Introduction to Drinfeld modules

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The goal of this note is to introduce Drinfeld modules and explain their application to explicitly class field theory of function fields.

1 Analytic theory

1.1 Inspiration from characteristic zero

Let \( \Lambda \) be a discrete \( \mathbb{Z} \)-submodule of \( \mathbb{C} \) of finite rank \( r \). We must have \( r \leq 2 \). Write \( \Lambda = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_r \).

\[ r = 0, \mathbb{C}/\Lambda \simeq \mathbb{G}_a(\mathbb{C}), \text{ additive group}; \]
\[ r = 1, \mathbb{C}/\Lambda \simeq \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*, \quad z \mapsto \exp(2\pi iz/\omega), \text{ multiplicative group}; \]
\[ r = 2, \mathbb{C}/\Lambda \simeq E(\mathbb{C}), \quad z \mapsto (P(z), P'(z)), \text{ elliptic curve}. \]

1.2 Characteristic \( p \) analogue

Throughout this note, we keep the following notations.

\( F_q \): a finite field of \( q \)-elements of characteristic \( p \);
\( X \): a geometrically connected smooth projective curve over \( F_q \);
\( K \): the function field of \( X \);
\( \infty \): a fix closed point of \( X \) with residue field \( F_\infty \) and degree \( d_\infty = \dim_{F_q}(F_\infty) \);
\( A = \Gamma(X - \{\infty\}, \mathcal{O}_X) \);
\( K_\infty \): the completion of \( K \) at the point \( \infty \);
C: the completion of an algebraic closure \( \overline{K_\infty} \) of \( K_\infty \).

We have a one-to-one correspondence between the set of closed points of \( X \) and the set of discrete valuations on \( K \). For any \( x \in |X| \), let \( v_x \) be the corresponding discrete valuation on \( K \). Then

\[
A = \{ a \in K | v_x(a) \geq 0 \text{ for any } x \in |X| - \{\infty\} \}.
\]

There is a homomorphism \( \deg : K^* \rightarrow \mathbb{Z} \) such that \( \deg(a) = \dim_{\mathbb{F}_q}(A/aA) \) for any \( 0 \neq a \in A \). By the product formula, \( -d_\infty v_\infty(a) = \deg(a) \) for any \( a \in K^* \). Actually, we can define \( \deg(I) \) to be \( \dim_{\mathbb{F}_q}(A/I) \) for any nonzero ideal \( I \) of \( A \).

**Lemma 1.1.** \( A \) is discrete in \( K_\infty \) and the quotient \( K_\infty/A \) is compact.

**Proof.** For any \( n > 0 \), applying \( R\Gamma(X, \bullet) \) to the short exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(n_\infty) \rightarrow \mathcal{O}_X(n_\infty)/\mathcal{O}_X \rightarrow 0,
\]

we have an exact sequence

\[
0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X(n_\infty)) \rightarrow H^0(X, \mathcal{O}_X(n_\infty)/\mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X(n_\infty)) \rightarrow 0.
\]

By taking direct limit and using the fact \( H^1(X, \mathcal{O}_X(n_\infty)) = 0 \) for \( n \gg 0 \), we get an exact sequence

\[
0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow A \rightarrow K_\infty/\mathcal{O}_\infty \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0,
\]

where \( \mathcal{O}_\infty \) is the discrete valuation ring of \( K_\infty \). Then

\[
0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow \mathcal{O}_\infty \rightarrow K_\infty/A \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0
\]

is also exact. Since \( H^1(X, \mathcal{O}_X) \) is finite dimensional over \( \mathbb{F}_q \), then \( K_\infty/A \) is compact. \( \square \)

**Definition 1.2.** A lattice in \( C \) is a discrete \( A \)-submodule of \( C \) of finite rank, where the rank of an \( A \)-module \( M \) is defined to be \( \dim_K(K \otimes_A M) \).

By the following lemma, we have \( \text{rank}_A(\Lambda) = \dim_{K_\infty}(K_\infty \Lambda) \) for any lattice \( \Lambda \) in \( C \).

**Lemma 1.3.** Let \( L \) be a local field and \( R \) a discrete subring of \( L \) such that \( L/R \) is compact. Let \( V \) be a finitely dimensional \( L \)-vector space with the canonical topology and let \( M \) be an \( R \)-submodule of \( V \). If \( M \) is discrete, then the canonical homomorphism \( L \otimes_R M \rightarrow LM \) is an isomorphism. The converse also holds if \( M \) is projective over \( R \). In both cases, \( M \) is finitely generated over \( R \) and \( \dim_F(F \otimes_R M) = \dim_L(LM) \), where \( F \) is the fraction field of \( R \).
Proof. Suppose $M$ is discrete. Choose an $L$-basis $m_1, \ldots, m_k$ of $LM$ with $m_i \in M$ and set $M_0 = \sum_{i=1}^k Rm_i$. Since $M$ is discrete, we can choose a neighborhood $U_1$ of 0 in $V$ such that $U_1 \cap M = 0$. There is a neighborhood $U$ of 0 in $V$ such that $U - U \subset U_1$. Then for any $x, y \in M$, $x - y \in U$ if and only if $x = y$. It followes that $(U + M_0)/M_0 \cap M/M_0 = 0$ and hence $M/M_0$ is discrete in $V/M_0$ and $LM/M_0$. Since $L/R$ is compact, $LM/M_0 = \sum_{i=1}^k (L/R)m_i$ is compact and $M/M_0$ is thus a finite set. We have

$$ \dim_L(L \otimes_R M) = \dim_F(F \otimes_R M) = \dim_F(F \otimes_R M_0) = k = \dim_L(LM). $$

Conversely, suppose $M$ is projective over $R$ and we have a canonical isomorphism $L \otimes_R M \simeq LM$. Then $M$ is finitely generated over $R$ and we can find an $R$-module $N$ such that $M \oplus N$ is a free $R$-module of finite rank. Hence $M \oplus N$ is discrete in $L \otimes_R (M \oplus N)$ and hence $M$ is discrete in $L \otimes_R M \simeq LM$. \hfill \Box

**Remark 1.4.** The rank of a lattice in $C$ can be arbitrary large since $[C : K_{\infty}] = +\infty$.

**Definition 1.5.** Let $R$ be a ring containing $F_q$. A polynomial $f \in R[z]$ is called $F_q$-linear if $f(z + w) = f(z) + f(w) \in R[z, w]$ and $f(az) = af(z) \in R[z]$ for any $a \in F_q$. We can also define $F_q$-linear power series.

**Lemma 1.6.** Let $f \in R[[z]]$. Then $f$ is $F_q$-linear if and only if $f = \sum_{i=0}^\infty a_i z^i$ for some $a_i \in R$.

**Proof.** The if part is trivial. For the only if part, suppose $f = \sum_{n=0}^\infty a_n z^n$ is $F_q$-linear. The equality $f(z + w) = f(z) + f(w)$ means that $a_n C_n = 0$ if $1 \leq i \leq n - 1$. If $n$ is not a power of $p$, we can find $1 \leq i \leq n - 1$ such that $p \nmid C_n$ and hence $a_n = 0$. Now suppose $n$ is a power of $p$. The equality $f(az) = af(z)$ means that $a_n (a^n - \alpha) = 0$ for any $\alpha \in F_q$. If $n$ is not a power of $q$, we can find $\alpha \in F_q$ such that $a_n (a^n - \alpha) \neq 0$ and hence $a_n = 0$. This prove the only if part. \hfill \Box

**Theorem 1.7.** Let $\Lambda$ be an $A$-lattice in $C$. There exists an $F_q$-linear entire power series $e_\Lambda(z) \in C[[z]]$ which defines an $F_q$-linear isomorphism $C/\Lambda \simeq C$.

**Proof.** Define

$$ e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right). $$

Since $\Lambda$ is discrete, then $e_\Lambda(z)$ is entire. Let’s prove $e_\Lambda(z)$ is $F_q$-linear.
Write $\Lambda = \bigcup \Lambda_i$ for some $\mathbb{F}_q$-subspace of $\Lambda$ of finite dimension and set $e_i(z) = z \prod_{0 \neq \lambda \in \Lambda_i} (1 - \frac{z}{\lambda})$. Then $e_\Lambda(z) = \lim_i e_i(z)$. To prove $e_\Lambda(z)$ is $\mathbb{F}_q$-linear, we need only to show this for $e_i(z)$. For any $a \in \mathbb{F}_q$, by comparing the degrees, roots and coefficients in $z$ of $e_i(az)$ and $ae_i(z)$, we have $e_i(az) = ae_i(z)$. Let $F(z, w) = e_i(z + w) - e_i(z) - e_i(w) \in \mathbb{C}[z]$. We can write $F(z, w) = \sum_{i=0}^{d-1} f_i z^i$ for some $f_i \in \mathbb{C}[w]$ of degree $< d$, where $d = \# \Lambda_i$. For any $\lambda \in \Lambda_i$, we have

$$F(z, \lambda) = e_i(z + \lambda) - e_i(z) - e_i(\lambda) = 0.$$  

This shows each $\lambda \in \Lambda_i$ is a root of $f_i(z)$ for any $i$. But $\deg f_i < d$, we must have $f_i = 0$ and hence $F(z, w) = 0$. This shows that $e_i(z)$ and hence $e_\Lambda(z)$ are $\mathbb{F}_q$-linear.

The entire series $e_\Lambda(z)$ define an $\mathbb{F}_q$-linear map $\mathbb{C} \to \mathbb{C}$ of analytic spaces with kernel $\Lambda$. By Weistrass representation theorem, $e_\Lambda(z) : \mathbb{C} \to \mathbb{C}$ is surjective. So we get an isomorphism $e_\Lambda(z) : \mathbb{C}/\Lambda \simeq \mathbb{C}$. \hfill \ensuremath{\Box}

**Corollary 1.8.** For any $a \in A$, there exists a unique polynomial $\phi_a \in \mathbb{C}[z]$ making the following diagram commute:

$$\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{a} & \mathbb{C}/\Lambda \\
\downarrow{e_\Lambda} & & \downarrow{e_\Lambda} \\
\mathbb{C} & \xrightarrow{\phi_a} & \mathbb{C}.
\end{array}$$

Moreover, $\phi_a$ is a $\mathbb{F}_q$-linear polynomial of degree $q^r \deg(a)$ where $r$ is the rank of the lattice $\Lambda$. For any $a, b \in A$, $\phi_a(\phi_b(z)) = \phi_{ab}(z)$.

**Proof.** Define

$$\phi_a(z) = az \prod_{0 \neq \lambda \in a^{-1}\Lambda} (1 - z/e_\Lambda(\lambda)).$$

Then $e_\Lambda(az)$ and $\phi_a(e_\Lambda(z))$ are two entire series with the same root set $a^{-1}\Lambda$ and with the same derivative $a$. So these two series only have simple roots and hence $e_\Lambda(az) = \phi_a(e_\Lambda(z))$. Moreover, $\phi_a(z)$ is $\mathbb{F}_q$-linear. The equality $\phi_a(\phi_b(z)) = \phi_{ab}(z)$ holds by the following commutative diagram

$$\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{a} & \mathbb{C}/\Lambda \\
\downarrow{e_\Lambda} & & \downarrow{e_\Lambda} \\
\mathbb{C} & \xrightarrow{\phi_a} & \mathbb{C}.
\end{array}$$

\hfill \ensuremath{\Box}
For any \( F_q \)-algebra \( R \), denote by \( \tau \) the \( q \)-th power map on \( R \) and by \( R\{\tau\} \) the twist polynomial ring with relation \( \tau r = r^q \tau \) for any \( r \in R \). We have a one-to-one correspondence

\[
R\{\tau\} \simeq \{ F_q \text{-linear polynomials in } R[z] \}, \quad f = \sum_i a_i \tau^i \mapsto f(z) = \sum_i a_i z^{q^i}.
\]

For any \( f = \sum a_i \tau^i \in R\{\tau\} \), define \( w(f) = \min \{ i | a_i \neq 0 \} \), \( \deg(f) = \max \{ i | a_i 
eq 0 \} \), c.t.(\( f \)) = \( a_0 \) and \( \text{l.c.}(f) = a_{\deg(f)} \).

Thus any lattice \( \Lambda \) in \( C \) defines a ring homomorphism \( \phi : A \to C\{\tau\} \) sending \( a \) to \( \phi_a \) whose constant term is \( a \). This leads the definition of Drinfeld modules in the next section.

2 Algebraic theory

In this section, fix a homomorphism \( \iota \) from \( A \) to a field \( L \). The characteristic \( \text{char}_A(L) \) of the \( A \)-field \( L \) is defined to be \( \text{ker}(\iota) \).

