chi_\gamma(E) := \mathrm{Tr}_E(\gamma).$$

- Let W be the Weyl group, $T^{reg} \subset T$ be the set of elements where W-action is free.

Conjecture 1 (Faltings)

All characters $\chi : R_k(\mathfrak{g}) \to \mathbb{C}$ are of the form

$$\chi_\gamma : R_k(\mathfrak{g}) \to \mathbb{C}, \quad E \mapsto \chi_\gamma(E) := \mathrm{Tr}_E(\gamma)$$

where $\gamma \in T^{reg}/W$.

Finite-dimensional proofs: $r = 2$

- **Beauville**: The basic distinction between the proofs using standard algebraic geometry, **which up to now work only in the case** $r = 2$, and proofs that use infinite-dimensional algebraic geometry to **mimic the heuristic approach of the physicists-these work for all** r.

- Compute $\chi(\Theta^k_{\mathcal{U}_C})$: Bertram-Szenes, Zagier, Donaldson-Witten.

- Thaddeus (1994): Stable pairs, linear systems and Verlinde formula (**Invent. Math.** 117, 317-353).

- Narasimhan-Ramadas (1993-1996): Factorization of generalized theta functions I, II (**Invent. Math.**, Topology): $|I| > 0$.

- Daskalopoulos-Wentworth (1993-1996): An analytic proof when $g \geq 2$ (**Math. Ann.** (1993), (1996)): $|I| > 0$.

Finite-dimensional proofs: $r > 2$

- Jeffrey-Kirwan (1998): Intersection theory on $SU_C(r, \mathcal{L})$ (**Ann. of Math.**): $(r, d) = 1$, $|I| = 0$.
- Marian-Oprea (2007): Counts of maps to Grassmannians and intersections on the moduli space of bundles (**J. Diff. Geom.**): $(r, d) = 1$, $|I| = 0$.
- Jeffrey (2001): The Verlinde formula for parabolic bundles (**J. of the LMS**): $(r, d) = 1$, $|I| > 0$, **weights** $\{\vec{a}(x)\}_{x \in I}$ **are very small !**
- Bismut-Labourie (1999): $\chi(\Theta_{\mathcal{U}_{C,\,\omega}})$ equals to the index of a Dirac operator when $k >> 0$ (**Surv. Differ. Geom.** 5, 97–311).
- E. Meinrenken: This result combined with recent vanishing results of (C. Teleman, Ann.of Math.) gives a new proof of the Verlinde formula **when** k **is sufficiently large**.

Degeneration method

- Degenerate $C_t \rightsquigarrow C_0 = X$ to a curve X with one node $x_0 \in X$.
- Need to prove: $\dim H^0(\mathcal{U}_{C_t,\omega_t}, \Theta_{\mathcal{U}_{C_t,\omega_t}}) = \dim H^0(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}})$.
- Let $\pi : \widetilde{X} \to X$ be the normalization, $\pi^{-1}(x_0) = \{x_1, x_2\}$.

Theorem 1 (Sun, 2000-2003, JAG and Ark. Mat.)

$$H^0(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}_{\widetilde{X},\omega^\mu}, \Theta_{\mathcal{U}_{\widetilde{X},\omega^\mu}})$$

$$H^0(\mathcal{U}_{X_1\cup X_2,\omega_1\cup\omega_2}, \Theta_{\mathcal{U}_{X_1\cup X_2,\omega_1\cup\omega_2}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}_{X_1,\omega_1^\mu}, \Theta_{\mathcal{U}_{X_1,\omega_1^\mu}}) \otimes H^0(\mathcal{U}_{X_2,\omega_2^\mu}, \Theta_{\mathcal{U}_{X_2,\omega_2^\mu}})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through* $0 \le \mu_r \le \cdots \le \mu_1 < k$.

