iQG and Hall algebras

Lecture 3

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Lecture 3 iQuantum Groups:
i-divided powers, Serre relations, and braid group symmetries

"8" = 1/9/21

①

Quantum Groups admits various remarkable structures & applications
[Drinfeld, Jimbo, Lusztig, ...]
Such as
(quasi) R matrix, canonical basis,
Hall algebra, categorification, braid group action

i-Quantum groups are generalizations of QG arising from quantum symmetric pairs [G. Letzter ~'99]

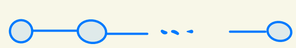
I. Symmetric pairs (SP) $(\mathfrak{g}, \mathfrak{g}^\theta)$, $\mathfrak{g}$: semisimple $\circlearrowright$ θ: involution

classification of (irred.) SP $\longleftrightarrow$ Satake diagrams
$\updownarrow$ real forms
$(I = I_\circ \sqcup I_\bullet, \tau)$ + conditions
└ diagram involution ($\tau = id$ is allowed)

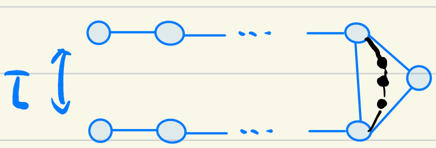
quasi-split if $I_\bullet = \phi$
split if $I_\bullet = \phi$ and $\tau = id$. [Satake = Dynkin]

Example (1) (split SP) $(\mathfrak{g}, \mathfrak{g}^\omega)$, where $\mathfrak{g} \circlearrowright \omega$ is Chevalley involution

e.g. $(sl_n, (sl_n)^{-tr=\omega} = so_n)$, $so_n = \langle f_i + e_i \mid i \in I \rangle$



$\pm e_i \leftrightarrow f_i$, $h_i \leftrightarrow -h_i$

(2) 

$sl_n \supseteq sl_n^{\omega\tau}$
$sl_n^{\omega\tau} \cong sl_p \oplus gl_q$ $(p+q=n)$ $\longleftrightarrow$ real forms $SU(p,q)$
$SU(p,q) \supseteq S(U_p \oplus U_q)$
quasi-split: $p = q, q+1$

(3) (SP of diagonal type) $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}^{diag})$

Will present a quantization of $(\mathfrak{g}, \mathfrak{g}^\theta) \rightsquigarrow (\mathcal{U}_v(\mathfrak{g}), \mathcal{U}'_v(\mathfrak{g}^\theta))$

$\overset{\parallel}{\text{D-J}}$ $\overset{\#}{\text{D-J}}$; compare Ex (1)

II iQuantum Groups (iQG) $\widetilde{\mathcal{U}}^\imath, \mathcal{U}^\imath$

$$\widetilde{\mathcal{U}} = \langle E_i, F_i, K_i, K_i' \rangle_{i \in I} \quad \text{Drinfeld double, e.g., } [E_i, F_i] = \frac{K_i - K_i'}{q_i - q_i^{-1}}$$

$\mathbin{\rotatebox{90}{$\subseteq$}}$ [Letzter] [Lu-W]

$$\widetilde{\mathcal{U}}^\imath = \langle B_i, \Bbbk_i := K_i K'_{\tau i} \ (i \in I_\circ), \mathcal{U}_{I_\bullet} \rangle$$

Here $B_i := F_i + T_{w_\bullet}(E_{\tau i}) K_i' \xrightarrow[\text{q.split}]{T_{w_\bullet} = 1} B_i = F_i + E_{\tau i} K_i'$

Fact $\mathcal{U}^\imath$ is a Coideal subalgebra, $\Delta : \widetilde{\mathcal{U}}^\imath \longrightarrow \mathcal{U}^\imath \otimes \widetilde{\mathcal{U}}$

From now on, assume $\widetilde{\mathcal{U}}^\imath$ is quasi-split

$$\widetilde{\mathcal{U}}^\imath = \langle B_i, \Bbbk_i \rangle_{i \in I}$$

Note: $\Bbbk_i \ (i = \tau i)$, & $\Bbbk_i \Bbbk_{\tau i} \ (i \neq \tau i)$ are central in $\widetilde{\mathcal{U}}^\imath$ ☺ $K_i K_i'$ central in $\widetilde{\mathcal{U}}$.