2.1 Basic definitions

**Definition 2.1.** A Drinfeld module over \( L \) is a ring homomorphism

\[
\phi : A \to L\{\tau\}, \quad a \mapsto \phi_a,
\]

such that c.t.(\( \phi_a \)) = \( \iota(a) \) for any \( a \in A \) and \( \phi_a \neq \iota(a) \) for some \( a \in A \).

Equivalently, a Drinfeld \( A \)-module over \( L \) is an \( A \)-module scheme over \( L \) whose underlying \( F_q \)-vector space scheme is isomorphic to \( G_{a,L} = \text{Spec} L[z] \) and the \( A \)-module action on \( G_{a,L} \) is given by the ring homomorphism \( \phi : A \to \text{End}_{F_q}(G_{a,L}) = L\{\tau\} \) satisfying the above conditions. So \( \phi \) defines a functor

\[
\phi : \text{Alg}_L \to \text{Mod}_A, \quad R \mapsto \phi(R),
\]

where \( \phi(R) = R \) as abelian groups and the \( A \)-module structure on \( \phi(R) \) is given by \( a.r = \phi_a(r) \) for any \( a \in A \) and \( r \in R \).

2.2 Rank and height

**Proposition 2.2.** Let \( \phi \) be a Drinfeld module over \( L \).

1. There exists a positive rational number \( r \) such that \( \deg(\phi_a) = r \deg(a) \) for any \( a \in A \).

2. Suppose \( p = \text{char}_A(L) \) is nonzero. Then there exists a positive rational number \( h \) such that \( w(\phi_a) = h \deg(p)v_p(a) \) for any \( a \in A \).
Proof. (1) Define $\mu(a) = -\deg(\phi_a)$ for any $a \in A$ and $\mu(0) = +\infty$. Then $\mu(ab) = \mu(a) + \mu(b)$ and $\mu(a + b) \geq \min\{\mu(a), \mu(b)\}$ for any $a, b \in A$. So we can extend $\mu$ to a nontrivial valuation $\tilde{\mu} : K \to \mathbb{Z} \cup \{+\infty\}$ on $K$. As $\tilde{\mu}(a) = -\deg(\phi_a) < 0$ for some $a \in A$, $\tilde{\mu}$ is the valuation on $K$ defined by $\infty \in X$. Then there exists a positive rational number $r$ such that $\deg(\phi_a) = r \deg(a)$ for any $a \in A$.

(2) Define $\nu(a) = w(\phi_a)$ for any $a \in A$ and $\nu(0) = +\infty$. Then $\nu(ab) = \nu(a) + \nu(b)$ and $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for any $a, b \in A$. So we can extend $\nu$ to a valuation $\tilde{\nu} : K \to \mathbb{Z} \cup \{+\infty\}$ on $K$. As $\tilde{\nu}(a) > 0$ for any $a \in p$, $\tilde{\nu}$ is the valuation on $K$ corresponding to $p$. So there exists a positive rational number $h$ such that $w(\phi_a) = h \deg(p) v_p(a)$ for any $a \in A$. \qed

Definition 2.3. The numbers $r$ and $h$ in Proposition 2.2 are called the rank and height of $\phi$, respectively.

To show $r$ and $h$ are positive integers, we need to study the torsion points of Drinfeld modules.

2.3 Torsion points

Definition 2.4. Let $\phi$ be a Drinfeld module over $L$ and let $a \in A$. For any $L$-algebra $R$, let

$$\phi[a](R) = \{r \in R | \phi_a(r) = 0\}$$

be the $a$-torsion submodule of the $A$-module $\phi(R)$. More generally, for any ideal $I$ of $A$, let $\phi[I](R) = \bigcap_{i \in I} \phi[i](R)$.

Actually, the functor $\phi[a] : \text{Alg}_L \to \text{Mod}_A$ is the $A$-module scheme $\phi[a] = \ker(\phi_a : \mathbb{G}_{a,L} \to \mathbb{G}_{a,L})$ which is represented by the finite scheme $\text{Spec } L[z]/(\phi_a(z))$ over $L$ of degree $q^r \deg(a)$.

If $I$ is a nonzero ideal of $A$, then the left ideal $\sum_{i \in I} L\{\tau\} \phi_i$ of $L\{\tau\}$ is generated by a unique monic polynomial $\phi_I$. Then the functor $\phi[I] : \text{Alg}_L \to \text{Mod}_A$ is represented by the finite scheme $\text{Spec } L[z]/(\phi_I(z))$ over $L$.

Lemma 2.5. Let $R$ be a Dedekind domain and $M$ an $R$-module.

(1) For any distinct maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of $R$ and any $e_1, \ldots, e_n \in \mathbb{N}$, we have

$$M[p_1^{e_1} \cdots p_n^{e_n}] = \bigoplus_{i=1}^n M[p_i^{e_i}].$$
(2) If \( M \) is a divisible \( R \)-module, then for any maximal ideal \( \mathfrak{p} \) of \( R \) and \( e \in \mathbb{N} \), \( M[\mathfrak{p}^e] \) is a free \( R/\mathfrak{p}^e \)-module of some rank \( r \) independent of \( e \). Moreover, \( M[\mathfrak{p}^\infty] := \bigcup_{e=1}^{\infty} M[\mathfrak{p}^e] \) is isomorphic to \( (K_\mathfrak{p}/\hat{R}_\mathfrak{p})^r \), where \( \hat{R}_\mathfrak{p} \) is the completion of \( R \) at \( \mathfrak{p} \) and \( L_\mathfrak{p} \) its fraction field.

Proof. (1) is obvious. The homomorphism \( M \to M_\mathfrak{p} \) induces an isomorphism \( M[\mathfrak{p}^e] \simeq M_\mathfrak{p}[\mathfrak{p}^e R_\mathfrak{p}] \).

For (2), we may assume that \( R \) is a discrete valuation ring. Fix a uniformizer \( \pi \) of \( R \) and choose a free \( R \)-module \( F \) of rank \( r \) and an isomorphism \( i_1 : \pi^{-1}F/F \simeq M[\pi] \) of \( R/\mathfrak{p} \)-modules. Let's construct an isomorphism \( i_e : \pi^{-e}F/F \simeq M[\pi^e] \) of \( R/\mathfrak{p}^e \)-modules by induction on \( e \). Given the isomorphism \( i_e : \pi^{-e}F/F \simeq M[\pi^e] \), using divisability of \( M \), there is an isomorphism \( i_{e+1} : \pi^{-e-1}F/F \simeq M[\pi^{e+1}] \) making the following diagram commutes:

\[
\begin{array}{cccccc}
0 & \rightarrow & \pi^{-1}F/F & \xrightarrow{i_1} & \pi^{-e-1}F/F & \xrightarrow{\pi} & \pi^{-e}F/F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M[\pi] & \xrightarrow{i_{e+1}} & M[\pi^{e+1}] & \xrightarrow{\pi} & M[\pi^e] & \rightarrow & 0.
\end{array}
\]

So \( i_{e+1} \) is an isomorphism. The family \( \{i_e\} \) is an isomorphism from the direct systems \( \{\pi^{-e}F/F\} \) to \( \{M[\pi^e]\} \) and hence \( M[\mathfrak{p}^\infty] = \lim_{\mathfrak{p}^e} \pi^{-e}F/F = (L_\mathfrak{p}/\hat{R}_\mathfrak{p})^r \). \( \square \)

**Proposition 2.6.** Let \( \phi \) be a Drinfeld module over an algebraically closed field \( L \) of rank \( r \) and height \( h \).

(1) If \( I \) is an ideal of \( A \) prime to \( \text{char}_A(L) \), then \( \phi(L)[I] \) is a free \( A/I \)-module of rank \( r \). In particular, \( r \) is a positive integer.

(2) Suppose \( \mathfrak{p} = \text{char}_A(L) \neq 0 \). Then for any positive integer \( e \in \mathbb{N} \), \( \phi(L)[\mathfrak{p}^e] \) is a free \( A/\mathfrak{p}^e \)-module of rank \( r - h \). In particular, \( h \) is a positive integer.

Proof. For any \( 0 \neq a \in A \), \( \phi_a : L \to L \) is surjective. Hence \( \phi(L) \) is \( A \)-divisible. By Lemma 2.5, we only need to show that for any maximal ideal \( \mathfrak{p} \) of \( A \), there exists a positive integer \( e \) such that \( \#\phi(L)[\mathfrak{p}^e] = q^{er \deg(p)} \) if \( \mathfrak{p} \neq \text{char}_A(L) \) and \( \#\phi(L)[\mathfrak{p}^e] = q^{e(r-h)\deg(p)} \) if \( \mathfrak{p} = \text{char}_A(L) \). Let \( e \) be the class number of \( A \). Then \( \mathfrak{p}^e = (a) \) for some \( a \in A \). We have \( \deg(a) = e \deg(p) \) and \( \deg(\phi_a) = er \deg(p) \). If \( \mathfrak{p} \neq \text{char}_A(L) \), then \( a \notin \mathfrak{p} \) and \( \phi_a(z) \) is a separable polynomial of degree \( q^{r \deg(a)} \), and thus \( \#\phi(L)[\mathfrak{p}^e] = \#\phi(L)[a] = q^{r \deg(a)} = q^{e \deg(p)} \). If \( \mathfrak{p} = \text{char}_A(L) \), then \( w(\phi_a) = hv_p(a) \deg(p) = eh \deg(p) \). In this case, \( \#\phi(L)[\mathfrak{p}^e] = \#\phi(L)[a] = q^{e(r-h)\deg(a)} = q^{e(r-h)\deg(p)} \). \( \square \)
2.4 Drinfeld modules and lattices in $C$

**Definition 2.7.** A morphism $f : \phi \to \psi$ of Drinfeld modules over $L$ is a polynomial $f \in L\{\tau\}$ such that $\psi_a f = f \phi_a$ for any $a \in A$. In other words, a morphism from $\phi$ to $\psi$ is an endomorphism $f$ of the additive group scheme over $L$ such that for any $a \in A$, the following diagram commutes:

\[
\begin{array}{ccc}
G_{a,L} & \xrightarrow{f} & G_{a,L} \\
\downarrow \phi_a & & \downarrow \psi_a \\
G_{a,L} & \xrightarrow{f} & G_{a,L}.
\end{array}
\]

We denote by $\text{Hom}(\phi, \psi)$ the set of morphisms from $\phi$ to $\psi$. A nonzero morphism of Drinfeld modules is called an isogeny.

**Proposition 2.8.** Isogenous Drinfeld modules have the same rank and height.

**Proof.** For any $f \in \text{Hom}(\phi, \psi)$, we have $\deg(\psi_a) + \deg(f) = \deg(f) + \deg(\phi_a)$ and hence $\deg(\psi_a) = \deg(\phi_a)$ for any $a \in A$. Then $\phi$ and $\psi$ have the same rank by definition. So is the height. \qed

**Definition 2.9.** A morphism from an $A$-lattice $\Lambda$ of $C$ to another one $\Lambda'$ of the same rank is an element $c \in C$ such that $c \Lambda \subset \Lambda'$.

**Theorem 2.10.** The functor from the categories of lattices in $C$ to the categories of Drinfeld modules over $C$ constructed in Corollary 1.8 defines an equivalence of categories. Moreover, any lattice and its corresponding Drinfeld module have the same rank.

**Proof.** (1) Given a lattice $\Lambda$ in $C$ of rank $r$, define

\[
e_{\Lambda}(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - \frac{z}{\lambda}),
\]

and for any $0 \neq a \in A$, define

\[
\phi_a(z) = az \prod_{0 \neq \lambda \in a^{-1}A/\Lambda} (1 - \frac{z}{e_{\Lambda}(\lambda)}).
\]

Then $\phi_a(z)$ is an $\mathbb{F}_q$-linear polynomial of degree $q^{r \deg(a)}$ which defines a polynomial $\phi_a \in C\{\tau\}$ of degree $r \deg(a)$. By Corollary 1.8, we get a Drinfeld module $\phi : A \to C\{\tau\}$ over $C$ of rank $r$.