Vanishing Theorems

Theorem 2 (Sun, 2000, JAG)

- *If* $g(C_t) \geq 2$*, then* $H^1(\mathcal{U}_{C_t,\omega_t}, \Theta_{\mathcal{U}_{C_t,\omega_t}}) = 0.$
- *If* $g(X) \geq 3$*, then* $H^1(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) = 0.$

Theorem 3 (Sun-Zhou, 2020, SCi. China)

- *For any ample line bundle* $\mathcal{L}$ *on* $\mathcal{U}_{C_t,\omega_t}$*, one has*

$$H^i(\mathcal{U}_{C_t,\omega_t}, \mathcal{L}) = 0 \quad \forall\ i > 0.$$

- *If* X *is irreducible,* $H^1(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) = 0.$
- *If* X *is reducible,* $H^1(\mathcal{U}_{X,\omega}, \mathcal{L}) = 0$ *for any ample line bundle* $\mathcal{L}$ *on* $\mathcal{U}_{X,\omega}$*.*

Recurrence relations of $D_g(r,d,\omega)$

Theorem 4 (Sun–Zhou, 2020, Sci. China)

Let $W_k = \{\, \lambda = (\lambda_1, ..., \lambda_r) \,|\, 0 = \lambda_r \le \lambda_{r-1} \le \cdots \le \lambda_1 \le k \,\}$ *and*

$$W_k' = \left\{ \lambda \in W_k \;\mid\; \left(\sum_{x \in I_1} \sum_{i=1}^{l_x} d_i(x) r_i(x) + \sum_{i=1}^{r} \lambda_i \right) \equiv 0 (\mathrm{mod}\ r) \right\}.$$

Then we have the following recurrence relation

$$D_g(r,d,\omega) = \sum_{\mu} D_{g-1}(r,d,\omega^{\mu})$$

$$D_g(r,d,\omega) = \sum_{\lambda \in W_k'} D_{g_1}(r,0,\omega_1^{\lambda}) \cdot D_{g_2}(r,d,\omega_2^{\lambda})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through* $0 \le \mu_r \le \cdots \le \mu_1 < k$ *and* ω^{μ}, ω_1^{λ}, ω_2^{λ} *are explicitly determined by* μ *and* λ.

Proof of $D_g(r,d,\omega) = V_g(r,d,\omega)$

$$V_g(r,d,\omega) := (-1)^{d(r-1)} \left(\frac{k}{r}\right)^g (r(r+k)^{r-1})^{g-1}$$

$$\sum_{\vec{v}} \frac{\exp\left(2\pi i \left(\frac{d}{r} - \frac{|\omega|}{r(r+k)}\right) \sum_{i=1}^r v_i\right) S_\omega\left(\exp 2\pi i \frac{\vec{v}}{r+k}\right)}{\prod_{i<j}\left(2\sin\pi\frac{v_i - v_j}{r+k}\right)^{2(g-1)}}$$

where $\vec{v} = (v_1, v_2, \ldots, v_r)$ runs through the integers

$$0 = v_r < v_{r-1} < \cdots < v_2 < v_1 < r+k.$$

Lemma 1

If $D_0(r,d,\omega) = V_0(r,d,\omega)$ holds, then

$$D_g(r,d,\omega) = V_g(r,d,\omega)$$

holds.

Proof of $D_g(r,d,\omega) = V_g(r,d,\omega)$

Proof.

$$D_g(r,d,\omega) = \sum_{\mu} D_{g-1}(r,d,\omega^{\mu}) = \sum_{\mu} V_{g-1}(r,d,\omega^{\mu})$$

where $|\omega^{\mu}| = |\omega| + k \cdot r$, $S_{\omega^{\mu}} = S_{\omega} \cdot S_{\mu} \cdot S_{\mu^*}$ and $\mu = (\mu_1, \dots, \mu_r)$ runs through the integers $0 \leq \mu_r \leq \cdots \leq \mu_1 < k$. Then it is enough to show

$$\sum_{\mu} S_{\mu}\left(\exp 2\pi i \frac{\vec{v}}{r+k}\right) \cdot S_{\mu^*}\left(\exp 2\pi i \frac{\vec{v}}{r+k}\right)$$

$$= \exp\left(2\pi i \frac{k}{r+k} \sum_{i=1}^{r} v_i\right) \frac{k(r+k)^{r-1}}{\prod_{i<j}\left(2\sin \pi \frac{v_i - v_j}{r+k}\right)^2}.$$

