iQG and Hall algebras

Lecture 1

January 4, 2021

Sichuan University

iQuantum Groups and Hall algebras

Lecture 1

1/4/21

I. Overview

"Quantum Groups" $\rightsquigarrow$ "Hall algebras"

- Lecture 1 Quantum Groups and Ringel-Hall algebras

  – Quantum groups $U = U_v(g)$ $\leftsquigarrow$ "Dynkin" diagrams

  $U = U^- U^0 U^+$

  – Hall algebras [Steinitz, Ringel, Lusztig, Green, Xiao, Peng, ...]

  $H(kQ)$ $\xleftarrow{k=\mathbb{F}_q}$ quivers Q (= diagrams w/ arrows)

  [Ringel] $H(kQ) \simeq U^+|_{v=\sqrt{q}}$, for Q Dynkin

- Lecture 2 [Lu] Hall algebra realization of QG

  (reformulating [Bridgeland])

  $\rightsquigarrow$ semi-derived Hall algebra (SDH)

  "i-quiver algebras" associated to $(Q \sqcup Q', \text{swap})$

- Lecture 3 i-Quantum groups & quantum symmetric pairs

  – i-divided powers – Serre presentation – braid group symmetries

- Lecture 4 [Lu] i-Hall algebra realization of iQG $\widetilde{U}^i$

  $\rightsquigarrow$ SDH

  i-quiver algebras associated to i-quivers Q ↺ involution τ

- Lecture 5 TBA

Quantum Groups (QG)

Let C be a (generalized) Cartan matrix, $C = (c_{ij})_{i,j\in I}$,

- $c_{ii} = 2$ · $c_{ij} = c_{ji} \leq 0$ for $i \neq j$ symmetric

$C \rightsquigarrow$ semisimple/Kac-Moody Lie algebra $\mathfrak{g}$ & simple roots α_i ($i \in I$).

Quantum integers & quantum binomial coefficients. Let v be an indeterminate.

$$[r] = \frac{v^r - v^{-r}}{v - v^{-1}} \qquad [r]! = [r]\cdots[2][1], \qquad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m][m-1]\cdots[m-r+1]}{[r]!}$$

$[r] = [r]_v$, $r \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}$

Drinfeld double quantum group $\tilde{U} = \tilde{U}_v(\mathfrak{g})$ is a $\mathbb{Q}(v)$-algebra

- generators E_i, F_i, K_i, K_i' ($i \in I$)
- relations K_i, K_j' ($\forall i,j$) commute

$$K_i E_j = v^{c_{ij}} E_j K_i \qquad K_i F_j = v^{-c_{ij}} F_j K_i$$

$$K_i' E_j = v^{-c_{ij}} E_j K_i' \qquad K_i' F_j = v^{+c_{ij}} F_j K_i'$$

$$E_i F_j - F_j E_i = \begin{cases} 0 & i \neq j \\ \dfrac{K_i - K_i'}{v - v^{-1}} & i = j \end{cases}$$

(Serre)

$$\sum_{r+s=1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix} E_i^s E_j E_i^r = 0 \qquad (S_E)$$

for $i \neq j$.

$$\sum_{r+s=1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix} F_i^s F_j F_i^r = 0 \qquad (S_F)$$

Remark. $K_i K_i'$ is central ($\forall i \in I$)

(Drinfeld–Jimbo) Quantum group $\mathcal{U} = \mathcal{U}_v(\mathfrak{g}) = \langle E_i, F_i, K_i, K_i^{-1} \rangle_{i\in I}$

is obtained from $\tilde{\mathcal{U}}$ by a central reduction:

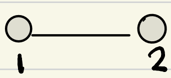
$$\tilde{\mathcal{U}} \Big/ \underbrace{\langle K_i K_i' - 1 \mid i \in I \rangle}_{\text{ideal}} \xrightarrow{\simeq} \mathcal{U}, \qquad K_i' \longmapsto K_i^{-1}, \quad E_i/F_i/K_i \longmapsto \text{Same notation}$$