Letzter's iQG $\mathcal{U}^\imath = \mathcal{U}^\imath_\varsigma$, depending on parameters $\varsigma = (\varsigma_i)_{i \in I}$, $\varsigma_i \in \mathbb{Q}(v)$

is a coideal subalgebra of $\mathcal{U}$, e.g. $B_i = F_i + \varsigma_i E_{\tau i} K_i^{-1}$

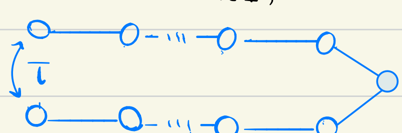
$(\mathcal{U}, \mathcal{U}^\imath)$ forms a quantum symmetric pair

$\mathcal{U}^\imath_\varsigma$ is obtained from $\widetilde{\mathcal{U}}^\imath$ via a central reduction

$$\mathcal{U}^\imath_\varsigma = \widetilde{\mathcal{U}}^\imath \Big/ \underset{\text{ideal}}{\langle \Bbbk_i - \varsigma_i \ (i = \tau i), \ \Bbbk_i \Bbbk_{\tau i} - \varsigma_i \varsigma_{\tau i} \ (i \neq \tau i) \rangle}$$

Shall focus on $\widetilde{\mathcal{U}}^\imath$

Example (1) (split iQG) $\widetilde{\mathcal{U}}^\imath = \langle B_i, \Bbbk_i \rangle_{i \in I}$, with $\Bbbk_i$ central ; $\mathcal{U}^\imath = \langle B_i \rangle_{i \in I}$.

(2) (quasi-split iQG)  (Type AIII)

(3) (QG $\subseteq$ iQG) $(\omega \otimes 1) \circ \Delta : \widetilde{\mathcal{U}} \hookrightarrow \widetilde{\mathcal{U}} \otimes \widetilde{\mathcal{U}}$ Coideal subalg

$\Rightarrow (\widetilde{\mathcal{U}} \otimes \widetilde{\mathcal{U}}, \widetilde{\mathcal{U}})$ is QSP assoc. to $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}^{\text{diag}})$

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III $\boxed{\text{i-Divided powers (iDP)}}$ [Bao-W, Berman-W, Lu-W]

Recall $\boxed{F_i^{(m)} = F_i^m / [m]_i!}$

$\tilde{U}^i$ iCanonical basis U^i ($vk_i = 1$)

$$\tilde{U}^i(sl_2) = \mathbb{Q}(v)[k_i][B_i], \text{ fix } i \text{ — a polynomial algebra.}$$

Define $B_{i,\bar{p}}^{(m)} \in$ ($\bar{p} \in \mathbb{Z}/2\mathbb{Z}$) iDP

∇. The monomial basis $\{B_i^{m}\}_{m \geq 0}$ is not "simple"

- $B_i^2/[2]_i$ $\cancel{\circlearrowleft}$ $L(2\lambda+1)_{\mathbb{Z}[v,v^{-1}]}$, $\lambda \in \mathbb{Z}_{\geq 0}$ $\circlearrowleft$ $\frac{B_i^2 - 1}{[2]_i}$ $(vk_i = 1)$

$$B_{i,\bar{1}}^{(m)} := \frac{1}{[m]_i!} \begin{cases} \prod_{j=1}^{\ell} \left(B_i^2 - vk_i [2j-1]_i^2\right) & \text{if } m = 2\ell \\ B_i \prod_{j=1}^{\ell} \left(B_i^2 - vk_i [2j-1]_i^2\right) & \text{if } m = 2\ell+1 \end{cases}$$

$$B_{i,\bar{0}}^{(m)} := \frac{1}{[m]_i!} \begin{cases} B_i \prod_{j=1}^{\ell} \left(B_i^2 - vk_i [2j]_i^2\right) & \text{if } m = 2\ell+1 \\ B_i^2 \prod_{j=1}^{\ell} \left(B_i^2 - vk_i [2j]_i^2\right) & \text{if } m = 2\ell+2 \end{cases}$$

or

$$B_i B_{i,\bar{p}}^{(r)} = \begin{cases} [r+1]_i B_{i,\bar{p}}^{(r+1)} & \text{if } \bar{p} \neq \bar{r} \\ [r+1]_i B_{i,\bar{p}}^{(r+1)} + k_i [r]_i B_{i,\bar{p}}^{(r-1)} & \text{if } \bar{p} = \bar{r} \end{cases}$$

$(\bar{p} \in \{\bar{0}, \bar{1}\})$

e.g. $B_{i,\bar{p}}^{(1)} = B_i$, $\quad B_{i,\bar{0}}^{(2)} = \frac{B_i^2}{[2]_i}$, $\quad B_{i,\bar{1}}^{(2)} = \frac{B_i^2 - vk_i}{[2]_i}$