To prove it, we essentially use the fact:

$$\sum_{g \in G} \chi_V(g)\overline{\chi_V(g)} = |G|.$$

Proof of $D_0(r,d,\omega) = V_0(r,d,\omega)$

Lemma 2

If $D_0(r,d,\omega) = V_0(r,d,\omega)$ *when* $|I| \leq 3$, *then* $D_0(r,d,\omega) = V_0(r,d,\omega)$.

Let $I = I_1 \cup I_2$ with $|I_1| = 2$, we have

$$D_0(r,d,\omega) = \sum_{\mu \in W_k} V_0(r,0,\omega_1^{\mu}) \cdot V_0(r,d,\omega_2^{\mu}) = \frac{(-1)^{d(r-1)}}{(r(r+k)^{r-1})^2}$$

$$\sum_{\vec{v},\vec{v'}} \frac{\exp\left(2\pi i\left(-\frac{|\omega_1|}{r(r+k)}\right)|\vec{v}|\right)}{\prod_{i<j}\left(2\sin\pi\frac{v_i-v_j}{r+k}\right)^{-2}} \cdot \frac{\exp\left(2\pi i\left(\frac{d}{r}-\frac{|\omega_2|}{r(r+k)}\right)|\vec{v'}|\right)}{\prod_{i<j}\left(2\sin\pi\frac{v'_i-v'_j}{r+k}\right)^{-2}} \cdot$$

$$S_{\omega_1}\left(\exp 2\pi i\frac{\vec{v}}{r+k}\right) \cdot S_{\omega_2}\left(\exp 2\pi i\frac{\vec{v'}}{r+k}\right) \cdot \sum_{\mu \in W_k} \exp 2\pi i\frac{-|\mu|\cdot|\vec{v}|}{r(r+k)}$$

$$\exp 2\pi i\frac{-|\mu^*|\cdot|\vec{v'}|}{r(r+k)} \cdot S_{\mu}\left(\exp 2\pi i\frac{\vec{v}}{r+k}\right) \cdot S_{\mu^*}\left(\exp 2\pi i\frac{\vec{v'}}{r+k}\right)$$

Proof of $D_0(r,d,\omega) = V_0(r,d,\omega)$

When $\vec{v} = \vec{v'}$, we have

$$\sum_{\mu \in W_k} S_\mu \left(\exp 2\pi i \frac{\vec{v}}{r+k} \right) \cdot S_{\mu^*} \left(\exp 2\pi i \frac{\vec{v}}{r+k} \right)$$

$$= \exp \left(2\pi i \frac{k}{r+k} |\vec{v}| \right) \cdot \frac{r(r+k)^{r-1}}{\prod_{i<j} \left(2 \sin \pi \frac{v_i - v_j}{r+k} \right)^2}$$

when $\vec{v} \neq \vec{v'}$, by using $\sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = 0$, we have

$$\sum_{\mu \in W_k} \exp 2\pi i \frac{-|\mu| \cdot |\vec{v}|}{r(r+k)} \cdot \exp 2\pi i \frac{-|\mu^*| \cdot |\vec{v'}|}{r(r+k)} \cdot$$

$$S_\mu \left(\exp 2\pi i \frac{\vec{v}}{r+k} \right) \cdot S_{\mu^*} \left(\exp 2\pi i \frac{\vec{v'}}{r+k} \right) = 0.$$

Computation of of $V_0(r,0,\{\omega_s,\lambda_y,\lambda_z\})$

Lemma 3

(1) When $|I|=0$, $V_0(r,0,\{\lambda_x\}_{x\in I})=1$;

(2) $V_0(r,0,\lambda_x)=1$ *if* $\lambda_x=\lambda_r(x)\omega_r$ *and zero otherwise;*