Facts
- $\exists$ variats of $\mathcal{U}$ with different $\mathcal{U}^0$ ($\approx$ different Lie covering groups)
- $\mathcal{U}/\tilde{\mathcal{U}}$ is a Hopf algebra
- (triangular decomposition) $\mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+ \simeq \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+$

Here $\mathcal{U}^+ = \langle E_i, i \in I \mid \text{Serre rel } (S_E) \rangle$, $\mathcal{U}^+ = \mathcal{U}_v(\mathfrak{n}^+)$

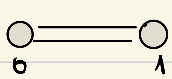
$\mathcal{U}^0 = \langle K_i^{\pm 1} \mid i \in I \rangle$, $\quad \mathcal{U}^- = \langle F_i \mid i \in I \rangle$

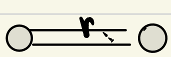
- $\mathcal{U}_v(\mathfrak{g}) \overset{v \mapsto 1}{\rightsquigarrow} \mathcal{U}(\mathfrak{g})$

Example 1 (A_2)  $\mathfrak{g} = \mathfrak{sl}_3 \supseteq \mathfrak{n}^+ = \langle e_1, e_2, e_{12} = [e_1 e_2] \rangle$

$$\mathfrak{n}^+ = \begin{bmatrix} 0 & * & * \\ & 0 & * \\ & & 0 \end{bmatrix}$$

$\mathcal{U}^+ = \mathcal{U}_v(\mathfrak{n}^+)$ has a PBW basis $E_1^a E_{12}^b E_2^c$ $(a,b,c \in \mathbb{Z}_{\geq 0})$, where $E_{12} = [E_1, E_2]_{v^{-1}}$

2 $(\hat{A}_1)$  affine Lie algebra $\mathfrak{g} = \hat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$

3.  $(r \geq 3)$ general Kac-Moody algebra

Cartan matrix $= \begin{bmatrix} 2 & -r \\ -r & 2 \end{bmatrix}$

$r=1$: positive definite

$r=2$: semi-definite

$r \geq 3$: indefinite

Quivers

A quiver Q consists of a pair $Q = (\underset{\text{vertex set}}{\overset{I}{\overset{\shortparallel}{Q_0}}}, \underset{\text{arrow set}}{Q_1})$

e.g. an arrow $h: i \longrightarrow j$

$s, t: Q_1 \to Q_0$ $\quad s(h) = i, \; t(h) = j$

Source / target
起/终点

e.g.
$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, shift left] \arrow[r, shift right] & \underset{2}{\circ} & \underset{3}{\circ} \arrow[l] \arrow[loop above] \end{tikzcd} \qquad I = \{1, 2, 3\}$$

A quiver is acyclic if there is no oriented cycles

"loop"

$\circ \rightleftarrows \circ$ $\quad$ ...

A quiver is called Dynkin if the underlying diagram $|Q|$ (with directions of arrows ignored) is ADE Dynkin diagram.

e.g. quivers with underlying A_3 Dynkin diagram $\circ - \circ - \circ$

$\circ \to \circ \to \circ \qquad \circ \to \circ \leftarrow \circ \qquad \circ \leftarrow \circ \to \circ \qquad \circ \leftarrow \circ \leftarrow \circ$

Take a ground field k. Will assume throughout: Q contains no loop.

A representation V of a quiver Q consists of
- a k-vector space V_i, $\forall i \in I$,
- a linear map $x_h: V_i \to V_j$, $\forall h \in Q_1$, $h: i \longrightarrow j$

V is nilpotent if $\exists N$ s.t. $x_{h_1} \cdots x_{h_n} = 0$ on V for all $h_1, \cdots, h_n$, $n > N$.

A morphism of representations $\underset{\overset{\shortparallel}{(f_i)}}{f}: V \longrightarrow W$ s.t.
$$\begin{tikzcd} V_i \arrow[r, "x_h"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowright"] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "x_h"] & W_j \end{tikzcd}$$

$\rightsquigarrow$ abelian category $\mathrm{rep}_k(Q)$ of f.-d. reps.