PBW expansion of iDP [Berman-W]

$$U^i \hookrightarrow U \curvearrowright \dot{U} = \bigoplus_{\lambda \in \mathbb{Z}} U \mathbb{1}_\lambda$$

modified QG

$B_{i,\bar{0}}^{(m)}$ depends on parity of m, e.g.,

$$B_{i,\bar{0}}^{(2m)} \mathbb{1}_{2\lambda} = \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} v^{2(a+c)(m-a-\lambda)-2ac-\binom{2c+1}{2}} \begin{bmatrix} m-c-a-\lambda \\ c \end{bmatrix}_{v^2} E_i^{(a)} F_i^{(2m-2c-a)} \mathbb{1}_{2\lambda}$$

$\in$ integral form $\dot{U}_{\mathbb{Z}[v,v^{-1}]}$

IV. Serre presentation of $\tilde{\mathcal{U}}^\imath$

Fact [Letzter]

$\exists$ filtration on $\tilde{\mathcal{U}}^\imath$ ($\simeq$ informally, view F_i as leading term of $\underset{\substack{\| \\ F_i + \tilde{E}_{\tau i} K_i'}}{B_i}$)

s.t. $\mathrm{gr}^F \tilde{\mathcal{U}}^\imath \cong \mathcal{U}^- = \langle F_i \rangle_{i \in I}$ over $\tilde{\mathcal{U}}^{\imath,0} := \langle k_i \mid i \in I \rangle$

$\Downarrow$

Expect relations among B_i's are deformation of Serre relations in $\mathcal{U}^-$

Recall q-Serre
$$\sum_{r=0}^{1-c} (-1)^r \begin{bmatrix} 1-c \\ r \end{bmatrix} F_i^{1-c-r} F_j F_i^r = 0 \qquad \text{Here } c = c_{ij} \ (i \neq j) :$$

or
$$\sum_{r=0}^{1-c} (-1)^r F_i^{(1-c-r)} F_j F_i^{(r)} = 0. \qquad F_i^{(r)} = \frac{F_i^r}{[r]!} \ \text{divided powers}$$

Shall fix $i \in I$ such that $\tau i = i$

Example [Letzter] on $\mathcal{U}^\imath \rightsquigarrow \tilde{\mathcal{U}}^\imath$

(1) $B_i^2 B_j - [2]_i B_i B_j B_i + B_j B_i^2 = v k_i B_j$ if $c_{ij} = -1$

(2) $B_i^3 B_j - [3]_i B_i^2 B_j B_i + [3]_i B_i B_j B_i^2 - B_j B_i^3 = [2]_i^2 v_i k_i (B_i B_j - B_j B_i)$, if $c_{ij} = -2$

(3) skipped $c_{ij} = -3$? $c_{ij} \leq -4$

Reformulation via iDP

Example (1) $c_{ij} = -1 \iff B_{i,\overline{p+1}}^{(2)} B_j - B_i B_j B_i + B_j B_{i,\overline{p}}^{(2)} = 0$ "classical looking!"

(2) $c_{ij} = -2 \iff B_{i,\overline{p}}^{(3)} B_j - B_i^2 B_j B_i + B_i B_j B_{i,\overline{p}}^{(2)} - B_j B_{i,\overline{p}}^{(3)} = 0$

$\vdots$

Theorem 1 [陈新红-卢明-ww, CLW18]

For (quasi-split) iQG $\widetilde{\mathbf{U}}^\imath$ (or $\mathbf{U}^\imath$) of $\forall$ Kac-Moody type, $\quad \forall j \neq i = \tau i$

i-Serre:
$$\boxed{\sum_{r+s=1-a_{ij}} (-1)^r B^{(r)}_{i,\overline{p+a_{ij}}} B_j B^{(s)}_{i,\overline{p}} = 0}$$

Proof uses PBW expansion formula of iDP + new q-binomial identities

This leads to a Serre presentation for (quasi-)split iQG of $\forall$ KM type.

Theorem 2 [CLW18] (generalization of [Letzter, Kolb, Balagovic-Kolb])

The quasi-split iQG $\widetilde{\mathbf{U}}^\imath$ has a presentation:

- generators B_i, k_i $\quad (i \in \mathbb{I})$
- relations: $\quad$ for $\ell \in \mathbb{I}$, $i \neq j \in \mathbb{I}$,