(2) Let $\phi$ be a Drinfeld module over $C$ of rank $r$. Choose $a \in A \setminus \mathbb{F}_q$ and write $\phi_a = \sum_{i=0}^{d} a_i \tau^i$. There exists a unique series $e_{\phi} = \sum_{i=0}^{\infty} e_i \tau^i \in C\{\tau\}$ with $e_0 = 1$ and $e_{\phi} a = \phi_a e_{\phi}$ by the equalities

\[
e_n(a^{q^n} - a) = a_d e_{n-d}^q + \cdots + a_1 e_{n-1}^q \quad (n \geq 0).
\]
As $d_\infty v_\infty(a) = -\deg(a) < 0$, we have

$$v_\infty(e_n) \geq \min\{v_\infty(a_0 e_n^{q^d}), \ldots, v_\infty(a_1 e_n^{q-1})\} - q^n v_\infty(a).$$

Thus there exists a positive real number $c$ such that for $n \gg 0$,

$$\frac{v_\infty(e_n)}{q^n} \geq \min\{\frac{v_\infty(e_{n-1})}{q^{n-1}}, \ldots, \frac{v_\infty(e_{d-1})}{q^{d-1}}\} + c.$$

This proves $\lim_{n \to \infty} \frac{v_\infty(e_n)}{q^n} = +\infty$ and hence $e_\phi(z)$ is an entire function. For any $b \in A$, we have

$$(e_\phi^{-1} \phi_b e_\phi) a = e_\phi^{-1} \phi_b \phi_a e_\phi = e_\phi^{-1} \phi_a \phi_b e_\phi = a(e_\phi^{-1} \phi_b e_\phi) \in C\{\{\tau\}\}.$$ 

If we write $e_\phi^{-1} \phi_b e_\phi = \sum_i b_i \tau^i$ for some $b_i \in C$, then $b_i(a q^i - a) = 0$ for any $i \geq 0$ and hence $b_i = 0$ for any $i \geq 1$. We must have $e_\phi^{-1} \phi_b e_\phi = b$ and $e_\phi b = \phi_b e_\phi$ for any $b \in A$. Let $\Lambda$ be the kernel of the $F_q$-linear map $e_\phi : C \to C$. Then $\Lambda$ is a discrete $A$-submodule of $C$. The isomorphism $e_\phi : C/\Lambda \simeq C$ induces an isomorphism $a^{-1}\Lambda/\Lambda \simeq \ker(e_\phi : C \to C)$ which is a free $A/aA$-module of rank $r$ by Proposition 2.6. To show $\Lambda$ is a lattice, we only need to show it is a finitely generated $A$-module. By Lemma 1.3, it is sufficient to show $\dim_{K_\infty}(K_\infty\Lambda) < +\infty$. If not, we can find infinitely many elements $\lambda_1, \lambda_2, \ldots$ in $\Lambda$ which are linearly independent over $K_\infty$. Set $\Lambda_r = \sum_{i=1}^r K_\infty \lambda_i \cap \Lambda$ for each $i$. By Lemma 1.3, $\Lambda_r$ is a finitely generated $A$-module of rank $r$. The natural monomorphism $a^{-1}\Lambda_r/\Lambda_r \to a^{-1}\Lambda/\Lambda$ implies $\#(a^{-1}\Lambda/\Lambda) > \#(a^{-1}\Lambda_r/\Lambda_r) = \#(A/aA)^r$, which contradicts to $a^{-1}\Lambda/\Lambda \simeq (A/aA)^r$. It follows that $\Lambda$ is a lattice in $C$ of rank $r$.

(3) Let $\Lambda_1$ and $\Lambda_2$ be two lattices in $C$ of the same rank $r$, and let $c$ be a nonzero element in $C$ such that $c\Lambda_1 \subset \Lambda_2$. As $\Lambda_1 \subset c^{-1}\Lambda_2$, consider

$$f(z) = cz \prod_{0 \neq \lambda \in c^{-1}\Lambda_2/\Lambda_1} (1 - z/e_{\Lambda_1}(\lambda)).$$

Then $f(z)$ is an $F_q$-linear polynomial. Comparing the roots and coefficients of the entire series $e_{\Lambda_2}(cz)$ and $f(e_{\Lambda_1}(z))$, they must be equal. Let $\phi$ and $\psi$ be the Drinfeld modules over $C$ corresponding to $\Lambda_1$ and $\Lambda_2$, respectively. Then $f \in \Hom(\phi, \psi)$.

(4) Given a nonzero morphism $f : \phi \to \psi$ of Drinfeld modules over $C$. Let $\Lambda$ and $W$ be their corresponding lattices. We have $e_{\Lambda} a = \phi_a e_{\Lambda}$, $e_{W} a = \psi_a e_{W}$ and $f \phi_a = \psi_a f$ for any $a \in A$. Then $(e_{W}^{-1} f \phi_a) a = a(e_{W}^{-1} f \phi_a) \in C\{\{\tau\}\}$. We must have $e_{W}^{-1} f \phi_a = c \in C^\times$ and then $c \Lambda \subset W$. 

2.5 Endomorphism ring of Drinfeld modules

Given a Drinfeld module \( \phi \) over \( L \) of rank \( r \), denote by \( \text{End}(\phi) \) the ring of endomorphisms of \( \phi \). More precisely,

\[
\text{End}(\phi) = \{ P \in L\{\tau\} | P\phi_a = \phi_a P \text{ for any } a \in A \}.
\]

The ring homomorphism \( A \rightarrow \text{End}(\phi) \) by sending \( a \) to \( \phi_a \) gives an \( A \)-module structure on \( \text{End}(\phi) \).

**Proposition 2.11.** (1) \( \text{End}(\phi) \) is a projective \( A \)-module of rank \( \leq r^2 \).

(2) If \( r = 1 \), the above ring homomorphism \( A \rightarrow \text{End}(\phi) \) is an isomorphism.

**Proof.** Fix some \( a \in A \setminus \mathbb{F}_q \) and \( a \notin \text{char}(L) \). Claim that \( \text{End}(\phi) \otimes_A A/(a) \rightarrow \text{End}_A(\phi[a](\overline{L})) \) is injective.

Indeed, suppose that \( P \in \text{End}(\phi) \) give rise to the trivial endomorphism on \( \phi[a](\overline{L}) \). Write \( P = Q\phi_a + R \) for some \( Q, R \in L\{\tau\} \) with \( \deg(R) < \deg(\phi_a) \). Hence \( R \) acts trivial on \( \phi[a](\overline{L}) \). Since \( a \notin \text{char}(L) \), by Proposition 2.6 \( \#\phi[a](\overline{L}) = q^{r \deg(a)} \). As \( \deg(R(z)) < \deg(\phi_a(z)) = q^{r \deg(a)} \), we must have \( R = 0 \) and hence \( P = Q\phi_a \). One can easily check that \( Q \in \text{End}(\phi) \). This proves the claim.

Define \( \delta : \text{End}(\phi) \rightarrow \mathbb{Z} \cup \{+\infty\} \) by \( \delta(P) = -\deg(P) \). The mapping \( \delta \) satisfies

1. \( \delta(P) = \infty \) if and only if \( P = 0 \).
2. \( \delta(PQ) = \delta(P) + \delta(Q) \) for any \( P, Q \in \text{End}(\phi) \).
3. \( \delta(P + Q) \geq \min\{\delta(P), \delta(Q)\} \) for any \( P, Q \in \text{End}(\phi) \).
4. \( \delta(aP) = rd_\infty v_\infty(a) + \delta(P) \) for any \( a \in A \) and \( P \in \text{End}(\phi) \).

Denote \( M = \text{End}(\phi) \). The mapping \( \delta \) thus gives rise to a norm on the \( K_\infty \)-vector space \( K_\infty \otimes_A M \). Note that \( \text{End}(\phi) \) is discrete in \( K_\infty \otimes_A M \).

Suppose \( \dim_K(K \otimes_A M) = \infty \). Choose infinitely many \( P_1, P_2, \ldots \in \text{End}(\phi) \) which are linearly independent over \( K \). Let \( V_n = \sum_{i=1}^{n} K_\infty P_i \) and \( M_n = V_n \cap M \). By Lemma 1.3, \( M_n \) is a projective \( A \)-module of rank \( n \). The canonical monomorphism \( a^{-1}M_n/M_n \rightarrow a^{-1}M/M \) implies that \( \#(a^{-1}M/M) \geq \#(a^{-1}M_n/M_n) = q^{n \deg(a)} \) for each \( n \). This contradicts to the claim that \( \#(a^{-1}M/M) \leq q^{2 \deg(a)} \). Hence \( \dim_K(K \otimes_A M)A \leq r^2 \) and (1) holds.
If \( r = 1 \), End(\( \phi \)) is an invertible \( A \)-module. The monomorphism \( A \to \text{End}(\phi) \) induces an isomorphism \( K \simeq K \otimes_A \text{End}(\phi) \). So End(\( \phi \)) can be viewed as a subring of \( K \) which is integral over \( A \). But \( A \) is integrally closed in \( K \), we must have \( A = \text{End}(\phi) \). \( \Box \)

### 3 Carlitz module and cyclotomic function fields

In this section, we will construct the cyclotomic extensions of the rational function field \( \mathbb{F}_q(t) \) by the Carlitz module.

Let \( \phi \) be a Drinfeld module over an \( A \)-field \( L \) of rank \( r \). Fix an algebraic closure \( \overline{L} \) of \( L \). Recall that \( \phi[I](L) = \{ x \in \overline{L} | \phi_i(x) = 0 \text{ for any } i \in I \} \) for any nonzero ideal \( I \) of \( A \). Let \( L_I \) be the field extension of \( L \) by adding \( \phi[I](L) \). For any \( \sigma \in \text{Gal}(\overline{L}/L) \), \( \sigma \) preserves \( \phi[I](L) \) and \( L_I/L \) is thus a finite normal extension.

Suppose \( I \) is prime to \( \text{char}_A(L) \). Then \( I^e = (a) \) for some positive integer \( e \) and some \( a \in A \) with \( \iota(a) \neq 0 \). In other words, \( \phi_a(z) \in L[z] \) is separable and \( L/(a)/L \) is separable. So \( L_I/L \) is Galois and we also have a canonical monomorphism

\[
\chi : \text{Gal}(L_I/L) \hookrightarrow \text{Aut}_A(\phi[I]) \simeq \text{GL}_r(A/I) .
\] (3.1)

In particular, \( L_I/L \) is an abelian extension if \( r = 1 \).

In the remainder of this section, suppose \( A = \mathbb{F}_q[t] \) and consider the Carlitz module

\[
C : A \to K\{\tau\}, \ t \mapsto t + \tau
\]

over \( K = \mathbb{F}_q(t) \). For any \( 0 \neq a \in A \), let \( C[a] = \{ \lambda \in C | C_a(\lambda) = 0 \} \) and \( K_a = K(C[a]) \). Then \( C[a] \) is a free \( A/aA \)-module of rank one.

**Theorem 3.1.** (1) \( K_a/K \) is an abelian Galois extension of Galois group \( (A/aA)^\times \).

(2) For any maximal ideal \( p \) of \( A \), \( K_a/K \) is ramified at \( p \) if and only if \( a \in p \).

(3) Let \( \mathcal{O}_a \) be the integral closure of \( A \) in \( K_a \) and let \( \lambda \) be a generator of the \( A \)-module \( C[a] \).

We have \( \mathcal{O}_a = A[\lambda] \).

**Proof.** First suppose \( a = p^e \) for some positive integer \( e \) and some monic irreducible polynomial \( p(z) \) of degree \( d \). The composition \( A \xrightarrow{C} A\{\tau\} \to A/pA\{\tau\} \) defines a Drinfeld module \( \overline{C} : A \to A/pA\{\tau\} \) over \( A/pA \) of rank 1 and height 1. So \( \overline{C}_{p^e} = \tau^{de} \in A/pA\{\tau\} \) and hence \( C_{p^e} - \tau^{de} \in pA\{\tau\} \). Define
\[
\phi_{p^e}(z) = C_{p^e}(z)/C_{p^{e-1}}(z). \text{ Then } \phi_{p^e}(z) = C_p(C_{p^{e-1}}(z))/C_{p^{e-1}}(z) \in A[z] \text{ and } \phi_{p^e}(z) \equiv z^{d_{p^e}} - q^{d_{(e-1)}} \pmod{pA[z]}. \text{ The constant term of } \phi_{p^e}(z) \text{ is } p. \text{ In other words, } \phi_{p^e}(z) \text{ is an Eisenstein polynomial over } A \text{ with respect to the prime ideal } pA \text{ and so it is irreducible over } K. \text{ For any generator } \lambda \text{ of the } A\text{-module } C[p^e], \text{ we have } C_{p^e}(\lambda) = 0 \text{ but } C_{p^{e-1}}(\lambda) \neq 0. \text{ Thus } \phi_{p^e}(z) \text{ is the minimal polynomial over } K \text{ of any generator of } C[p^e] \text{ and } K_{p^e} = K(\lambda). \text{ So for any } 0 \neq b \in A \text{ prime to } p, \text{ we have an isomorphism of fields }

\[
\sigma_b : K_{p^e} \simeq K_{p^e} \text{ by } \sigma_b(\lambda) = C_b(\lambda).
\]

This proves that

\[
\chi : \text{Gal}(K_{p^e}/K) \simeq \text{Aut}_A(C[p^e]) \simeq (A/(p^e))^\times.
\]

Moreover, \( K_{p^e}/K \) is totally ramified at \( pA \).