Path algebra kQ :

linear basis : paths in Q e.g. $\overset{1}{\circ} \to \overset{2}{\circ} \to \overset{3}{\circ}$

multiplication: concatenation

$\{ e_1, e_2, e_3,$ e_i = lazy paths at i = ⓘ   length 0

$1\to 2,\ 2\to 3,$ 1

$1 \to 2 \to 3 \}$ 2

$\dim kQ = 6$

Facts 1) Q acyclic $\Leftrightarrow$ kQ is finite-dimensional

2). $\mathrm{rep}^{\mathrm{nil}}_k(Q) \simeq \mathrm{mod}^{\mathrm{nil}}(kQ)$

$(V_i)_{i\in I} \longmapsto V = \bigoplus_{i\in I} V_i$

$(M_i)_{i\in I} \longleftarrow M$ $(M_i = e_i M)$

3) Simple modules are $S(i)$, $i \in I$. $S(i)_i \simeq k$ at vertex i, $S(i)_j = 0$, $j \neq i$.

Grothendieck group $[\mathrm{rep}_k(Q)] \cong \mathbb{Z}^I$, $[V] \longmapsto \underline{\dim} V := (\dim V_i)_{i\in I}$

$\Vert\wr$ "Lie th'y" $\downarrow$

root lattice $\mathbb{Z}I$ $\sum_i \dim V_i\, \alpha_i$

4) $\mathrm{rep}_k(Q)$ is hereditary, i.e., $\mathrm{Ext}^{i\geq 2}_Q = 0$. $\exists$ projective resolution $0 \to P' \to P \to V \to 0$

5) Euler form : $\forall V, W \in \mathrm{rep}_k(Q)$, $\underline{\dim} V = \vec{v}$, $\underline{\dim} W = \vec{w}$

$\langle\ ,\ \rangle : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$

$$\langle \vec{v}, \vec{w} \rangle = \langle V, W \rangle = \dim \mathrm{Hom}_Q(V,W) - \dim \mathrm{Ext}^1_Q(V,W)$$

$$= \sum_{i\in I} v_i w_i - \sum_{h\in Q_1} v_{s(h)} w_{t(h)}$$

Symmetric form $(\vec{v}, \vec{w}) := \langle \vec{v}, \vec{w}\rangle + \langle \vec{w}, \vec{v}\rangle$

quadratic form $q(\vec{v}) = \langle \vec{v}, \vec{v} \rangle$

Gabriel's Theorem

(1) A connected quiver Q is of "finite representation type" (i.e., $\exists$ finitely many indecomposables in $\mathrm{rep}_k Q$)

$\iff$ Q is Dynkin

(2) For Q Dynkin, $\leadsto \mathfrak{g}$, the indecomposables $\overset{1:1}{\longleftrightarrow}$ $\Phi^+ = \{\text{positive roots for } \mathfrak{g}\}$

$V = (V_i) \longmapsto \sum_{i\in I} \dim V_i \, \alpha_i$

Sketch. Q is Dynkin $\iff$ the quadratic form $q(\cdot)$ is positive definite

reflection functors $\Downarrow$ ⇑ Alg geom of orbits

(2) $\Longrightarrow$ Q finite rep type

Example

$$\underset{1}{0} \longrightarrow \underset{2}{0} \longrightarrow \underset{3}{0} \longrightarrow \cdots \longrightarrow \underset{n}{0}$$

[+ chang arrows

$\mathfrak{g} = \mathfrak{sl}_{n+1}$ $\quad \Phi^+ = \{\beta_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \le i \le j \le n\}$

$\updownarrow 1:1$

$$I_{ij}: 0 \to \cdots 0 \to \underset{i}{k} \longrightarrow \underset{i+1}{k} \longrightarrow \cdots \longrightarrow \underset{j}{k} \longrightarrow 0 \cdots \to 0$$

All indecomposables

$\leadsto$ Kac-Moody algebra $\mathfrak{g}$

Remark [Kac] For general quiver Q with no loops,

$\exists$ indecomposable nilpotent rep of Q of dimension vector $\vec{v} = (v_i)_{i\in I}$