- $k_i k_\ell = k_\ell k_i$
- $k_i B_\ell = v^{c_{\tau i,\ell} - c_{i,\ell}} B_\ell k_i$
- $\displaystyle\sum_{r=0}^{1-c_{ij}} (-1)^r B_i^{(r)} B_j B_i^{(1-c_{ij}-r)} = 0$ if $j \neq \tau i \neq i$; here $B_i^{(r)} = \dfrac{B_i^r}{[r]!}$ $(\tau i \neq i)$
- $$\sum_{r=0}^{1-c_{ij}} (-1)^{r+c_{i,\tau i}} B_i^{(r)} B_{\tau i} B_i^{(1-c_{ij}-r)} = \frac{1}{v_i - v_i^{-1}} \Big(v_i^{c_{i,\tau i}} (v_i^{-2}; v_i^{-2})_{-c_{i,\tau i}} B_i^{(-c_{i,\tau i})} k_i - (v_i^{2}; v_i^{2})_{-c_{i,\tau i}} B_i^{(-c_{i,\tau i})} k_{\tau i} \Big), \quad \text{if } \tau i \neq i \qquad (*)$$
- $\displaystyle\sum_{r=0}^{1-c_{ij}} (-1)^r B^{(r)}_{i,\overline{c_{ij}+p}} B_j B^{(1-c_{ij}-r)}_{i,\overline{p}} = 0.$

Here $(a;x)_n = (1-a)(1-ax)\cdots(1-ax^{n-1})$

Remark
- For $c_{i,\tau i} = 0$, (*) becomes $B_{\tau i} B_i - B_i B_{\tau i} = \dfrac{k_i - k_{\tau i}}{v - v^{-1}} \approx [E_i, F_i] = \cdots$
- For split ADE, only relations
$$\begin{cases} c_{ij} = -1: & B_i^2 B_j - [2] B_i B_j B_i + B_j B_i^2 = v k_i B_j \\ c_{ij} = 0: & B_i B_j - B_j B_i = 0 \\ \cdot\, k_i \text{ central} & \end{cases}$$
- quasi-split ADE, "similar"

V. $\boxed{\text{Lusztig-Serre relations}}$

Recall, in QG $\mathcal{U}$,

Serre relation: $\boxed{\sum_{r+s=1-a_{ij}} (-1)^r F_i^{(r)} F_j F_i^{(s)} = 0}$ $(i \neq j \in I)$

Lusztig-Serre (= higher order Serre)

Set $f^-_{i,j;n,m,e} = \sum_{r+s=m} (-1)^r v_i^{er(1-nc_{ij}-m)} F_i^{(r)} F_j^{(n)} F_i^{(s)}$

$(e = \pm 1,\ m \in \mathbb{Z},\ n \geq 0)$ $i \neq j$ fixed.

$\boxed{f^-_{n,m,e} = 0, \quad \text{for } m \geq 1 - n a_{ij}}$ (L-S)

(L-S) simplifies, for $m = 1 - nc_{ij}$:

L-S "minimal degree" $\boxed{\sum_{r+s=1-na_{ij}} (-1)^r F_i^{(r)} F_j^{n} F_i^{(s)} = 0}$ $\overset{n=1}{\rightsquigarrow}$ Serre

Proof of (L-S) is based on recursive relations:

$$\begin{cases} \left[E_i, f^-_{i,j;n,m,e}\right] = [nc_{ij}+m-1]\, K_{-ei}\, f^-_{i,j,n,m+1,e} \\ \left[f^-_{i,j;n,m,e}, F_i\right]_{q^*} = [m+1]\, f^-_{i,j,n,m+1,e} \end{cases}$$

Braid group symmetries $\exists$ $T_i \in \mathrm{Aut}(\mathcal{U}_i)$ such that ... +

$$\boxed{T_i(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v_i^r F_i^{(r)} F_j F_i^{(s)} \qquad (j \neq i)}$$

$$\| \quad f^-_{1,-c_{ij},1}$$

e.g. $T_i(F_j) = F_j F_i - v_i F_i F_j$, for $c_{ij} = -1$

Next goal: braid group symmetries of $\widetilde{\mathcal{U}}^\imath$ $\Leftarrow$ Lusztig-Serre for $\widetilde{\mathcal{U}}^\imath$

Theorem 3 [CLW'19] (Serre-Lusztig relations of minimal degree)

$$(\imath\text{-}LS_{min}) \qquad \sum_{r+s=1-\underset{n}{\wedge}c_{ij}} (-1)^r B^{(r)}_{i,\overline{p+c_{ij}}} B_j^{n} B^{(s)}_{i,\overline{p}} = 0 \qquad \text{for } \underset{\underset{j}{\#}}{i}=\tau i,\ n\geq 1$$

$\rightsquigarrow$ iDP ✓

Proof uses the PBW expansion formula for $B^{(m)}_{i,\overline{p}}$ and q-binomial identities