Let’s compute the discriminant \( \delta = d(1, \lambda, \ldots, \lambda^{\phi(p^e)-1}) \) where \( \phi(b) \neq \#(A/bA)^\times \) for any \( b \in A \).

By the definition of discriminant,

\[
\pm \delta = \pm \det(\sigma \lambda^i)_{\sigma \in \text{Gal}(K_{p^e}/K)} = \prod_{0 \leq i < \phi(p^e)} (C_x(\lambda) - C_y(\lambda)).
\]

Differenting both sides of \( C_{p^e}(z) = C_{p^{e-1}}(z)\phi_{p^e}(z) \) and substituting \( z = \lambda \), we have \( p^e = C_{p^{e-1}}(\lambda)\phi'_{p^e}(\lambda) \).

Differenting \( \phi_{p^e}(z) = \prod_{y \in (A/p^eA)^\times} (z - C_y(\lambda)) \) and substituting \( z = C_x(\lambda) \), we have

\[
\phi'_{p^e}(C_x(\lambda)) = \prod_{y \in (A/p^eA)^\times, y \neq x} (C_x(\lambda) - C_y(\lambda)).
\]

Then

\[
\pm \delta = \prod_{x \in (A/pA)^\times} \phi'_{p^e}(C_x(\lambda))
= \prod_{\sigma \in \text{Gal}(K_{p^e}/K)} \sigma(\phi'_{p^e}(\lambda)) = N_{K_{p^e}/K}(\phi'_{p^e}(\lambda))
= N_{K_{p^e}/K}(p^e)/N_{K_{p^e}/K}(C_{p^{e-1}}(\lambda))
= N_{K_{p^e}/K}(p^e)/N_{K_{p^e}/K}(N_{K_{p^e}/K}(C_{p^{e-1}}(\lambda)))
= \pm p^{\phi(p^e)-1}(eq^{d_{(e-1)}}).
\]

Let \( w \in O_{p^e} \). Then \( w = \sum_{i=0}^{\phi(p^e)-1} a_i \lambda^i \) for some \( a_i \in K \). Hence

\[
\text{Tr}_{K_{p^e}/K}(w\lambda^j) = \sum_{i=0}^{\phi(p^e)-1} a_i \text{Tr}_{K_{p^e}/K}(\lambda^{i+j}) \in A \text{ for any } 0 \leq j < \phi(p^e).
\]

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Set $T = (\text{Tr}_{K^p/K}(\lambda^{i+j}))_{0 \leq i,j < \phi(p^e)}$, $a = (a_0, \ldots, a_{\phi(p^e)-1})$ and $b = (\text{Tr} w, \ldots, \text{Tr}(w\lambda^{\phi(p^e)-1}))$. We have $b = aT$ and $bT^* = da$. This shows $\delta a_i \in A$. Since $\delta$ is a power of $p$, we have $p^e w = \sum_{i=0}^{\phi(p^e)-1} b_i \lambda^i$ for some $n \in \mathbb{N}$ and $b_i \in A$ such that at least one $b_i$ not divided by $p$. Let $i_0$ be the smallest integer such that $v_p(b_{i_0}) = 0$. Since $v_p(\lambda) = 1/\phi(p^e)$, we have $v_p(b_{i_0} \lambda^{i_0}) < v_p(b_i \lambda^i)$ for any $i \neq i_0$. So

$$n \leq v_p(p^e w) = v(\sum_{i=0}^{\phi(p^e)-1} b_i \lambda^i) = v(b_{i_0} \lambda^{i_0}) = i_0/\phi(p^e) < 1.$$ 

We must have $n = 0$ and then $w \in A[\lambda]$. So $O_{p^e} = A[\lambda]$ and $1, \lambda, \ldots, \lambda^{\phi(p^e)-1}$ is an integral basis of $O_{p^e}/A$. Hence $\delta O_{p^e}/A$ is a power of $p$. As a consequence, $K_{p^e}/K$ is unramified at any prime ideal of $A$ not equal to $pA$. We prove the theorem for $a = p^e$.

For general $a$, write $a = p_{e_1}^e \cdots p_{e_t}^e$ for some pairwise different irreducible polynomials $p_i$ and some $e_i \in \mathbb{N}$. We prove our theorem by induction on $t$. Let $b = p_{e_1}^e \cdots p_{e_{t-1}}^e$ and $\lambda$ a generator of $C[a]$. Then $C_b(\lambda)$ is a generator of $C[p_{e_t}^e]$ and $C_{p_{e_t}^e}(\lambda)$ is a generator of $C[b]$. By induction, our theorem holds for $b$ and $p_{e_t}^e$. Choose $f, g \in A$ such that $fb + gp_{e_t}^e = 1$. We have $\lambda = C_f(C_b(\lambda)) + C_g(C_{p_{e_t}^e}(\lambda))$ and thus $K_a = K_b \cdot K_{p_{e_t}^e}$. Now $K_b \cap K_{p_{e_t}^e} = K$, because $K_b$ is unramified at $p_i A$ and $K_{p_{e_t}^e}$ is totally ramified at $p_i A$. As a consequence,

$$[K_a : K] = [K_b : K] \cdot [K_{p_{e_t}^e} : K] = \phi(b)\phi(p_{e_t}^e) = \phi(a).$$

So the monomorphism $\chi : \text{Gal}(K_a/K) \hookrightarrow (A/\pi A)^\times$ given in (3.1) is an isomorphism.

\begin{corollary}
For any $b \in A$ prime to $a$, there exists a unique $\sigma_b \in \text{Gal}(K_a/K)$ such that $\sigma_b(\lambda) = C_b(\lambda)$ for any generator $\lambda$ of $C[a]$. In particular, if $b$ is a monic irreducible polynomial furthermore, $\sigma_b = (bA, K_a/K)$.
\end{corollary}

\section{Reduction theory}

\subsection{Drinfeld modules over rings}

We can also define Drinfeld modules over arbitrary $A$-algebras or even $A$-schemes. In such generalizing, the underlying $\mathbb{F}_q$-vector space scheme need only be locally isomorphic to $\mathbb{G}_a$, so it should be the $\mathbb{F}_q$-vector space scheme associated to a line bundle on the base scheme.

For simplicity, let $R$ be an $A$-algebra with $\text{Pic} R = 0$. This holds if $R$ is a principle ideal domain. Then a Drinfeld module over $R$ is a ring homomorphism

$$\phi : A \to R[\tau], \quad a \to \phi_a$$

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such that \( \text{c.t.}(\phi_a) = a \in R \) and \( \text{l.c.}(\phi_a) \in R^\times \) for any \( 0 \neq a \in A \) and \( \phi_a \neq a \) for some \( a \in A \). Then for any maximal ideal \( m \) of \( R, \phi \mod m \) yields a Drinfeld module over \( R/m \) of the same rank.

## 4.2 Reduction theory of Drinfeld modules

Let \( R \) be a discrete valuation ring with fraction field \( L \), maximal ideal \( m \) and residue field \( F \). Let \( \nu : K^\times \to \mathbb{Z} \) be the discrete valuation.

**Definition 4.1.** Let \( \phi \) be a Drinfeld module over \( L \) of rank \( r \).

1. We say \( \phi \) has integral coefficients if \( \phi(A) \subset R\{\tau\} \) and the composition \( A \xrightarrow{\phi} R\{\tau\} \to F\{\tau\} \) defines a Drinfeld module over \( F \) of rank \( 0 < r_1 \leq r \).

2. We say \( \phi \) has stable reduction if it is isomorphic to a Drinfeld module \( \psi \) over \( L \) which has integral coefficients.

3. We say \( \phi \) has good reduction if \( \phi \) is isomorphic to a Drinfeld module \( \psi \) over \( L \) such that \( \psi(A) \subset R\{\tau\} \) and \( \text{l.c.}(\psi_a) \in R^\times \) for any \( 0 \neq a \in A \).

4. We say \( \phi \) has potentially stable (resp. good) reduction if there exists a finite extension \( (L', v') \) of \( (L, v) \) such that \( \phi \) has stable (resp. good) reduction on \( L' \).

**Lemma 4.2.** Let \( \phi \) and \( \psi \) be two Drinfeld modules over \( L \) of the same rank. If \( \phi \) and \( \psi \) have integral coefficients, then for any isomorphism \( c : \phi \simeq \psi \), we have \( c \in R^\times \).

**Proof.** Choose \( a \in A \setminus \mathbb{F}_q \) such that \( \deg(\phi_a \mod m) > 0 \). Write \( \phi_a = \sum_i a_i \tau^i \) for some \( a_i \in R \). There exists \( n > 0 \) such that \( a_n \in R^\times \) and \( a_i \in m \) for any \( i > m \). As \( \psi_a = c \phi_a c^{-1} \in R\{\tau\} \), we have \( c^{-1}a_n \in R \). This implies \( c^{-1} \in R \). Similarly, \( \psi = c^{-1} \phi c \) implies \( c \in R \). This proves \( c \in R^\times \).

**Corollary 4.3.** If \( \phi \) has stable reduction which is isomorphic to a Drinfeld module \( \psi \) having integral coefficients, then the isomorphic class of \( \psi \mod m \) does not depend on the choice of \( \psi \).

**Lemma 4.4.** Let \( \phi \) be a Drinfeld module over \( K \). Then \( \phi \) has stable reduction on some finite extension \( L' \) of \( K \).

**Proof.** Choose \( a_1, \ldots, a_n \in A \) which generates \( A \) as an \( \mathbb{F}_q \)-algebra. Write each \( \phi_a = \sum_j a_{ij} \tau^j \) for some \( a_{ij} \in L \) and set \( c = \min_{i,j \geq 1} \frac{v(a_{ij})}{q^j-1} \). Let \( n \) be the denominator of the rational number \( c \). Let \( L' \) be a totally ramified extension of \( L \) of index \( n \) and let \( \alpha \in L' \) with \( v(\alpha) = c \). Put \( \psi_a = \alpha \phi_a \alpha^{-1} \) for any \( a \in A \). Then \( \psi_{a_i} = \sum_j a_{ij} \alpha^{1-q^i} \tau^j \in R'(\tau) \) for any \( 1 \leq i \leq n \) and \( a_{ij} \alpha^{1-q^i} \in R^\times \) for some
1 ≤ i ≤ n and j ≥ 1 where \( R' \) is the valuation ring of \( L' \). This shows that \( \psi : A \to L'\{\tau\} \) has integral coefficients. In other words, \( \phi \) has stable reduction over \( L' \).

**Corollary 4.5.** Let \( \phi \) be a Drinfeld module over \( L \) of rank 1. If there exists \( a \in A \setminus \mathbb{F}_q \) such that \( \text{l.c.}(\phi_a) \in R^\times \), then \( \phi \) is a Drinfeld module over \( R \). In particular, \( \phi \) has good reduction.

**Proof.** By Lemma 4.4, there exists a finite ramified extension \( L' \) of \( L \) and \( \alpha \in L' \) such that \( \alpha \phi_\alpha^{-1}(A) \subset R'\{\tau\} \) and the composition \( A \xrightarrow{\alpha \phi_\alpha^{-1}} R'\{\tau\} \to R'/m'\{\tau\} \) defines a rank one Drinfeld module over \( R' \). In particular, \( \phi_b \in R\{\tau\} \) and \( \text{l.c.}(\phi_a) \in R^\times \) for any \( b \in R \). In other words, \( \phi \) is a Drinfeld module over \( R \).

## 5 Class field theory

Let \( I \) be the group of fractional \( A \)-ideals in \( K \), \( P \) the group of principal fractional \( A \)-ideals in \( K \), and \( \text{Pic}A = I/P \) the ideal class group of \( A \). In this section, fix an \( A \)-field \( L \).

### 5.1 Rank one Drinfeld modules over \( C \)

**Proposition 5.1.** We have bijections

\[
\text{Pic}A \simeq \{\text{rank 1 lattices in } C\}/\text{homothety} \simeq \{\text{rank 1 Drinfeld modules over } C\}/\text{isomorphism}.
\]

**Proof.** We need only to consider the first map. For injectivity, let \( I \) and \( I' \) be two fractional ideals of \( K \) such that they are homothety in \( C \). That is \( I = cI' \) for some \( c \in C \). We must have \( c \in K^\times \).