$\iff \alpha = \sum_{i\in I} v_i \alpha_i \in \Phi^+$, positive roots for $\mathfrak{g}$

(recall $\Phi^+ = \Phi^+_{re} \sqcup \Phi^+_{im}$)

$\exists !$ indecomp of Q of dim $\vec{v}$ $\iff$ $\alpha = \sum v_i \alpha_i \in \Phi^+_{re}$

Let $\boxed{k = \mathbb{F}_q}$ and $\boxed{\mathbf{v} = \sqrt{q}}$

The (Ringel-) Hall algebra of kQ, $\mathcal{H}(kQ)$, has

- linear basis $[M]$, $M \in \mathrm{Iso}(kQ)$ iso-classes
- Multiplication:

$$[M] \diamond_R [N] = \mathbf{v}^{\langle M,N \rangle} \sum_{L \in \mathrm{Iso}(kQ)} F^L_{M,N} [L], \qquad (\diamond_R) \qquad \text{Ringel}$$

twisting by Euler form

where $\boxed{F^L_{M,N} := \left| \{ N' \subseteq L \mid N' \simeq N,\ L/N' \simeq M \} \right|}$

Recall

Riedtman-Peng formula $\quad F^L_{M,N} = \dfrac{|\mathrm{Ext}^1_Q(M,N)_L|}{|\mathrm{Hom}_Q(M,N)|} \cdot \dfrac{|\mathrm{Aut}(L)|}{|\mathrm{Aut}(M)| \cdot |\mathrm{Aut}(N)|}$

Here $\mathrm{Ext}^1(M,N)_L \subseteq \mathrm{Ext}^1(M,N)$ denotes the subset parametrizing the extensions with middle term $\cong L$.

A variant of multiplication in $\mathcal{H}(kQ)$ is

$$[M] \diamond_B [N] = \mathbf{v}^{\langle M,N \rangle} \sum_{L \in \mathrm{Iso}(kQ)} \frac{|\mathrm{Ext}^1_Q(M,N)_L|}{|\mathrm{Hom}_Q(M,N)|} [L]. \qquad (\diamond_B) \qquad \text{Bridgeland}$$

Remark (1) $(\mathcal{H}(kQ), \diamond_R)$, with or without Euler form twisting, is an associative algebra with unit $[0]$. ☹ bilinear!

(2) $(\mathcal{H}(kQ), \diamond_B) \cong (\mathcal{H}(kQ), \diamond_R)$

$[M]_B / |\mathrm{Aut}(M)| \longmapsto [M]$

(3) In $(\mathcal{H}(kQ), \diamond_R)$, $\exists$ a "Hopf pairing" $\{[M], [N]\} = \dfrac{\delta_{[M][N]}}{|\mathrm{Aut}(M)|}$.

So the map in (2) is "passing to dual basis" wrt $\{\cdot, \cdot\}$.

[PBW, canonical basis in $(\mathcal{H}(kQ), \diamond_R) \longleftrightarrow$ dual PBW, canonical basis in $(\mathcal{H}(kQ), \diamond_B)$

(4) Q: Dynkin. [Gabriel] $\Rightarrow$ $\mathcal{H}(kQ)$ has the size of $\mathcal{U}(\mathfrak{n}^+)$ or $\mathcal{U}^+$

Multiplication $\diamond = \diamond_R$ below $\qquad q = \underline{v}^2$

Example (rank 1) $\qquad \circ \qquad$ Simple module S $\qquad$ Write $[nS] = [S^{\oplus n}]$

$$\mathrm{Hom}_Q(S,S) = k, \qquad \mathrm{Ext}^1_Q(S,S) = 0, \qquad \langle S,S\rangle = 1$$

$$[(r-1)S]\diamond_R[S] = \underline{v}^{r-1}\#Gr(1,r)\,[rS] \qquad \text{Grassmannian } \# = \frac{q^r-1}{q-1}$$

