Lecture 2 Semi-derived Hall algebras and quantum groups

Contents:

§1 Double framed quiver algebras

- k : field
- A : a finite dimensional algebra
- $\mathrm{mod}(A)$: finite dimensional (left) A-mod
- $\mathrm{Proj}(A)$: finitely generated proj A-modules.
- $\mathrm{Inj}(A)$: finitely generated injective A-modules

§1.1 Double framed quiver algebras

- $R_2 := k\left(1 \underset{\varepsilon_2}{\overset{\varepsilon_1}{\rightleftarrows}} 2\right) / (\varepsilon_1\varepsilon_2, \varepsilon_2\varepsilon_1)$

- R_2 is self-injective, i.e., projective modules coincides with injective modules

· Q: an (acyclic) quiver

· DTQA of Q is $\Lambda = kQ \otimes R_2$

How to describe Λ via bound quiver?

· Q^{dbl}: 2 copies of Q, i.e., $Q \sqcup Q'$.
· $Q^{\#}$: constructed from Q^{dbl} by adding 2 opposite arrows between twin vertices
$I^{\#}$: some quadratical relations

· $\Lambda \cong kQ^{\#}/I^{\#}$

Example (1) $Q: 1 \xrightarrow{\alpha} 2$

$$Q^{\#}: \begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, shift right, "\varepsilon_1"'] & 2 \arrow[d, shift right, "\varepsilon_2"'] \\ 1' \arrow[r, "\alpha'"'] \arrow[u, shift right, "\varepsilon_1'"'] & 2' \arrow[u, shift right, "\varepsilon_2'"'] \end{tikzcd} \qquad I^{\#}: \begin{array}{l} \alpha'\varepsilon_1 = \varepsilon_2\alpha \\ \alpha\varepsilon_1' = \varepsilon_2'\alpha' \\ \varepsilon_i\varepsilon_i' = 0 = \varepsilon_i'\varepsilon_i \end{array}$$

(2) $Q: 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$

$$Q^{\#}: \begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, shift right, "\varepsilon_1"'] & 2 \arrow[d, shift right, "\varepsilon_2"'] & 3 \arrow[l, "\beta"'] \arrow[d, shift right, "\varepsilon_3"'] \\ 1' \arrow[r, "\alpha'"'] \arrow[u, shift right, "\varepsilon_1'"'] & 2' \arrow[u, shift right, "\varepsilon_2'"'] & 3' \arrow[l, "\beta'"] \arrow[u, shift right, "\varepsilon_3'"'] \end{tikzcd}$$

§1.2 Gorenstein algebras

· A is called to be Gorenstein if $\mathrm{inj.dim}_A A < \infty$ & $\mathrm{inj.dim}\, A_A < \infty$

[Zaks Lemma] If A is Gorenstein, then $\mathrm{inj.dim}_A A = \mathrm{inj.dim}\, A_A$

$$\mathrm{inj.dim}_A A = \mathrm{inj\,dim}\, A_A$$

If this common value $\leq d$, we call A to be d-Gorenstein.

Example. (1) Λ is 1-Gorenstein

(2) 1-Gorenstein algebras $\supseteq$ hereditary algebras, cluster-tilted alg, self-injective algebras, etc.

§1.3 Modules with finite proj dim (Cartan Part)

- $\mathrm{Proj}^{<\infty}(A)$: modules with finite proj dim
- $\mathrm{Proj}^{\leq m}(A)$: modules with proj.dim $\leq m$.

Similarly, $\mathrm{Inj}^{<\infty}(A)$, $\mathrm{Inj}^{\leq m}(A)$

- For a d-Gorenstein alg A,

$$\mathrm{Proj}^{<\infty}(A) = \mathrm{Proj}^{\leq d}(A) = \mathrm{Inj}^{\leq d}(A) = \mathrm{Inj}^{<\infty}(A)$$

$$\mathrm{Proj}^{<\infty}(\Lambda) = \mathrm{Proj}^{\leq 1}(\Lambda) = \mathrm{Inj}^{\leq 1}(\Lambda) = \mathrm{Inj}^{<\infty}(\Lambda)$$

- $\Lambda = kQ^{\#}/I^{\#}$, $\forall\, i \in Q_0$, define

$$\mathbb{H}_i := k\left(i \underset{\varepsilon_i'}{\overset{\varepsilon_i}{\rightleftarrows}} i' \right) / (\varepsilon_i \varepsilon_i', \varepsilon_i' \varepsilon_i)$$