For surjectivity, take a lattice \( \Lambda \) in \( C \) of rank 1 and \( 0 \neq \lambda \in \Lambda \). Replacing \( \Lambda \) by \( \lambda^{-1}\Lambda \), we may assume that \( 1 \in \Lambda \). The injective homomorphism \( \Lambda \to K \otimes A \Lambda = K \) implies that \( \Lambda \) is a fractional ideal of \( K \).

**Proposition 5.2.** Every rank 1 Drinfeld module \( \phi \) over \( C \) is isomorphic to one defined over \( K_\infty \).

**Proof.** Let \( \Lambda \) be the corresponding lattice in \( C \) to \( \phi \). By Proposition 5.1, we may assume \( \Lambda \subset K \subset K_\infty \). By the construction of \( e_\Lambda(z) \) in Theorem 1.7 and \( \phi_a(z) \) in Corollary 1.8, we have \( e_\Lambda(z) \in K_\infty[[z]] \) and \( \phi_a \in K_\infty\{\tau\} \) for any \( a \in A \).
5.2 The action of ideals on Drinfeld modules

Let \( \phi \) be a Drinfeld module over \( L \) of rank \( r \) and height \( h \). For any nonzero ideal \( I \) of \( A \), the left ideal \( \sum_{i \in I} L(\tau) \phi_i \) of \( L(\tau) \) is generated by a unique monic polynomial \( \phi_I \). The scheme \( \text{Spec } L[z]/(\phi_I(z)) \) represents the functor

\[
\phi[I] : \text{Alg}_L \to \text{Mod}_A, \ R \mapsto \phi(R)[I].
\]

We have \( \#\phi[I](\mathcal{T}) = q^{\deg(\phi_I) - w(\phi_I)} \).

**Lemma 5.3.**

1. \( \deg(\phi_I) = r \deg(I) \).
2. \( w(\phi_I) = 0 \) if \( 0 = \text{char}_A(L) \) and \( w(\phi_I) = \text{hv}_p(I) \deg(p) \) if \( 0 \neq p = \text{char}_A(L) \).

**Proof.** First claim that there exists an ideal \( J \) of \( A \) prime to \( I \) such that \( J \not\subseteq p \) and \( IJ = (a) \) for some \( a \in A \).

Indeed, choose \( a_q \in q^{v_p(I)} \backslash q^{v_p(I) + 1} \) for each maximal ideal \( q \) of \( A \) dividing \( I \) or \( q = p \). By strong approximation theorem, there exists \( a \in K^x \) such that \( v_q(a - a_q) > v_q(I) \) for any maximal ideal \( q \) of \( A \) dividing \( I \) or \( q = p \) and \( v_q(a) \geq 0 \) otherwise. Thus \( a \in I \) and \( v_q(a) = v_q(I) \) when \( q \mid I \) or \( q = p \). Take \( J = aI^{-1} \). Then \( J \) is an ideal of \( A \) satisfying the required conditions.

So we have an isomorphism \( \phi[a] \simeq \phi[I] \oplus \phi[J] : \text{Alg}_L \to \text{Mod}_A \) of functors and hence

\[
\text{Spec } L[z]/(\phi_a(z)) = \text{Spec } L[z]/(\phi_I(z)) \times_L \text{Spec } L[z]/(\phi_J(z)) = \text{Spec } L[z]/(\phi_I(z)) \otimes_L L[z]/(\phi_J(z)).
\]

So \( \deg(\phi_a(z)) = \deg(\phi_I(z)) \cdot \deg(\phi_J(z)) \) and \( \deg(\phi_a) = \deg(\phi_I) + \deg(\phi_J) \). By counting elements of both sides of \( \phi[a](\mathcal{T}) = \phi[I](\mathcal{T}) \oplus \phi[J](\mathcal{T}) \), we have \( q^{\deg(\phi_a) - w(\phi_a)} = q^{\deg(\phi_I) - w(\phi_I)} q^{\deg(\phi_J) - w(\phi_J)} \) and hence \( \deg(\phi_a) - w(\phi_a) = \deg(\phi_I) - w(\phi_I) + \deg(\phi_J) - w(\phi_J) \). So \( w(\phi_a) = w(\phi_I) + w(\phi_J) \).

As \( l.c.(\phi_a) \phi(a) = \phi_a \), the lemma holds for \( (a) \) by the definitions of rank and height. By Proposition 2.6, we have \( \#\phi[J](\mathcal{T}) = q^{r \deg(J)} \). Choose positive integer \( n \) such that \( J^n = (b) \) for some \( b \in A \). Then \( (b) \) is a separable polynomial over \( L \) and so is \( \phi_I(z) \). This implies that \( \#\phi[J](\mathcal{T}) = \deg(\phi_I(z)) \) and hence \( \deg(\phi_J) = r \deg(J) \) and \( w(\phi_J) = 0 = \text{hv}_p(J) \deg(p) \).

**Lemma 5.4.** Let \( I \) be a nonzero ideal of \( A \). For any \( a \in A \), \( \phi_I \phi_a \in L(\tau) \phi_I \) and \( \phi_I \phi_a = (I \ast \phi)_a \phi_I \) for a unique \( (I \ast \phi)_a \in L(\tau) \). Then

\[
I \ast \phi : A \to L(\tau), \ a \mapsto (I \ast \phi)_a
\]

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is a Drinfeld module over $L$ and $\phi_1 : \phi \to I \ast \phi$ is an isogeny.

**Proof.** Since $\phi_1$ is a generator of $\sum L(\tau) \phi_i$, then $\phi_I = \sum f_I \phi_i$ for some $f_I \in L(\tau)$. Hence $\phi_I \phi_a = \sum f_I \phi_I \phi_a = \sum f_I \phi_a \phi_i$ and hence $\phi_I \phi_a = (I \ast \phi)_a \phi_I$ for a unique $(I \ast \phi)_a \in L(\tau)$. Obviously, $I \ast \phi : A \to L(\tau)$, $a \mapsto (I \ast \phi)_a$ is a ring homomorphism. By $\phi_I \phi_a = (I \ast \phi)_a \phi_I$, the constant term of $(I \ast \phi)_a$ is $\iota(a)^{q^{\deg(p)}}$. To show $I \ast \phi$ is a Drinfeld module, we need only to show that $\iota(a)^{q^{\deg(p)}} = \iota(a)$. If $w(\phi_a) = 0$, there is nothing to prove. Otherwise, by Lemma 5.3 we have $\operatorname{char}_A(L) = 0$ and $p \neq \operatorname{char}_A(L) \neq 0$ and $w(\phi_a) = hv_p(a) \deg(p) > 0$. In this case, $\iota(a)^{q^{\deg(p)}} = \iota(a)$ and hence $\iota(a)^{q^{\deg(p)}} = \iota(a)$. □

**Lemma 5.5.** (1) For any two nonzero ideals $I$ and $J$ of $A$, we have $(IJ) \ast \phi = J \ast (I \ast \phi)$.

(2) For any $0 \neq a \in A$, we have $(a) \ast \phi = u^{-1} \phi u$ where $u = \operatorname{l.c.}(\phi_a)$.

**Proof.** We have

$$L(\tau) \phi_{IJ} = \sum_{i \in I, j \in J} L(\tau) \phi_i \phi_j = \sum_{j \in J} L(\tau) \phi_I \phi_j = \sum_{j \in J} (I \ast \phi)_j \phi_I = L(\tau) (I \ast \phi)_J \phi_I$$

and then $\phi_{IJ} = (I \ast \phi)_J \phi_I$. For any $b \in A$, we have

$$(IJ) \ast \phi_b \phi_{IJ} = \phi_{IJ} \phi_b = (I \ast \phi)_J \phi_I \phi_b = (I \ast \phi)_J (I \ast \phi)_b \phi_I = (J \ast (I \ast \phi))_b (I \ast \phi)_J \phi_I = (J \ast (I \ast \phi))_b \phi_{IJ}$$

So $(IJ) \ast \phi_b = (J \ast (I \ast \phi))_b$ for any $b \in A$ and hence $(IJ) \ast \phi = J \ast (I \ast \phi)$.

If $I = (a)$ for some $a \in A$, then $\phi_a = u \phi_I$. For any $b \in A$,

$$(I \ast \phi)_b u^{-1} \phi_a = (I \ast \phi)_b \phi_I = \phi_I \phi_b = u^{-1} \phi_a \phi_b = u^{-1} \phi_b \phi_a$$

and $I \ast \phi_b = u^{-1} \phi_b u$. Then $u^{-1}$ defines an isomorphism $\phi \to I \ast \phi$. □

If $\operatorname{l.c.}(\phi_a)$ has an $q^{-\deg(a)}$-th root $v$ in $L$, define the action of the fractional ideal $(a^{-1})$ on $\phi$ to be $(a^{-1}) \ast \phi := v \phi v^{-1}$. Then $(a) \ast (a^{-1}) \ast \phi = \phi$. For any nonzero ideal $I$ of $A$, the action of the fractional ideal $a^{-1}I$ on $\phi$ is given by $(a^{-1}I) \ast \phi := I \ast ((a^{-1}) \ast \phi)$.

**Corollary 5.6.** Fix a perfect subfield $L_0$ of $L$. Let $\mathcal{X}$ be the set of Drinfeld modules $\phi$ over $L$ such that $\operatorname{l.c.}(\phi_a) \in L_0$ for each $a \in A$. The operation $\ast$ defines an action of the group $\mathcal{I}$ on $\mathcal{X}$. It induces an action of $\operatorname{Pic}A$ on the set of isomorphic classes of Drinfeld modules in $\mathcal{X}$.

**Proposition 5.7.** Let $\mathcal{X}(C)$ be the set of isomorphic classes of Drinfeld modules over $C$ of rank one. Then $\mathcal{X}(C)$ is a principle homogeneous space under the action of $\operatorname{Pic}A$. 

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Proof. Suppose $\phi$ is a Drinfeld module over $C$ of rank one. Let $\Lambda$ and $I*\Lambda$ be the corresponding lattices of $\phi$ and $I*\phi$, respectively. By Theorem 5.4, we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}/\Lambda & \longrightarrow & \mathbb{C}/(I*\Lambda) \\
\downarrow e_\Lambda & & \downarrow e_{I*\Lambda} \\
\phi(C) & \longrightarrow & (I*\phi)(C)
\end{array}
$$

of $A$-modules whose vertical arrows are isomorphisms. Since $\ker(I*\phi)$ is the $I$-torsion submodule of $\phi(C)$, we have $I*\Lambda = I^{-1}\Lambda$ and our assertion holds. \qed

5.3 Sgn-normalized Drinfeld modules

Recall that $F_\infty$ is the residue field of $\infty \in X$ and $d_\infty = \dim_{F_q}(F_\infty)$.

Definition 5.8. A sgn function on $K_\infty^\times$ is a homomorphism $sgn: K_\infty^\times \to F_\infty^\times$ such that $sgn|_{F_\infty^\times} = id$.

There are exactly $q^{d_\infty} - 1$ sgn functions on $K_\infty^\times$. From now on, fix a sgn function $sgn: K_\infty^\times \to F_\infty^\times$ and a uniformizer $\pi \in K_{\infty}$ with $sgn(\pi) = 1$.

Let $U_1 = \{ x \in K_\infty | v_\infty(x - 1) > 0 \}$. Then $sgn(U_1) = 1$ because $U_1$ is a pro-p-group. The uniformizer $\pi \in K_\infty$ defines an isomorphism $K_\infty \simeq F_\infty((\pi))$. Any $a \in K_\infty^\times$ can be uniquely written as $a = \zeta \pi^n u$ for some $\zeta \in F_\infty^\times$, $n \in \mathbb{Z}$ and $u \in U_1$, then $sgn(a) = \zeta$.

Definition 5.9. A rank one Drinfeld module $\phi$ over $L$ is called sgn-normalized if there exists an $F_q$-algebra homomorphism $\eta: F_\infty \to L$ such that $l.c.(\phi_a) = \eta(sgn(a))$ for any $0 \neq a \in A$.

Example 5.10. Suppose $A = F_q[t]$ and $sgn(t) = 1$. The sgn-normalized Drinfeld module over $L$ is just the Carlitz module given by $C: A \to L\{\tau\}$, $t \mapsto t + \tau$.

Theorem 5.11. (1) Every rank one Drinfeld module $\phi$ over $C$ is isomorphic to a sgn-normalized Drinfeld module.

(2) The set of sgn-normalized Drinfeld modules over $C$ isomorphic to $\phi$ is a principle homogeneous space under $F_\infty^\times / F_q^\times$.