(3) $V_0(r,0,\{\lambda_x,\lambda_y\})=1$ *if* $\lambda_x=\lambda_y^*$ *and zero otherwise;*

(4) Let $Y(\lambda_y,\omega_s)$ *be the set of partitions its Young diagrams are obtained from* λ *by adding* s *boxes with no two in the same row. Then*

$$V_0(r,0,\{\omega_s,\lambda_y,\lambda_z\})=\begin{cases}1 & \textit{when } \lambda_z^*\in Y(\lambda_y,\omega_s)\\ 0 & \textit{when } \lambda_z^*\notin Y(\lambda_y,\omega_s)\end{cases}$$

where $\omega_s:=(\overbrace{1,\ldots,1}^{s},\overbrace{0,\ldots,0}^{r-s})\quad(1\le s\le r)$.

Computation of of $D_0(r, 0, \{\omega_s, \lambda_y, \lambda_z\})$

Lemma 4

(1) When $|I| = 0$, $\mathcal{U}_{\mathbb{P}^1}(r, 0, \{\lambda_x\}_{x\in I})$ *consists one point;*

(2) When $|I| = 1$, $\mathcal{U}_{\mathbb{P}^1}(r, 0, \lambda_x)$ *consists one point if* $\lambda_x = \lambda_r(x)\omega_r$ *and is empty otherwise;*

(3) When $|I| = 2$, $\mathcal{U}_{\mathbb{P}^1}(r, 0, \{\lambda_x, \lambda_y\})$ *consists one point if* $\lambda_x = \lambda_y^*$ *and is empty otherwise;*

(4) When $|I| = 3$, $\mathcal{U}_{\mathbb{P}^1}(r, 0, \{\omega_s, \lambda_y, \lambda_z\})$ *(*$1 \le s \le r-1$*) consists one point if* $\lambda_z^* \in Y(\lambda_y, \omega_s)$ *and is empty otherwise.*

Corollary 1

$D_0(r, 0, \{\lambda_x\}_{x\in I}) = V_0(r, 0, \{\lambda_x\}_{x\in I})$ *when* $|I| \le 2$*, and*

$$D_0(r, 0, \{\omega_s, \lambda_y, \lambda_z\}) = V_0(r, 0, \{\omega_s, \lambda_y, \lambda_z\}).$$

Proof of of $D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) = V_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\})$

- For any partition $\lambda = (\lambda_1, ..., \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, let

$$s(\lambda) = \max\{\, i \,|\, \lambda_i - \lambda_{i+1} > 0 \,\}, \quad m(\lambda) = \sum_{i=1}^{s(\lambda)} \lambda_i$$

  and $s(\lambda) = 0$ if $\lambda_1 = \lambda_2 = \cdots = \lambda_r$.

- When $s(\lambda_x) = 0$, λ_x defines trivial parabolic structure at $x \in I$ and the proof reduces to the case of $|I| = 2$.

- Thus we can assume $s(\lambda_x) > 0$ and will prove

$$D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) = V_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\})$$

  by induction of $m(\lambda_x) - s(\lambda_x)$.

Proof of of $D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) = V_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\})$

- When $m(\lambda_x) - s(\lambda_x) = 0$, λ_x must be $\omega_{s(\lambda_x)}$ and we are done by Corollary 1.

- Assume $D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) = V_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\})$ holds for any λ_y and λ_z when $m(\lambda_x) - s(\lambda_x) < N$