Example:
$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, shift right, "\varepsilon_1"'] & 2 \arrow[d, shift right, "\varepsilon_2"'] \\ \cdot \arrow[u, shift right, "\varepsilon_1'"'] & \cdot \arrow[u, shift right, "\varepsilon_2'"'] \end{tikzcd}$$

Example:

$$\begin{tikzcd} \cdot \arrow[d, shift right, "\varepsilon_1"'] & \cdot \arrow[d, shift right, "\varepsilon_2"'] \\ 1' \arrow[u, shift right, "\varepsilon_1'"'] \arrow[r, "\alpha'"] & 2' \arrow[u, shift right, "\varepsilon_2'"'] \end{tikzcd}$$

$\mathbb{H}_1$ $\qquad$ $\mathbb{H}_2$

- $\mathbb{H} := \coprod_{i \in Q_0} \mathbb{H}_i \cong \overbrace{R_2 \times \cdots \times R_2}^{\# Q_0}$

For $\mathbb{H}_i$, define $\mathbb{H}_i$-modules

$$\mathbb{K}_i := k \underset{0}{\overset{1}{\rightleftarrows}} k \quad (2\text{-dimensional})$$

$$\mathbb{K}_i' := k \underset{1}{\overset{0}{\rightleftarrows}} k$$

viewed as Λ-modules

Then $\mathbb{K}_i, \mathbb{K}_i' \in \operatorname{Proj}^{\leq 1}(\Lambda) = \operatorname{Inj}^{\leq 1}(\Lambda)$

$K_0(\operatorname{Proj}^{\leq 1}(\Lambda))$: Grothendieck group of $\operatorname{Proj}^{\leq 1}(\Lambda)$

free abelian group generated by $\mathbb{K}_i, \mathbb{K}_i'$.

- $\operatorname{res}_{\mathbb{H}} : \operatorname{mod}(\Lambda) \longrightarrow \operatorname{mod}(\mathbb{H})$ restriction

- $\operatorname{Proj}^{\leq 1}(\Lambda) = \{ K \mid \operatorname{res}_{\mathbb{H}}(K) \in \operatorname{Proj} \mathbb{H} \}$

§1.4 Euler form

- $\langle -, - \rangle_Q$: Euler form of $\operatorname{mod} kQ$

$$\langle L, M \rangle_Q = \dim \operatorname{Hom}(L, M) - \dim \operatorname{Ext}^1(L, M).$$

$$\langle S_i, S_j \rangle_Q = \delta_{ij} - \#\{ i \longrightarrow j \}.$$

- $\langle -, - \rangle_\Lambda$: defined for $M, N \in \operatorname{mod}(\Lambda)$ with one of them in $\operatorname{Proj}^{\leq 1}(\Lambda) = \operatorname{Inj}^{\leq 1}(\Lambda)$.

$$\langle L, M \rangle_\Lambda = \dim \operatorname{Hom}_\Lambda(L, M) - \dim \operatorname{Ext}^1_\Lambda(L, M).$$

$\langle L, M\rangle_\Lambda = \dim \mathrm{Hom}_\Lambda(L,M) - \dim \mathrm{Ext}^1_\Lambda(L,M)$.

· res : $\mathrm{mod}(\Lambda) \longrightarrow \mathrm{mod}(kQ^{dbl}) = \mathrm{mod}\, kQ \sqcup Q'$

· $\langle \mathbb{K}_i, M\rangle_\Lambda = \langle S_i, \mathrm{res} M\rangle_{Q\sqcup Q'}$

· $\langle M, \mathbb{K}_i\rangle_\Lambda = \langle \mathrm{res} M, S_{i'}\rangle_{Q\sqcup Q'}$

· If $U, V \in \mathrm{Proj}^{\leq 1}(\Lambda)$.

$\langle U, V\rangle_\Lambda = \frac{1}{2}\langle \mathrm{res}\, U, \mathrm{res}\, V\rangle_{Q\sqcup Q'}$.

§2 Semi-derived Hall algebras

§2.1 Hall algebras

· A : finite dim algebra over $k = \mathbb{F}_q$.

· $v = \sqrt{q}$

· $\mathrm{Iso}(\mathrm{mod}(A)) = \{[M] \mid \text{iso classes of } \mathrm{mod}(A)\}$

· $H(A)$: (Ringel-) Hall algebra of $\mathrm{mod}(A)$.

$$H(A) = \bigoplus_{[M]} \mathbb{Q}(v)[M].$$

$$[L] \diamond [M] = \sum_{[N]} \frac{|\mathrm{Ext}^1(L,M)_N|}{|\mathrm{Hom}(L,M)|}[N]$$

where $\mathrm{Ext}^1(L,M)_N \subseteq \mathrm{Ext}^1(L,M)$ formed by

$0 \to M \to N \to L \to 0$.