Proof. (1) Extend $\phi: A \to C\{\tau\}$ to a ring homomorphism from $K$ to the ring $C\{\{\tau^{-1}\}\}$ of twist Laurent series which is still denoted by $\phi$. For any $a \in A$, we have $-\deg(\phi_a) = v_{\tau^{-1}}(\phi_a) = d_\infty v_\infty(a)$. So we can extend $\phi: K \to C\{\{\tau^{-1}\}\}$ to a continuous homomorphism $K_\infty \to C\{\{\tau^{-1}\}\}$ denoted by $\phi$ again. Choose $a \in C$ such that $a^{1-q^{d_\infty}} = l.c.(\phi_{\tau^{-1}})$. Replacing $\phi$ by $a^{-1}\phi a$, we
may assume \( l.c.(\phi_{a^{-1}}) = 1 \). Define \( \eta : \mathbb{F}_\infty \to L \) by \( \eta(c) = l.c.(\phi_c) \) for any \( c \in \mathbb{F}_\infty^\times \) and \( \eta(0) = 0 \). If we write any \( 0 \neq a \in A \) as \( a = c\pi^nu \) for some \( c \in \mathbb{F}_\infty^\times \), \( n \in \mathbb{Z} \) and \( u \in U_1 \), then we have

\[
l.c.(\phi_a) = l.c.(\phi_c\phi_n^a\phi_u) = l.c.(\phi_c) = \eta(c) = \eta(\text{sgn}(a)).
\]

So \( \phi \) is sgn-normalized.

(2) We may assume that \( \phi \) is sgn-normalized. Let \( \alpha \in C^\times \). Then \( \alpha^{-1}\phi\alpha \) is sgn-normalized if and only if \( 1 = l.c.(\alpha^{-1}\phi_{a^{-1}}) = \alpha^{q^{\deg(L)}-1} \) \( \) if and only if \( \alpha \in \mathbb{F}_\infty^\times \). By Proposition 5.20, \( \text{Aut}(\phi) = \mathbb{A}^\times = \mathbb{F}_q^\times \) and then \( \alpha^{-1}\phi\alpha = \phi \) implies \( \alpha \in \mathbb{F}_\infty^\times \). This proves (2).

\[\square\]

**Definition** 5.12. Let \( \mathcal{X}^+(L) \) be the set of sgn-normalized Drinfeld modules over \( L \). Let \( \mathcal{P}^+ \) be the subgroup of \( \mathcal{I} \) generated by \( (c) \) for those \( c \in K^\times \) such that \( \text{sgn}(c) = 1 \) and let \( \text{Pic}^+A = \mathcal{I}/\mathcal{P}^+ \).

**Proposition** 5.13. The set \( \mathcal{X}^+(L) \) is stable under \( \mathcal{I} \). For any \( \phi \in \mathcal{X}^+(L) \), \( \text{Stab}_\mathcal{I}(\phi) = \mathcal{P}^+ \).

**Proof.** By definition, there exists \( \eta : \mathbb{F}_\infty \to L \) such that \( l.c.(\phi_a) = \eta(\text{sgn}(a)) \) for any \( a \in A \). For any nonzero ideal \( I \) of \( A \), \( (I\ast\phi)_a\phi_I = \phi_I\phi_a \) implies \( l.c.(I\ast\phi)_a = l.c.(\phi)_{q^{\deg(I)}(\phi_I)} = l.c.(\phi)_{q^{\deg(I)}} = \eta(\text{sgn}(a))^{q^{\deg(I)}} \). This shows \( I\ast\phi \in \mathcal{X}^+(L) \). By Corollary 5.6, \( \mathcal{X}^+(L) \) is stable under \( \mathcal{I} \).

Now let \( I \in \mathcal{I} \) such that \( I\ast\phi = \phi \). Then \( I = b^{-1}J \) for some \( b \in A \) and some ideal \( J \) of \( A \). Hence \( \phi = I\ast\phi = (b^{-1})\ast(J\ast\phi) \) and \( (b)\ast\phi = J\ast\phi \). The composition \( \phi \xrightarrow{\phi_J} J\ast\phi = (b)\ast\phi \xrightarrow{\deg((b))} \phi \) is an endomorphism of \( \phi \). By Proposition 5.20, \( \text{End}(\phi) = A \) and hence \( l.c.(\phi_b)\phi_J = \phi_c \) for some \( c \in A \). Set \( J' = J + (c) \). Then \( \phi_J = \phi_{J'} = l.c.(\phi_c)^{-1}\phi_c \) and by Lemma 5.3, we have \( \deg J = \deg J' = \deg c \) and hence \( J = (c) \). By \( l.c.(\phi_b)\phi_J = \phi_c \), we have \( \eta(\text{sgn}(b)) = l.c.(\phi_c) = l.c.(\phi_b) = \eta(\text{sgn}(b)) \) and hence \( \text{sgn}(b^{-1}c) = 1 \). So \( I = (b^{-1}c) \in \mathcal{P}^+ \).

\[\square\]

**Theorem** 5.14. The action of \( \mathcal{I} \) on Drinfeld modules makes \( \mathcal{X}^+(C) \) a principle homogeneous space under \( \text{Pic}^+A \).

**Proof.** By Proposition 5.13, \( \mathcal{X}^+(C) \) is a disjoint union of principle homogeneous spaces under \( \text{Pic}^+A \). So we need only to check that \( \#\mathcal{X}^+(C) = \#\text{Pic}A \). By Proposition 5.1 and Theorem 5.11, we have \( \#\mathcal{X}^+(C) = \#\text{Pic}A \cdot \#\mathbb{F}_\infty^\times / \mathbb{F}_q^\times \). On the other hand, the short exact sequence

\[
1 \to \mathcal{P}/\mathcal{P}^+ \to \mathcal{I}/\mathcal{P}^+ = \text{Pic}^+A \to \mathcal{I}/\mathcal{P} = \text{Pic}A \to 1
\]

and the isomorphism \( \mathcal{P}/\mathcal{P}^+ \cong \mathbb{F}_\infty^\times / \mathbb{F}_q^\times \) induced by \( \text{sgn} \) show that \( \#\text{Pic}^+A = \#\text{Pic}A \cdot \#\mathbb{F}_\infty^\times / \mathbb{F}_q^\times \). \[\square\]
5.4 The narrow Hilbert class field

Fix $\phi \in \mathcal{X}^+(C)$. Define

$$H^+ = K(\text{all coefficients of } \phi_a \text{ for any } a \in A).$$

Then $\phi$ is a Drinfeld module over $H^+$, so is $I \ast \phi$ for any $I \in \mathcal{I}$. By Theorem 5.14, these are objects in $\mathcal{X}^+(C)$. So $H^+$ is independent of the choice of $\phi$, which is called the narrow Hilbert class field of $(A, \text{sgn})$.

**Theorem 5.15.** (1) The field $H^+$ is a finite abelian extension of $K$.

(2) The extension $H^+/K$ is unramified outside $\infty \in X$.

(3) We have $\text{Gal}(H^+/K) \simeq \text{Pic}^+ A$.

**Proof.** (1) The group $\text{Aut}(C/K)$ of automorphisms of $C$ fixing $K$ acts on $\mathcal{X}^+(C)$, so it maps $H^+$ to itself. Also, $H^+$ is finitely generated over $K$. These imply that $H^+$ is a finite normal extension of $K$. By Proposition 5.2, $\phi$ is isomorphic to Drinfeld module $\psi$ over $K_\infty$. Extend $\psi : A \to K_\infty\{\{r^{-1}\}\}$ to $\psi : K_\infty \to K_\infty\{\{r^{-1}\}\}$ as in the proof of Theorem 5.11 and let $c \in C$ such that $c^{-1} - q^{d_\infty} = \text{l.c.}(\psi_{\pi^{-1}}) \in K_\infty$. Then $c^{-1}\psi c$ is a sgn-normalized Drinfeld module over a finite separable extension $K_\infty(c)$ of $K_\infty$ isomorphic to $\phi$. The completion $K_\infty$ of a global field $K$ is a separable extension of $K$, hence $H^+$ is separable over $K$. The automorphism group of $\mathcal{X}^+(C)$ as a principal homogeneous space under $\text{Pic}^+ A$ is equal to $\text{Pic}^+ A$, so we have a monomorphism $\chi : \text{Gal}(H^+/K) \to \text{Aut}\mathcal{X}^+(C) \simeq \text{Pic}^+ A$. So $\text{Gal}(H^+/K)$ is a finite abelian group.

(2) Let $B^+$ be the integral closure of $A$ in $H^+$. Let $\mathfrak{P}$ be a nonzero prime ideal of $B^+$ lying above $p$ of $A$. Let $F_{\mathfrak{P}} = B^+/\mathfrak{P}$. By Corollary 4.5, each $\phi \in \mathcal{X}^+(H^+) = \mathcal{X}^+(C)$ is a Drinfeld module over the localization $B^+_{\mathfrak{P}}$, so there is a reduction map $\rho : \mathcal{X}^+(H^+) \to \mathcal{X}^+(F_{\mathfrak{P}})$. By Proposition 5.13, $\text{Pic}^+ A$ acts faithfully on the source and target. Moreover, the map $\rho$ is $\text{Pic}^+ A$-equivariant, and by Theorem 5.14 $\mathcal{X}^+(H^+)$ is a principal homogeneous space under $\text{Pic}^+ A$, so $\rho$ is injective. If some $\sigma \in \text{Gal}(H^+/K)$ belongs to the inertia group at $\mathfrak{P}$, then $\sigma$ acts trivially on $\mathcal{X}^+(F_{\mathfrak{P}})$, so $\sigma$ acts trivially on $\mathcal{X}^+(H^+)$ and $\sigma = 1$. Thus $H^+/K$ is unramified at $\mathfrak{P}$.

(3) Let $D_{\mathfrak{P}} = \{\sigma \in \text{Gal}(H^+/K) | \sigma(\mathfrak{P}) = \mathfrak{P}\}$. By (2), $D_{\mathfrak{P}} \simeq \text{Gal}(F_{\mathfrak{P}}/F_p)$. The Frobenius element in $\text{Gal}(F_{\mathfrak{P}}/F_p)$ defines an element $\text{Frob}_p \in \text{Gal}(H^+/K)$. For any $\tilde{\phi} \in \mathcal{X}^+(F_{\mathfrak{P}})$, we have $\tilde{\phi}_p = \tau^{d_\mathfrak{P}} \cdot p$ by Lemma 5.3. For any $a \in A$, the equality $(p \ast \tilde{\phi})_a \tilde{\phi}_p = \tilde{\phi}_p \tilde{\phi}_a$ implies that $(p \ast \tilde{\phi})_a = \text{Frob}_p \tilde{\phi}_a$ and hence $p \ast \tilde{\phi} = \text{Frob}_p \tilde{\phi}$.
Since $\rho : X^+(H^+) \to X^+(\mathbb{F}_p)$ is injective and Pic$^+A$-equivariant, then the action of Frobp and $p$ on $X^+(H^+)$ coincide. Thus $\chi : \text{Gal}(H^+/K) \to \text{Pic}^+A$ maps Frobp to the class of $p$ in Pic$^+A$. Such class generates Pic$^+A$, so $\chi$ is surjective. \hfill \Box

5.5 Hilbert class field

By the short exact sequence

$$1 \to \mathcal{P}/\mathcal{P}^+ \to \text{Pic}^+A \to \text{Pic}A \to 1,$$

the extension $K \subset H^+$ decomposes into two abelian extensions $K \xrightarrow{\text{Pic}A} H \xrightarrow{\mathcal{P}/\mathcal{P}^+} H^+$ with Galois group as shown. The surjective map $X^+(C) \to X(C)$ is compatible with the epimorphism of groups Pic$^+A \to \text{Pic}A$. By Proposition 5.2, each element of $X(C)$ is represented by a Drinfeld module over $K^\infty$, so the decomposition group $D_\infty$ of $H^+/K$ at $\infty \in X$ acts trivially on $X(C)$. So $D_\infty \subset \mathcal{P}/\mathcal{P}^+$. In other words, $\infty$ splits completely in $H/K$. The Hilbert class field $H_A$ of $A$ is defined as the maximal unramified extension of $K$ in which $\infty$ splits completely. Thus $H \subset H_A$. Class field theory shows that Pic$A \simeq \text{Gal}(H_A/K)$. So $H_A = H$.

5.6 Ray class fields

In this section, we generalize the construction to obtain all the abelian extensions of $K$, even the ramified ones. Fix notations as follows.

$m$: a nonzero ideal of $A$.

$\mathcal{I}_m$: the subgroup of $\mathcal{I}$ generated by maximal ideals of $A$ not dividing $m$.

$\mathcal{P}_m$: the subgroup of $\mathcal{I}$ generated by $(c)$ for those $c \in K^\times$ with $c \equiv 1 \pmod{m}$.