- For any λ_x with $m(\lambda_x) - s(\lambda_x) = N$, let $\lambda_x' = \lambda_x - \omega_{s(\lambda_x)}$

$$
\begin{aligned}
&D_0(r,0,\{\omega_{s(\lambda_x)},\lambda_x',\lambda_y,\lambda_z\}) \\
&= \sum_{\mu^*\in W_k'} D_0(r,0,\{\omega_{s(\lambda_x)},\lambda_x',\mu^*\}) \cdot D_0(r,0,\{\mu,\lambda_y,\lambda_z\}) \\
&= \sum_{\mu\in Y(\lambda_x',\omega_{s(\lambda_x)})} D_0(r,0,\{\mu,\lambda_y,\lambda_z\}) \\
&= D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) + \sum_{\mu\in Y(\lambda_x',\omega_{s(\lambda_x)})\setminus\{\lambda_x\}} V_0(r,0,\{\mu,\lambda_y,\lambda_z\}).
\end{aligned}
$$

Proof of of $D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) = V_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\})$

$$
\begin{aligned}
& D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) + \sum_{\mu\in Y(\lambda'_x,\omega_{s(\lambda_x)})\setminus\{\lambda_x\}} V_0(r,0,\{\mu,\lambda_y,\lambda_z\}) \\
&= D_0(r,0,\{\omega_{s(\lambda_x)},\lambda'_x,\lambda_y,\lambda_z\}) \\
&= \sum_{\mu\in W'_k} D_0(r,0,\{\omega_{s(\lambda_x)},\lambda_y,\mu\}) \cdot D_0(r,0,\{\lambda'_x,\lambda_z,\mu^*\}) \\
&= \sum_{\mu\in W'_k} V_0(r,0,\{\omega_{s(\lambda_x)},\lambda_y,\mu\}) \cdot V_0(r,0,\{\lambda'_x,\lambda_z,\mu^*\}) \\
&= V_0(r,0,\{\omega_{s(\lambda_x)},\lambda'_x,\lambda_y,\lambda_z\}) \\
&== V_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) + \sum_{\mu\in Y(\lambda'_x,\omega_{s(\lambda_x)})\setminus\{\lambda_x\}} V_0(r,0,\{\mu,\lambda_y,\lambda_z\}).
\end{aligned}
$$

Thus we have

$$D_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}) = V_0(r,0,\{\lambda_x,\lambda_y,\lambda_z\}).$$

Globally F-regular varieties

- Let M be a variety over a perfect field k of $char(k) = p > 0$,
$$F : M \to M$$
be the (absolute) Frobenius map and $F^e : M \to M$ be the e-th iterate of Frobenius map.

- When M is normal, for any (weil) divisor $D \in Div(M)$,
$$\mathcal{O}_M(D)(V) = \{\, f \in K(M) \,|\, div_V(f) + D|_V \geq 0 \,\}, \quad \forall\, V \subset M$$
is a reflexive subsheaf of constant sheaf $K = k(M)$

Definition 1

A normal variety M over a perfect field is called stably Frobenius D-split if
$$\mathcal{O}_M \to F^e_* \mathcal{O}_M(D)$$
is split for some $e > 0$.

Globally F-regular varieties: Projective case

Definition 2

A normal variety M over a perfect field is called globally F-regular if M is stably Frobenius D-split for any effective divisor D.

Proposition 1

Let M be a projective variety over a perfect field. Then the following statements are equivalent.

(1) *M is normal and is stably Frobenius D-split for any effective D;*

(2) *M is stably Frobenius D-split for any effective Cartier D;*

(3) *For any ample line bundle $\mathcal{L}$, the section ring of M*

$$R(M, \mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(M, \mathcal{L}^n)$$

is strongly F-regular.

Globally F-regular varieties: Theory of Characteristic 0

Definition 3

*A variety M over a field of characteristic zero is said to be of globally F-regular type if its "**modulo p reduction of M**" are globally F-regular for a dense set of p.*

Proposition 2 (K. E. Smith)

Let M be a projective variety over a field of characteristic zero. If M is of globally F-regular type, then we have

(1) *M is normal, Cohen-Macaulay with rational singularities. If M is $\mathbb{Q}$-Gorenstein, then M has log terminal singularities.*

(2) *For any nef line bundle $\mathcal{L}$ on M, we have $H^i(M, \mathcal{L}) = 0$ when $i > 0$. In particular, $H^i(M, \mathcal{O}_M) = 0$ whenever $i > 0$.*

Moduli spaces: globally F-regular type

Definition 4

Let C be a smooth projective curve, $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$ and

$$\det : \mathcal{U}_{C,\omega} \to J_C^d, \quad E \mapsto \det(E).$$

For any $L \in J_C^d$, the fiber $\mathcal{U}_{C,\omega}^L := \det^{-1}(L)$ is called **moduli spaces of semi-stable parabolic bundles with fixed determinant**, *which is normal with at most rational singularities.*

Theorem 5 (Sun-Zhou, 2020, Math. Ann.)