[Ringel 90'] $H(A)$ is an associative

alg with unit $[0]$

[Kinger 90] $H(A)$ is an associative alg with unit $[0]$.

- Hall number $F_{LM}^{N} := |\{X \subseteq N \mid X \cong M, N/X \cong L\}|$.

Riedtmann – Peng Lemma

$$F_{LM}^{N} = \frac{|\mathrm{Ext}^1(L,M)_N|}{|\mathrm{Hom}(L,M)|} \frac{|\mathrm{Aut}(N)|}{|\mathrm{Aut}(L)|\cdot|\mathrm{Aut}(M)|}$$

- Denote $[[M]] := \frac{[M]}{|\mathrm{Aut}(M)|}$.

$$[[L]] \diamond [[M]] = \sum_{[[N]]} F_{LM}^{N} [[N]].$$

§2.2 Semi-derived Hall algebras

[L.–Peng 16', L.–Wang 19']

- A: 1–Gorenstein algebra

$$\mathrm{Proj}^{\leq 1}(A) = \mathrm{Inj}^{\leq 1}(A).$$

In $H(A)$, define

$$I_A := \left([M] - [K \oplus L] \;\middle|\; \exists\, 0 \to K \to M \to L \to 0,\ K \in \mathrm{Proj}^{\leq 1}(A) \right).$$

two sided ideal of $H(A)$.

- Semi-derived Hall alg

$$SDH(A) := \left(H(A)/I_A\right)\left[[K]^{-1} \mid K \in \mathrm{Proj}^{\leq 1}(A)\right].$$

- In $SDH(A)$, $\forall\, K \in \mathrm{Proj}^{\leq 1}(A)$, $M \in \mathrm{mod}(A)$.

$$[M] \diamond [K] = \sum_{[\cdot]} \frac{|\mathrm{Ext}^1(M,K)_N|}{|\cdots|} [N]$$

$$[M] \diamond [K] = \sum_{[N]} \frac{|\mathrm{Ext}^1(M,K)_N|}{|\mathrm{Hom}(M,K)|} [N]$$

$0 \to K \to N \to M \to 0$

$$= \frac{|\mathrm{Ext}^1(M,K)_N|}{|\mathrm{Hom}(M,K)|} [K \oplus M]$$

$$= q^{-\langle M,K\rangle_\Lambda} [K \oplus M].$$

$$[K] \diamond [M] = q^{-\langle K,M\rangle_\Lambda} [K \oplus M]$$

§2.3 Twisted SDH for Λ.

- $\Lambda = kQ^\# / I^\#$.

- $\widetilde{SDH}(\Lambda)$:

$$[L] * [M] = v^{\langle \mathrm{res} L, \mathrm{res} M\rangle_{Q \sqcup Q'}} [L] \diamond [M]$$

$$= v^{\langle \mathrm{res} L, \mathrm{res} M\rangle_{Q \sqcup Q'}} \sum_{[N]} \frac{|\mathrm{Ext}^1(L,M)_N|}{|\mathrm{Hom}(L,M)|} [N]$$

- quantum torus $\tilde{T}(\Lambda)$: subalg of $\widetilde{SDH}(\Lambda)$ generated by $[K]^{\pm 1}$, $K \in \mathrm{proj}^{\leq 1}(\Lambda)$.

- $\tilde{T}(\Lambda)$ is the group alg of $K_0(\mathrm{Proj}^{\leq 1}(\Lambda))$

$$\tilde{T}(\Lambda) \cong \mathbb{Q}(v)[[K_i]^{\pm 1}, [K_i']^{\pm 1}]$$

(Cartan part)

Proof. $[K] * [K']$

$$= v^{\langle \mathrm{res} K, \mathrm{res} K'\rangle_{Q \sqcup Q'}} [K] \diamond [K']$$

$$= v^{\langle \mathrm{res} K, \mathrm{res} K'\rangle_{Q \sqcup Q'}} q^{-\langle K,K'\rangle_\Lambda} [K \oplus K']$$

$$= v^{\langle \operatorname{res} k, \operatorname{res} k' \rangle_{Q \sqcup Q'}} q^{-\langle k, k' \rangle_\Lambda} [k \oplus k']$$

$$= v^{\langle \operatorname{res} k, \operatorname{res} k' \rangle_{Q \sqcup Q'}} q^{-\frac{1}{2}\langle \operatorname{res} k, \operatorname{res} k' \rangle_{Q \sqcup Q'}} [k \oplus k']$$

$$\overset{v = \sqrt{q}}{=} [k \oplus k']$$

· Hall basis $SD\tilde{H}(\Lambda)$ is a free (left) $\tilde{T}(\Lambda)$-module with a basis $[M]$,