$\mathcal{P}_m^+$: the subgroup of $\mathcal{I}$ generated by $(c)$ for those $c \in K^\times$ with $c \equiv 1 \pmod{m}$ and $\text{sgn}(c) = 1$.

Pic$^+_mA := \mathcal{I}_m/\mathcal{P}_m$, the ray class group modulo $m$ of $A$.

Pic$^+_nA := \mathcal{I}_m/\mathcal{P}_m^+$, the narrow ray class group modulo $m$ of $A$.

$X^+_m(C) := \{(\phi, \lambda) | \phi \in X^+(C) \text{ and } \lambda \text{ generates the } A/m\text{-module } \phi|m|(C)\}$.

Here $c \equiv 1 \pmod{m}$ means that $c$ is quotient $b/c$ of two elements of $A$ relative prime to $m$ such that $a \equiv b \pmod{m}$.  

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Lemma 5.16. We have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & 
\rightarrow & (I_m \cap P^+)/P^+_m & 
\rightarrow & (I_m \cap P)/P_m & 
\rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 
\rightarrow & P_m/P^+_m & 
\rightarrow & I_m/P^+_m & 
\rightarrow & I_m/P_m & 
\rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 
\rightarrow & P/P^+ & 
\rightarrow & I/P^+ & 
\rightarrow & I/P & 
\rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

with exact rows and lines. Moreover, we have canonical isomorphisms \(P_m/P^+_m \simeq P/P^+ \simeq \mathbb{F}_q^\times\) and \((I_m \cap P^+)/P^+_m \simeq (I_m \cap P)/P_m \simeq (A/m)^\times\).

Proof. The second and third lines are obviously exact. By the snake lemma, to prove exactness of lines and rows in the above diagram, we need only to show that \(P_m/P^+_m \rightarrow P/P^+\) is an isomorphism and \(I_m/P_m \rightarrow I/P\) is surjective.

(1) Recall in Theorem 5.14 that the sgn function induces an isomorphism \(P/P^+ \simeq \mathbb{F}_q^\times\). Obviously, the sgn function induces a monomorphism \(P_m/P^+_m \rightarrow \mathbb{F}_q^\times\). To show it is surjective, we need find \(c \in 1 + m\) such that sgn(c) = \(\alpha\) for any \(\alpha \in \mathbb{F}_q^\times\). Choose \(x \in K^\times\) with sgn(x) = \(\alpha\). Then \(v_\infty(x - a/b) > v_\infty(x)\) for some \(a, b \in A\). We have \(a/bx \in U_1\) and hence

\[
\text{sgn}(ab^{d^{\infty - 2} - 1}) = \text{sgn}(a/b)\text{sgn}(b)^{d^{\infty - 1}} = \text{sgn}(a/b) = \text{sgn}(x)\text{sgn}(a/bx) = \text{sgn}(x) = \alpha.
\]

Take \(0 \neq y \in m\) and set \(c = 1 + ab^{d^{\infty - 2} - 1}x^{d^{\infty - 1}}\). Then \(c \equiv 1 \pmod{m}\) and sgn(c) = \(\alpha\).

(2) The surjectivity of \(I_m/P_m \rightarrow I/P\) is equivalent to \(I = I_mP\). Let \(I\) be a nonzero ideal of \(A\). For each maximal ideal \(p\) of \(A\) dividing \(Im\), choose \(a_p \in p^{v_p(I)} \setminus p^{v_p(I)+1}\). By strong approximation theorem, there exists \(a \in K^\times\) such that \(v_p(a - a_p) > v_p(I)\) for any maximal ideal \(p\) dividing \(Im\) and \(v_p(a) \geq 0\) for any \(p \nmid Im\). Take \(J = aI^{-1}\). Then \(J\) is an ideal of \(A\) prime to \(m\) and \(I = aJ^{-1} \in I_mP\).

(3) It remains to show \((A/m)^\times \simeq (I_m \cap P)/P_m\). Define a map \(\mu : I_m \cap P^+ \rightarrow (A/m)^\times\) as follows. Any element of \(I_m \cap P^+\) is of the form \((c)\) for some \(c \in K^\times\) with sgn(c) = \(1\) and \(c \in I_m\). So there exist ideals \(I\) and \(J\) of \(A\) prime to \(m\) such that \((c) = IJ^{-1}\). Then \(I^n = (a)\) for some positive integer \(n\) and some \(a \in A\) prime to \(m\). As \((c) = I^n(I^{n-1}J)^{-1} = (a)(I^{n-1}J)^{-1},\)
we have \((ac^{-1}) = I^{n-1}J\) and then \(ac^{-1} \in A\) prime to \(m\). Define \(\mu((c)) = (a \mod m) \cdot (ac^{-1} \mod m)^{-1} \in (A/m)^\times\). Obviously, \(\mu\) is a well defined homomorphism of groups. If \(\mu((c)) = 1\), then \(a \equiv ac^{-1} \pmod m\) and hence \((c) = \mathcal{P}_m^+\). It follows that \(\ker(\mu) = \mathcal{P}_m^+.\) Given \(x \in A\) prime to \(m\), we can find \(y \in m\) such that \(\deg(y) > \deg(x)\) and \(\text{sgn}(y) = 1\). Then \(\text{sgn}(x+y) = \text{sgn}(y) = 1\), \((x + y) \in \mathcal{P}_m^+\) and \(\mu((x+y)) = x \mod m \in (A/m)^\times\). This shows that \(\mu\) is surjective and hence it induces an isomorphism \((\mathcal{I}_m \cap \mathcal{P}^+)/\mathcal{P}_m^+ \simeq (A/m)^\times\). \(\square\)

**Lemma 5.17.** If \(m\) is prime to \(\text{char}_A(L)\), let

\[
\mathcal{X}_m^+(L) = \{(\phi, \lambda) | \phi \in \mathcal{X}_m^+(L) \text{ and } \lambda \text{ generates the } A/m\text{-module } \phi[m](L)\}.
\]

Then we have an action of \(\mathcal{I}_m\) on \(\mathcal{X}_m^+(L)\) such that the stabilizer of each \((\phi, \lambda)\) is \(\mathcal{P}_m^+\).

**Proof.** Let \((\phi, \lambda) \in \mathcal{X}_m^+(L)\) and let \(I\) be an ideal of \(A\) prime to \(m\). The isogeny \(\phi_I : \phi \to I \ast \phi\) induces an \(A\)-linear map \(\phi_I^*: \phi[m](L) \to (I \ast \phi)[m](L)\) with source and target are free \(A/m\)-modules of rank one. As \(I\) is prime to \(m\), \(\phi_I^*\) is injective and hence bijective. So \(\phi_I^*(\lambda)\) is a generator of \((I \ast \phi)[m](L)\). Define \(I \ast (\phi, \lambda) = (I \ast \phi, \phi_I^*(\lambda))\), which can be extended to an action of \(\mathcal{I}_m\) on \(\mathcal{X}_m^+(L)\).

Suppose \(I \ast (\phi, \lambda) = (\phi, \lambda)\) for some \(I \in \mathcal{I}_m\). By Theorem 5.14, \(I = (c)\) for some \(c \in K^\times\) with \(\text{sgn}(c) = 1\). As \((c) \in \mathcal{I}_m\), then \((c) \cap A\) is an ideal of \(A\) prime to \(m\). Choose \(x \in (1 + m) \cap (c) \cap A\) and take \(a = x^{d_w-1}\). Then \(a \in A\) and \(\text{sgn}(a) = 1\) and \(a = cb\) for some \(b \in A\). Hence \(a \in 1 + m\) and \(\text{sgn}(b) = 1\). The equality \(\phi_I^*(\lambda) = \lambda\) means that \(\phi_a(\lambda) = \phi_b(\lambda)\), and hence \(a - b \in m\). This shows that \(I = (c) \in \mathcal{P}_m^+\) and \(\text{Stab}_{\mathcal{I}_m}(\phi, \lambda) = \mathcal{P}_m^+\). \(\square\)

**Theorem 5.18.** Fix \((\phi, \lambda) \in \mathcal{X}_m^+(C)\). Define the narrow ray class field \(H_m^+\) modulo \(m\) of \((A, \text{sgn})\) to be \(H^+(\lambda)\).

1. The action of \(\mathcal{I}_m\) on \(\mathcal{X}_m^+(C)\) makes it to be a principle homogeneous space under \(\text{Pic}_m^+ A\).

2. The field \(H_m^+\) is independent of the choice of \((\phi, \lambda)\), and the extension \(H_m^+/K\) is finite abelian, unramified at each prime of \(A\) not dividing \(m\).

3. We have \(\text{Gal}(H_m^+/K) \simeq \text{Pic}_m^+ A\).

4. Let \(H_m\) be the subfield of \(H_m^+\) fixed by \(\mathcal{P}_m^+/\mathcal{P}_m^+\). Then \(H_m/K\) splits at \(\infty\) and \(\text{Gal}(H_m/K) = \text{Pic}_m A\).

**Proof.** By Lemma 5.17, \(\mathcal{X}_m^+(C)\) is a disjoint of principle homogeneous spaces under \(\text{Pic}_m^+ A\). To prove (1), we need only to show that \(#\text{Pic}_m^+ A = #\mathcal{X}_m^+(C)\). By Theorem 5.14, \(#\mathcal{X}_m^+(C) = #\mathcal{X}^+(C)\).
\[(A/m)^* = \#Pic^+_A \cdot \#(A/m)^*\]. By Lemma 5.16, \(#Pic^+_mA = \#Pic^+_A \cdot \#(A/m)^*\). So (1) holds.

(2) For any \(I \in I_m\), \(I*(\phi, \lambda) = (I*\phi, \phi^*_I(\lambda))\). So \(H^+_m\) is independent of the choice of \((\phi, \lambda)\). The group \(\text{Aut}(C/K)\) also acts on \(X^+_m(C)\), so \(H^+_m\) is stable under \(\text{Aut}(C/K)\). This shows that \(H^+_m/K\) is a finite Galois extension. The automorphism group of \(X^+_m(C)\) as a principle homogeneous space under \(\text{Pic}^+_mA\) is equal to \(\text{Pic}^+_mA\). So we have a monomorphism

\[\chi : \text{Gal}(H^+_m/K) \to \text{Aut}(X^+_m(C)) \simeq \text{Pic}^+_mA.\]

Thus \(H^+_m/K\) is a finite abelian extension.

Let \(B\) be the integral closure of \(A\) in \(H^+_m\), and let \(\mathfrak{p}\) be a maximal ideal of \(B\) lying above a maximal ideal \(\mathfrak{m}\) of \(A\) not dividing \(\mathfrak{m}\). By Corollary 4.5, for each \((\phi, \lambda) \in X^+_m(H^+_m) = X^+_m(C)\), \(\phi\) is a Drinfeld module over the localization \(B_\mathfrak{p}\). So there is a reduction map \(\rho : X^+_m(H^+_m) \to X^+_m(\mathbb{F}_\mathfrak{p})\) of principle homogeneous spaces under \(\text{Pic}^+_mA\). By (1), \(\rho\) is injective. If some \(\sigma \in \text{Gal}(H^+_m/K)\) belongs to the inertia group at \(\mathfrak{p}\), then \(\sigma\) acts trivially on \(X^+_m(\mathbb{F}_\mathfrak{p})\). Hence \(\sigma\) acts trivially on \(X^+_m(H^+_m)\) and \(\sigma = 1\). Thus \(H^+_m/K\) is unramified at \(\mathfrak{p}\).

(3) The Frobenius element in \(\text{Gal}(\mathbb{F}_\mathfrak{p}/\mathbb{F}_\mathfrak{q})\) defines an element \(\text{Frob}_{\mathfrak{p}} \in \text{Gal}(H^+_m/K)\). For any \(\bar{\phi} \in X^+_m(\mathbb{F}_\mathfrak{p})\), we have \(\bar{\phi}_{\mathfrak{p}} = \tau^{\text{deg}\mathfrak{p}}\bar{\phi}\) by Lemma 5.3. For any \(a \in A\), the equality \((\mathfrak{p} * \bar{\phi})_a \bar{\phi}_{\mathfrak{p}} = \bar{\phi}_{\mathfrak{p}} \bar{\phi}_a\) implies that \((\mathfrak{p} * \bar{\phi})_a = \text{Frob}_{\mathfrak{p}} \bar{\phi}_a\), and hence \(\mathfrak{p} * \bar{\phi} = \text{Frob}_{\mathfrak{p}} \bar{\phi}\).