$$\boxed{[mS]\diamond_R[nS] = \underline{v}^{mn}\,\#Gr(n,m+n)\,[(m+n)S]} \qquad \#Gr(n,m+n) = q^{-mn}\frac{|GL(m+n)|}{|GL(m)|\,|GL(n)|}$$

$$[S]^{\diamond_R n} = \underline{v}^{1+2+\cdots+(n-1)}\#Gr(1,2)\cdot\#Gr(1,3)\cdots\#Gr(1,n)\,[nS] = \underline{v}^{\binom{n}{2}}\frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}[nS] \quad (*)$$

$$= \underline{v}^{n(n-1)}\,[n]_{\underline{v}}!\,[nS]$$

$$|Aut(nS)| = |GL_n(q)| = q^{\binom{n}{2}}(q-1)(q^2-1)\cdots(q^n-1) \qquad |Aut(S)| = q-1$$

$$[mS]\diamond_B[nS] = \underline{v}^{mn}\,\#Gr(n,m+n)\cdot\frac{|GL(m,q)|\cdot|GL(n,q)|}{|GL(m+n,q)|}[(m+n)S] = \underline{v}^{-mn}[(m+n)S]$$

$$[S]^{\diamond_B n} = \frac{|Aut(S)|^n}{|Aut(nS)|}[S]^{\diamond_R n} \overset{(*)}{=} \underline{v}^{-\binom{n}{2}}[nS]$$

Example (A_2) $\qquad \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \qquad$ Simples $\underbrace{S_1, S_2,}_{\text{all indecomp. by [Gabriel]}}$ $I_{12} := (k \longrightarrow k)$

$$\begin{array}{c} k \longrightarrow k \\ \uparrow \\ 0 \longrightarrow k \\ \| \\ S_2 \end{array}$$

$$\mathrm{Hom}_Q(S_i,S_j) = \begin{cases} k & i=j \\ 0 & i\neq j \end{cases} \qquad \mathrm{Ext}^1_Q(S_i,S_j) = \begin{cases} k & (i,j)=(1,2) \\ 0 & \text{else} \end{cases}$$

$$\langle S_i,S_i\rangle = 1, \qquad \langle S_1,S_2\rangle = -1, \qquad \langle S_2,S_1\rangle = 0$$

Use $\diamond_R$ to do computations $\qquad [S_1]\diamond_R[S_2] = \underline{v}^{-1}\left(\overset{?}{1}\,[S_1\oplus S_2] + \overset{?}{1}\,[I_{12}]\right)$

$$[S_2]\diamond_R[S_1] = \underline{v}^{0}\left(\underset{?}{1}\,[S_1\oplus S_2] \cancel{+ \underset{?}{0}\,[I_{12}]}\right) = [S_1\oplus S_2]$$

Recall Serre: $\quad E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0 \qquad \underset{i}{\circ}\!-\!\!\!-\!\underset{j}{\circ}$

$i=1,\ j=2$

$$[S_2]\diamond_R[S_1]^{\diamond_R 2} = \underline{v}^2(\underline{v}+\underline{v}^{-1})[S_2]\diamond_R[2S_1] = \underline{v}^2(\underline{v}+\underline{v}^{-1})[2S_1\oplus S_2] \qquad (1)$$

Exercise (2)
$$[S_1]^{\diamond_R 2}\diamond_R[S_2] = \underline{v}^2(\underline{v}+\underline{v}^{-1})[2S_1]\diamond_R[S_2] = \underline{v}^2(\underline{v}+\underline{v}^{-1})\cdot\underline{v}^{-2}\big([2S_1\oplus S_2]+[S_1\oplus I_{12}]\big)$$
$$= (\underline{v}+\underline{v}^{-1})\big([2S_1\oplus S_2]+[S_1\oplus I_{12}]\big)$$

(☺ $\mathrm{Hom}(S_2, 2S_1\oplus S_2)\simeq k\simeq \mathrm{Hom}(S_2, S_1\oplus I_{12})$)

(3) $$[S_1]\diamond_R[S_2]\diamond_R[S_1] = [S_1]\diamond_R[S_1\oplus S_2] = \underline{v}^{-1}\big(\overset{?}{\underline{v}(\underline{v}+\underline{v}^{-1})}[2S_1\oplus S_2] + \overset{?}{1}\cdot[S_1\oplus I_{12}]\big)$$