$[M] \in \operatorname{mod} kQ \sqcup Q' = \operatorname{mod} kQ \times \operatorname{mod} kQ'$

$\subseteq \operatorname{mod}(\Lambda)$.

· $\tilde{H}(kQ)$: twisted Hall alg of $\operatorname{mod}(kQ)$

· $\operatorname{mod} kQ$ and $\operatorname{mod} kQ'$ are full subcat of $\operatorname{mod}(\Lambda)$ closed under extensions.

· $\tilde{H}(kQ) \longrightarrow SD\tilde{H}(\Lambda)$

$\quad [M] \longmapsto [M]$

e.g. $Q: 1 \xrightarrow{\alpha} 2$ $\qquad Q^\#$:

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, shift right, "\varepsilon_1"'] & 2 \arrow[d, shift right, "\varepsilon_2"'] \\ 1' \arrow[r, "\alpha'"'] \arrow[u, shift right, "\varepsilon_1'"'] & 2' \arrow[u, shift right, "\varepsilon_2'"'] \end{tikzcd}$$

· Triangular decomposition

$$SD\tilde{H}(\Lambda) \cong \tilde{H}(kQ) \otimes \tilde{T}(\Lambda) \otimes \tilde{H}(kQ').$$

§3. Realising the whole QGs

§3.1 quantum groups.

Q: a quiver

- Symmetric Euler form

$$(L,M)_Q := \langle L,M\rangle_Q + \langle M,L\rangle_Q$$

- Cartan matrix $C = (c_{ij})$

- $c_{ij} = (S_i, S_j)_Q$

EX: $1 \xrightarrow{\alpha} 2$ $\qquad C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

- Drinfeld double QG $\tilde{U}$ is a $\mathbb{Q}(v)$-algebra generated by E_i, F_i, K_i, K_i' $(i \in Q_0)$, such that

(QG1) K_i, K_i' invertible,

$$[K_i, K_j] = 0 = [K_i, K_j'] = [K_i', K_j']$$

(QG2)

$$K_i E_j = v^{c_{ij}} E_j K_i$$

$$K_i F_j = v^{-c_{ij}} F_j K_i$$

$$K_i' E_j = v^{-c_{ij}} E_j K_i'$$

$$K_i' F_j = v^{c_{ij}} F_j K_i'$$

(QG3) $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i'}{v - v^{-1}}$.

(QG4) Serre relations:

(QG4) Serre relations:

$$\sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix} E_i^r E_j E_i^{1-c_{ij}-r} = 0,$$

$$\sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix} F_i^r F_j F_i^{1-c_{ij}-r} = 0.$$

§ 3.2 Realisation

[Bridgeland's Thm reformulated]

[L.–Peng 16', L.–Wang 19']

$\exists$ inj morphism

$$\Psi : \tilde{U} \longrightarrow S\tilde{D}\mathcal{H}(\Lambda)$$

$$E_i \longmapsto \frac{v}{q-1}[S_i']$$

$$F_i \longmapsto -\frac{1}{q-1}[S_i]$$

$$K_i \longmapsto [\mathbb{K}_i]$$

$$K_i' \longmapsto [\mathbb{K}_i']$$

If Q is of Dynkin type, then Ψ is an isomorphism.

Proof. (QG1) follows by

$$\tilde{T}(\Lambda) \cong \mathbb{Q}(v)[[\mathbb{K}_i]^{\pm 1}, [\mathbb{K}_i']^{\pm 1}]$$