For any data ω, the moduli spaces $\mathcal{U}_{C,\omega}^L$ is **of globally F-regular type**.

Corollary 2

For any ample line bundle $\mathcal{L}$ on $\mathcal{U}_{C,\omega}$, we have $H^i(\mathcal{U}_{C,\omega}, \mathcal{L}) = 0, \ \forall\, i > 0$.

Vanishing Theorem for node curve X

Definition 5

Let $\pi : \widetilde{X} \to X$ *be the normalization of* X, $\pi^{-1}(x_0) = \{x_1, x_2\}$. *A generalized parabolic sheaf (GPS)* (E, Q) *consist:*

- *A parabolic sheaf* E *determined by* $\omega = (r, d, \{\vec{n}(x), \vec{a}(x)\}_{x \in I}, k)$,
- *A* r*-dimensional quotient* $E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0$.
- (E, Q) *is semi-stable if* $\forall\, E' \subset E$, E/E' *torsion free outside* $\{x_1, x_2\}$

$$pardeg(E') - dim(Q^{E'}) \leq rk(E') \frac{pardeg(E) - dim(Q)}{r}$$

Normalization of $\mathcal{U}_{X,\omega}$: The moduli space $\mathcal{P}$

- $\mathcal{P} = \{\,$semi-stable GPS $(E,Q) = (E, E_{x_1}\oplus E_{x_2} \to Q \to 0)\,\}$, which is called **moduli space of GPS** (*generalized parabolic sheaf*).

- $\phi : \mathcal{P} \to \mathcal{U}_{X,\omega}$ is defined by $\phi(E,Q) = F$, where F is given by

$$0 \to F \to \pi_* E \to_{x_0} Q \to 0$$

- $\phi : \mathcal{P} \to \mathcal{U}_{X,\omega}$ is the normalization of $\mathcal{U}_{X,\omega}$ such that

$$\phi^* : \mathrm{H}^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \hookrightarrow \mathrm{H}^1(\mathcal{P}, \Theta_{\mathcal{P}}).$$

- There exist a flat morphism $Det : \mathcal{P} \to J^d_{\widetilde{X}}$, let

$$\mathcal{P}^L = Det^{-1}(L).$$

Globally F-regular type of $\mathcal{P}^L$

Theorem 6 (Sun-Zhou, 2020, Math. Ann.)

The moduli space $\mathcal{P}^L$ of semi-stable generalized parabolic sheaves with fixed determinant L is of globally F-regular type.

Corollary 3

$H^i(\mathcal{P}^L, \mathcal{L}) = 0$ *for any* $i > 0$ *and nef line bundles* $\mathcal{L}$ *on* $\mathcal{P}^L$ *and*

$$H^i(\mathcal{P}, \Theta_{\mathcal{P}}) = 0 \quad \forall\, i > 0.$$

Corollary 4

Let X be a projective curve with at most one node and $\mathcal{U}_{X,\omega}$ be the moduli space of parabolic sheaves on X with any given data ω. Then

$$H^1(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) = 0.$$

Sketch of Proof: When $|I|$ is large enough

- Recall $\widetilde{\mathcal{R}}'_I := \widetilde{\mathcal{R}}' = \mathrm{Grass}_r(\widetilde{\mathcal{F}}_{x_1} \oplus \widetilde{\mathcal{F}}_{x_2}) \to \widetilde{\mathcal{R}}_I = \times_{x \in I} Flag_{\vec{n}(x)}(\widetilde{\mathcal{F}}_x)$, $\mathcal{P}^L = \widetilde{\mathcal{R}}^{ss}_{I,\omega}//SL(V)$ is determined by $\omega = (r, d, \{\vec{n}(x), \vec{a}(x)\}_{x \in I}, k)$.