Since \(\rho : X^+_m(H^+) \to X^+_m(\mathbb{F}_\mathfrak{p})\) is injective and \(\text{Pic}^+_A\)-equivariant, it follows that the actions of \(\text{Frob}_{\mathfrak{p}} \in \text{Gal}(H^+_m)\) and \(\mathfrak{p} \in I_m\) on \(X^+_m(H^+_m)\) coincide. Thus \(\chi : \text{Gal}(H^+_m/K) \to \text{Pic}^+_mA\) sends \(\text{Frob}_{\mathfrak{p}}\) to the class of \(\mathfrak{p}\) in \(\text{Pic}^+_mA\). Such class generates \(\text{Pic}^+_mA\), so \(\chi\) is surjective.

(4) Let \(X_m(C)\) be the set of isomorphic classes in \(X^+_m(C)\). Then \(X_m(C)\) is a principle homogeneous space under \(\text{Pic}^+_mA\). The surjective map \(X^+_m(C) \to X_m(C)\) is compatible with the epimorphism of groups \(\text{Pic}^+_mA \to \text{Pic}_mA\). By Proposition 5.2, each element of \(X(C)\) is represented by a Drinfeld module over \(K_\infty\), so the decomposition group \(D_\infty\) of \(H^+_m/K\) at \(\infty\) acts trivially on \(X_m(C)\). So \(D_\infty \subset \mathcal{P}_m/\mathcal{P}_m^+\). In other words, \(\infty\) splits completely in \(H^+_m/K\). The equality \(\text{Gal}(H_m/K) = \text{Pic}^+_mA\) holds by Lemma 5.16.

5.7 The maximal abelian extension of \(K\)

In this subsection, we construct the maximal abelian extension \(K^{ab}\) of \(K\).
Theorem 5.19. Let \( K^{ab,\infty} = \bigcup_m H_m \) when \( m \) runs over all nonzero ideals of \( A = \Gamma(X - \{\infty\}, O_X) \) and let \( K_c := \bigcup_{n \geq 1} F_{q^n} K \) be the constant extension of \( K \).

1. Then \( K^{ab,\infty} \) is the maximal abelian extension of \( K \) in which \( \infty \) splits completely.

2. Choose another closed point \( \infty' \) of \( X \). Then \( K^{ab} \) is the compositum of \( K_c, K^{ab,\infty} \) and \( K^{ab,\infty'} \).

Before proving the theorem, first recall the class field theory for function fields.

For any closed point \( p \) of \( X \), denote by \( K_p \) the completion of \( K \) at \( p \), \( O_p \) the discrete valuation ring of \( K_p \) and \( v_p \) the discrete valuation. Define the idèle group of \( K \) to be

\[ \mathbb{A}_K^\times = \{(a_p) \in \prod_{p \in |X|} K_p^\times \mid a_p \in O_p^\times \text{ for almost all } p\}. \]

For any effective divisor \( D = \sum n_p p \) of \( X \), let \( U_D = \prod_{p \in |X|} U_p^{(n_p)} \), where \( U_p^{(0)} = O_p^\times \) and \( U_p^{(n_p)} = \{ a \in K_p \mid v_p(a - 1) \geq n_p \} \) if \( n_p > 0 \). Equip the idèle group a canonical topology by taking a basic system of neighborhoods of \( 1 \in \mathbb{A}_K^\times \) to be the sets \( U_D \) where \( D \) runs over all the effective divisors of \( X \). Therefore \( \mathbb{A}_K^\times \) is a locally compact group. The inclusion \( K \subset K_p \) defines the diagonal embedding \( K^\times \to \mathbb{A}_K^\times \) which makes \( K^\times \) to be a discrete subgroup of \( \mathbb{A}_K^\times \). We call the quotient group \( C_K = \mathbb{A}_K^\times / K^\times \) the idèle class group of \( K \). For any finite field extension \( L/K \), we have the norm map

\[ N_{L/K}: \mathbb{A}_L^\times \to \mathbb{A}_K^\times, \quad N_{L/K}((a_p))_p = \prod_{p \mid \mathfrak{p}} N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(a_{\mathfrak{p}}). \]

The thrust of class field theory is that there exists a continuous homomorphism

\[ (\bullet, K^{ab}/K): \mathbb{A}_K^\times \to \text{Gal}(K^{ab}/K), \]

which satisfies the following properties:

(i) \( (\bullet, K^{ab}/K) \) has dense image and its kernel is \( K^\times \).

(ii) For each \( p \in |X| \), \( (\bullet, K^{ab}/K) \) is compatible with the local reciprocity map for \( K_p \). In particular, if \( \pi_p \in K_p \) is a uniformizer, then \( (\pi_p, K^{ab}/K) \) is a Frobenius element for \( p \).

(iii) For any finite abelian extension \( L/K \), \( (\bullet, K^{ab}/K) \) induces an isomorphism

\[ \mathbb{A}_L^\times / K^\times N_{L/K}(\mathbb{A}_L^\times) \cong \text{Gal}(L/K). \]
(iv) The map $L \mapsto N_L := K^\times N_{L/K}(\mathbb{A}_K^\times)$ is a one-to-one correspondence between finite abelian extensions of $K$ and open subgroups of $\mathbb{A}_K^\times$ of finite index containing $K^\times$. Moreover, $N_{LL'} = N_L \cap N_{L'}$ and $N_{L \cap L'} = N_L N_{L'}$ for any two finite abelian extensions $L, L'$ of $K$.

Observe that any open subgroup of $\mathbb{A}_K^\times$ contains $U_D$ for some effective divisor $D$ of $X$. To specify an open subgroup of finite index in $C_K$, it suffices to give an effective divisor $D$ of $X$ and an open subgroup $N$ of $\mathbb{A}_K^\times$ of finite index containing $K^\times U_D$. The corresponding abelian extension $K_N/K$ should have these properties:

(a) $K_N/K$ is unramified outside $\text{Supp}(D)$.

(b) There is an isomorphism $\mathbb{A}_K^\times/N \simeq \text{Gal}(K_N/K)$, which carries a uniformizer at $p \notin \text{Supp}(D)$ to the Frobenius element $\text{Frob}_p \in \text{Gal}(K_N/K)$.

The ray class field $K_D$ is the compositum of all finite extensions obtained this way. Then $\text{Gal}(K_D/K)$ is isomorphic to the profinite completion of the ray class group $C_D := \mathbb{A}_K^\times/K^\times U_D$.

Suppose $\infty \notin \text{Supp}(D)$. The divisor $D = \sum_p n_p p$ gives an ideal $m$ of $A$ such that $v_p(m) = n_p$ for any $p \neq \infty$. Let $\pi_\infty \in K_\infty$ be a uniformizer.

**Lemma 5.20.** Suppose $\infty \notin \text{Supp}(D)$. We have $\mathbb{A}_K^\times/K^\times U_D \pi_\infty^{n_\infty} \simeq \text{Pic}_m A$. In particular, $K^\times U_D \pi_\infty^{n_\infty}$ is a subgroup of $\mathbb{A}_K^\times$ of finite index. Any open subgroup of $\mathbb{A}_K^\times$ of finite index containing $K^\times U_D$ must contains $K^\times U_D \pi_\infty^{n_\infty}$ for some positive integer $n$.

**Proof.** Let

$$U'_D = \{(a_p) \in \mathbb{A}_K^\times | v_p(a_p - 1) \geq n_p \text{ for any } p \in \text{Supp}(D)\}.$$ 

By the weak approximation theorem, we have $\mathbb{A}_K^\times = K^\times U'_D$ and hence

$$\mathbb{A}_K^\times/K^\times U_D \pi_\infty^{n_\infty} = K^\times U'_D/K^\times U_D \pi_\infty^{n_\infty} \simeq U'_D/(U'_D \cap K^\times U_D \pi_\infty^{n_\infty}) \simeq U'_D/(((K^\times \cap U'_D)U_D \pi_\infty^{n_\infty})).$$

Any $p \in |X| - \{\infty\}$ defines a maximal ideal of $A$ which is still denoted by $p$. The canonical homomorphism

$$U'_D \to \mathcal{I}_m, \ (a_p) \mapsto \prod_{p \neq \infty} p^{v_p(a_p)}$$

induces an isomorphism

$$U'_D/((K^\times \cap U'_D)U_D \pi_\infty^{n_\infty}) \simeq \mathcal{I}_m/\mathcal{P}_m = \text{Pic}_m A.$$
Let $N$ be an open subgroup of $\mathbb{A}^\times_K$ of finite index containing $K^\times U_D$ and let $\mathcal{N} = N/K^\times U_D$. So $\mathcal{N}$ is a subgroup of $C_D$ of finite index. The short exact sequence
\[ 1 \to \pi_n^\infty \to C_D \to \text{Pic}_m A \to 1 \]
sows that $\mathcal{N} \cap \pi_n^\infty = \pi_n^{nZ}$ for some $n > 0$ and hence $K^\times U_D \pi_n^Z \subset N$.

**Corollary 5.21.** If $\infty \notin \text{Supp}(D)$, then the subgroup $K^\times U_D \pi^Z_\infty \subset \mathbb{A}^\times_K$ gives the extension $H_m/K$ defined in section 5.6.

**Proof.** By Theorem 5.18, $H_m$ is unramified outside $\text{Supp}(D)$ and splits at $\infty$. The assertion follows by the following commutative diagram
\[
\begin{array}{ccc}
\mathbb{A}^\times_K & \overset{(\bullet, H_m/K)}{\longrightarrow} & \text{Gal}(H_m/K) \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{A}^\times_K \big/ K^\times U_D \pi^Z_\infty & \overset{\sim}{\longrightarrow} & \text{Pic}_m A.
\end{array}
\]

**Lemma 5.22.** If $\infty \notin \text{Supp}(D)$, then the ray class field $K_D$ is the compositum of $H_m$ and $K_c$.

**Proof.** Consider the degree map
\[ \deg : \mathbb{A}^\times_K \to \mathbb{Z}, \quad \deg((a_p)) = \sum_{p \in \mathcal{X}} v_p(a_p) \deg(p). \]
Then $\deg(K^\times U_0) = 1$ and the inverse image of $n\mathbb{Z}$ in $\mathbb{A}^\times_K$ gives the constant extension $K_n := K\cdot \mathbb{F}_q^n$ of $K$ of degree $n$. Let $L$ be a finite extension of $K$ containing in $K_D$. By Lemma 5.20, we may assume $\mathcal{N}_L = K^\times U_D \pi^Z_\infty$ for some $n \geq 1$. Then $\mathcal{N}_L \supset K^\times U_D \pi^Z_\infty \cap \deg^{-1}(nd_\infty \mathbb{Z})$ and hence $L \subset H_m K_{nd_\infty}$.

**Lemma 5.23.** For any two effective divisors $D = \sum p a_p$ and $D' = \sum p a'_p$ of $X$, let $\min(D, D') = \sum p \min(a_p, a'_p)$ and $\max(D, D') = \sum p \max(a_p, a'_p)$. Then
\[ K_D \cap K_{D'} = K_{\min(D, D')} \] and $K_D \cdot K_{D'} = K_{\max(D, D')}$.

**Proof.** We may assume $\infty \notin \text{Supp}(D + D')$. Obviously, $K_D \cap K_{D'} \supset K_{\min(D, D')}$. Let $L$ be a finite extension of $K$ containing in $K_D \cap K_{D'}$. By Lemma 5.20, there exists $n \geq 1$ such that
\(\mathcal{N}_L \supset K^\times U_D \pi^n\) and \(\mathcal{N}_L \supset K^\times U_{D'} \pi^{n'}\). Hence \(\mathcal{N}_L \supset K^\times U_{\min(D,D')} \pi^{n}\) and \(L \subset K_{\min(D,D')}\). This proves \(K_D \cap K_{D'} \subset K_{\min(D,D')}\). The proof of \(K_D \cdot K_{D'} = K_{\max(D,D')}\) is similar.

We are ready to prove Theorem 5.19.

Recall that \(K^{ab} = \bigcup_{E} K_E\) when \(E\) runs over all effective divisors of \(X\). To prove \(K^{ab} = K_c K^{ab,\infty} K^{ab,\infty'}\), it suffices to show that \(K_E \subset K_c K^{ab,\infty} K^{ab,\infty'}\) for each \(E\). Write \(E = D + D'\) for some effective divisors \(D\) and \(D'\) such that \(\text{Supp}(D) \cap \text{Supp}(D') = \emptyset\), \(\infty \notin \text{Supp}(D)\) and \(\infty' \notin \text{Supp}(D')\). By Lemma 5.23, \(K_E = K_D K_{D'}\) and by Lemma 5.22, \(K_D \subset K^{ab,\infty} K_c\) and \(K_{D'} \subset K^{ab,\infty'} K_c\). This completes the proof of Theorem 5.19.