(QG2). $[\mathbb{K}_i] * [S_j] = v^{\langle S_i \oplus S_i', S_j \rangle_{Q \sqcup Q'}}$

$$q^{-\langle \mathbb{K}_i, S_j \rangle_\Lambda} [\mathbb{K}_i \oplus S_j]$$

$$= v^{\langle S_i, S_j \rangle_Q} q^{-\langle S_i, S_j \rangle_Q}$$

$$[\mathbb{K}_i \oplus S_j]$$

$$= v^{-\langle S_i, S_j\rangle_Q} [\mathbb{K}_i \oplus S_j]$$

$$[S_j] * [\mathbb{K}_i] = v^{\langle S_j,\, S_i \oplus S_i' \rangle_{Q \sqcup Q'}} \, q^{-\langle S_j, \mathbb{K}_i\rangle_{\chi}} [\mathbb{K}_i \oplus S_j]$$

$$= v^{\langle S_j, S_i\rangle_Q} \, q^{-\langle S_j, S_i'\rangle_{Q \sqcup Q'}} [\mathbb{K}_i \oplus S_j]$$

$$= v^{\langle S_j, S_i\rangle_Q} [\mathbb{K}_i \oplus S_j]$$

$$[\mathbb{K}_i] * [S_j] = v^{-\langle S_i, S_j\rangle_Q - \langle S_j, S_i\rangle_Q} [S_j] * [\mathbb{K}_i]$$

$$= v^{-c_{ij}} [S_j] * [\mathbb{K}_i]$$

(QG3). $[S_i] * [S_j'] = [S_i \oplus S_j']$

$= [S_j'] * [S_i]$, $\forall\, i \neq j$

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[d, shift right, "\varepsilon_1"'] & 2 \arrow[d, shift right, "\varepsilon_2"'] \\ 1 \arrow[r, "\alpha'"'] \arrow[u, shift right, "\varepsilon_1'"'] & 2 \arrow[u, shift right, "\varepsilon_2'"'] \end{tikzcd}$$

$$[S_i] * [S_i'] = [S_i \oplus S_i'] + (q-1)[\mathbb{K}_i]$$

$$[S_i'] * [S_i] = [S_i \oplus S_i'] + (q-1)[\mathbb{K}_i']$$

$$[[S_i], [S_j']] = \delta_{ij}(q-1)\big([\mathbb{K}_i] - [\mathbb{K}_i']\big)$$

(QG4) Serre relations follow from

Ringel – Green thm (Wang, Lecture 1)

$$\tilde{H}(kQ) \longrightarrow SD\tilde{H}(\Lambda).$$

e.g. $1 \xrightarrow{\alpha} 2$ $\qquad P_1 = k \xrightarrow{1} k$

$$[S_1] * [S_2] = v^{\langle S_1, S_2\rangle_Q}\big([S_1 \oplus S_2] + (q-1)[P_1]\big)$$

$$[S_1]*[S_2] = v^{\langle S_1, S_2\rangle_Q}\big([S_1\oplus S_2] + (q-1)[P_1]\big)$$

$$= v^{-1}\big([S_1\oplus S_2] + (q-1)[P_1]\big)$$

$$[S_2]*[S_1] = [S_1\oplus S_2].$$

$$[S_1]*[S_2] - v^{-1}[S_2]*[S_1] = v^{-1}(q-1)[P_1]$$

$$[S_1]*[P_1] = v^{\langle S_1, P_1\rangle_Q}[S_1\oplus P_1]$$

$$= [S_1\oplus P_1]$$

$$[P_1]*[S_1] = v^{\langle P_1, S_1\rangle_Q}\frac{1}{|\mathrm{Hom}(P_1,S_1)|}[S_1\oplus P_1]$$

$$= v^{-1}[P_1\oplus S_1]$$

$$[S_1]*[S_1]*[S_2] - (v+v^{-1})[S_1]*[S_2]*[S_1]$$
$$+[S_2]*[S_1]*[S_1]$$
$$= [S_1]*\big([S_1]*[S_2] - v^{-1}[S_2]*[S_1]\big)$$
$$- v\big([S_1]*[S_2] - v^{-1}[S_2]*[S_1]\big)*[S_1]$$
$$= [S_1]*v^{-1}(q-1)[P_1] - (q-1)[P_1]*[S_1]$$

$$= 0.$$

Outline of the proof of injectivity

$$\begin{tikzcd}
\tilde{\mathcal{U}} \arrow[r, "\Phi"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{SD}\tilde{\mathcal{H}}(\Lambda) \arrow[d, "\wr"] \\
\tilde{\mathcal{U}}^+ \otimes \tilde{\mathcal{U}}^0 \otimes \tilde{\mathcal{U}}^- \arrow[r, "R"'] & \tilde{\mathcal{H}}(kQ) \otimes \tilde{\mathcal{T}}(\Lambda) \otimes \tilde{\mathcal{H}}(kQ')
\end{tikzcd}$$

R is injective by Ringel – Green.

R is injective by Ringel – Green.