Proposition 3 (Sun, 2000, JAG)

There is ω_c such that $\mathcal{P}^L_{\omega_c} = \widetilde{\mathcal{R}}^{ss}_{I,\omega_c}//SL(V)$ is a Fano variety with only rational singularities (thus F-split type) if $(r-1)(g-1) + \frac{|I|}{2r} \geq 2$.

Proposition 4 (Sun, 2000, JAG)

For any $\omega = (r, d, I, \{\vec{n}(x), \vec{a}(x)\}_{x \in I}, k)$, we have

(1) $\mathrm{codim}(\widetilde{\mathcal{R}}'_I \setminus \widetilde{\mathcal{R}}'^{ss}_{I,\omega}) > (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$,

(2) $\mathrm{codim}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega} \setminus \{\mathcal{D}^f_1 \cup \mathcal{D}^f_2\} \setminus \widetilde{\mathcal{R}}'^{s}_{I,\omega}) \geq (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$.

- Let $\widetilde{U} = \mathcal{R}'^{ss}_{I,\omega} \cap \widetilde{\mathcal{R}}'^{ss}_{I,\omega_c}$, then $codim(\widetilde{\mathcal{R}}'^{ss}_{I,\omega} \setminus \widetilde{U}) \geq 2$.

Sketch of Proof: Increase the number $|I|$

- Add extra parabolic points $x \in J \subset \widetilde{X}$, the projection

$$p_I : \widetilde{\mathcal{R}}'_{I\cup J} \to \widetilde{\mathcal{R}}'_I$$

is $\mathrm{SL}(V)$-invariant. Choose $|J|$ such that $(r-1)(g-1)+\frac{|I\cup J|}{k+2r} \geq 2$.

- Choose the canonical weight ω_c on $\widetilde{\mathcal{R}}'_{I\cup J}$, consider

$$p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \supset \widetilde{U} = p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \cap \widetilde{\mathcal{R}}'^{ss}_{I\cup J,\omega_c} \to \mathcal{P}^L_{\omega_c}.$$

Then $p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \setminus \widetilde{U} = p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \cap (\widetilde{\mathcal{R}}'_{I\cup J,\omega_c} \setminus \widetilde{\mathcal{R}}'^{ss}_{I\cup J,\omega_c})$ has codimension at least $(r-1)(g-1)+\frac{|I\cup J|}{2r} \geq 2$.

- Let $U \subset \mathcal{P}^L_{\omega_c}$ be the image of $\widetilde{U}$, then p_I induces a morphism $f : U \to \mathcal{P}^L$ such that $f_*(\mathcal{O}_U) = \mathcal{O}_{\mathcal{P}^L}$.

Problem and discussions

Definition 6

Let X be a scheme and $Y \subset X$ a closed sub-scheme. The pair $(X, Y))$ is called of **compatible Frobenius split type** *if*

- *X is of Frobenius split type*
- *For almost p, there is a F-split $\varphi : F_*\mathcal{O}_{X_p} \to \mathcal{O}_{X_p}$ such that*

$$\varphi(F_*\mathcal{I}_{Y_p}) \subset \mathcal{I}_{Y_p}.$$

Problem 1

Are the pairs $(\mathcal{P}, \mathcal{D}_j(a))$ ($j = 1,\ 2$), $(\mathcal{D}_1(a), \mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1))$ of **compatible Frobenius split type** *?*

- If the answer of above problem is Yes, then, for any ample line bundle $\mathcal{L}$ on $\mathcal{U}_X$,

$$H^i(\mathcal{U}_X, \mathcal{L}) = 0 \quad \forall\ i > 0.$$

Thanks !