Next, remove "minimal degree" $\rightsquigarrow$ require $\imath$-analogue of $f_{i,j;n,m,e}$

Fact: $y_{n,m,e}$ has $f_{i,j;n,m,e}$ as leading term $\qquad \overset{!!}{y_{i,j;n,m,e,\overline{p}}}$

- (minimal degree) $y_{n,1-nc_{ij},e} =$ LHS $(\imath\text{-}LS_{min})$
- $\{y_{i,j;n,m,e}\}$ satisfy "hybrid" R.R.

$$[y_{n,m,e}, B_i]_{q^*} = [m+1]\, y_{n,m+1,e} - [m+nc_{ij}-1]\, q^* k_i\, y_{n,m+1,e}$$

Theorem 3b [CLW'19] (Serre-Lusztig relation)

$$(\imath\text{-}LS) \qquad y_{i,j;n,m,e,\overline{p}} = 0 \qquad \text{for } \underset{\underset{j}{\#}}{i}=\tau i,\ n\geq 1,\ m\geq 1-nc_{ij}$$

Definition below of $y_{n,m,p}$ depending on parity of $m-na_{ij}$ $\rightsquigarrow$ many examples

Definition of $y_{i,j;n,m,e,\overline{p}}$

For $m-na_{ij}$ even: $y_{i,j;n,m,e,\overline{p}}$ =

$$\sum_{u\geq 0}(q_ik_i)^u\left\{\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}+\overline{1}}}(-1)^r q_i^{-e((m+nc_{ij}-1)(r+u))}\begin{bmatrix}\frac{m+nc_{ij}}{2}\\ u\end{bmatrix}_{q_i^2} B_{i,\overline{p}}^{(r)}B_j^n B_{i,\overline{p+na_{ij}}}^{(s)}\right.$$

$$\left.+\sum_{\substack{r+s+2u=m\\ \overline{r}=\overline{p}}}(-1)^r q_i^{-e((m+nc_{ij}-1)(r+u))}\begin{bmatrix}\frac{m+nc_{ij}-2}{2}\\ u\end{bmatrix}_{q_i^2} B_{i,\overline{p}}^{(r)}B_j^n B_{i,\overline{p+na_{ij}}}^{(s)}\right\}.$$

For $m-na_{ij}$ ODD, use instead " $q_i^{-e((m+nc_{ij}-1)(r+u)\pm u)}$ " " $\begin{bmatrix}\frac{m+nc_{ij}-1}{2}\\ \end{bmatrix}$ "

The formula above greatly simplifies at $m=-nc_{ij}$: ☺ $\qquad \begin{bmatrix}0\\ u\end{bmatrix}=0,\ u>0 \qquad \begin{bmatrix}-1\\ u\end{bmatrix}=(-1)^u$

$$y_{i,j;n,-nc_{ij},e,\overline{p}}=\sum_{\substack{r+s=-nc_{ij}\\ \overline{r}=\overline{p}}}(-1)^r q_i^{er}B_{i,\overline{p}}^{(r)}B_j^n B_{i,\overline{p+na_{ij}}}^{(s)}$$

$$+\sum_{\substack{r+s+2u=-nc_{ij}\\ \overline{r}=\overline{p},\ u\geq 1}}(-1)^{r+u}q_i^{e(r+u)}(q_ik_i)^u B_{i,\overline{p}}^{(r)}B_j^n B_{i,\overline{p+na_{ij}}}^{(s)}$$

Conjecture [CLW19] For $i=\tau i$, $e=\pm 1$, $\exists\ T_{i,e}\in \mathrm{Aut}(\tilde{\mathcal{U}}^\imath)$ s.t.

$$\begin{cases} T_{i,e}(k_j)=(-q_i^{1+e}k_i)^{-c_{ij}}k_j \\ T_{i,e}(B_j)=\begin{cases} y_{i,j;1,-c_{ij},e,\overline{p}} & \text{if } j\neq i\\ (-q_i^{1+e}k_i)^{-1}B_i & j=i.\end{cases}\end{cases}$$

e.g. $c_{ij}=-1$

Remark It is more conceptual to talk about $T_{i,e}$ on $\tilde{\mathcal{U}}^\imath$ instead of $\mathcal{U}^\imath$

Theorem 4 [LW21] Conjecture holds (if $c_{i,\tau i}$ is even).

Proof uses iHall algebras & reflection functors

Next

Lecture 4 [Lu]

Hall algebra realization of iquantum groups (of ADE type)

Lecture 5 Hall algebra realization of iquantum groups (of Kac-Moody type)

\+ Discussions [further directions + open problems]