(☺ $|\mathrm{Hom}(S_1\oplus S_2, 2S_1\oplus S_2)| = \underset{q+1}{\underset{\shortparallel}{\underline{v}(\underline{v}+\underline{v}^{-1})}}$, $|\mathrm{Hom}(S_1\oplus S_2, S_1\oplus I_{12})| = 1$)

(1) + (2) + (3) $\Rightarrow$ $[S_1]^{\diamond_R 2}\diamond_R[S_2] - (\underline{v}+\underline{v}^{-1})[S_1]\diamond_R[S_2]\diamond_R[S_1] + [S_2]\diamond_R[S_1]^{\diamond_R 2} = 0$.

Example (rank 2 KM)

$$i \underset{s}{\overset{r}{\rightleftarrows}} j \qquad t := -c_{ij} = r+s$$

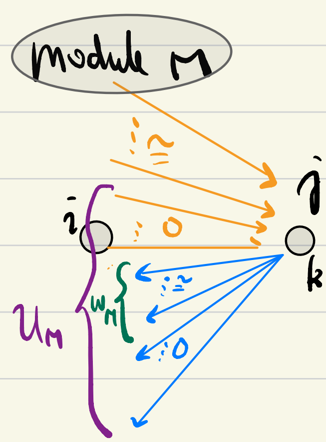
Serre $\Leftrightarrow$ $\displaystyle\sum_{a+b=1-c_{ij}} (-1)^b E_i^{(a)} E_j E_i^{(b)} = 0$, where $E_i^{(a)} = \dfrac{E_i^a}{[a]!}$ divided powers.

Let $[S_i]^{(a)} := \dfrac{[S_i]^{\diamond_R a}}{[a]!}$.

$[\![$ $[S_i]^{(a)}\diamond_R[S_j] = \underline{v}^{a(a-1)-ar}\sum[M]$, summed over $\{M \mid \exists N\subset M \text{ s.t. } N\simeq S_j,\ M/N\simeq S_i^{\oplus a}\}$

For a rep M of kQ of $\underline{\dim} = (t+1)d_i+d_j$, let
$$\begin{cases} U_M := \bigcap_{i\xrightarrow{h}j} \ker x_h \\ W_M := \sum_{i\xleftarrow{h}j} \mathrm{im}\, x_h \end{cases} \qquad \begin{cases} u_M = \dim U_M \\ w_M = \dim W_M \end{cases}$$

Then for $a+b = t+1$,
$$\underbrace{[S_i]^{(a)}\diamond_R[S_j]}\diamond_R[S_i]^{(b)} = \underline{v}^{-bs-ar+ab+a(a-1)+b(b-1)} \sum_{\substack{[M] \\ \underline{\dim}M=(t+1)d_i+d_j}} P_{M,b}[M],$$
where $P_{M,b}=0$ unless $W_M\subseteq U_M$ (☺ M is nilpotent), in which case
$$P_{M,b} = \#Gr(b-w_M, u_M-w_M) = \underline{v}^{(u_M-b)(b-w_M)}\begin{bmatrix} u_M-w_M \\ b-w_M\end{bmatrix}.$$



Hence, $\sum_{a+b=t+1} (-1)^b [S_i]^{(a)} \diamond_R [S_j] \diamond_R [S_i]^{(b)} = \sum_{\{[M] \mid w_M \le u_M,\ \dim M = (t+1)d_i + d_j\}} P_M [M]$,

where

$$P_M = \underline{v}^{(t+1)s - u_M w_M} \sum_{b=0}^{t+1} (-1)^b \underline{v}^{-d \cdot b} \begin{bmatrix} u_M - w_M \\ b - w_M \end{bmatrix}, \quad = 0 \text{ by Lemma (2) below} \quad d := 2s+1 - u_M - w_M$$

Note$_{(t\ge)}$ $u_M - 1 \ge s \ge w_M$, and thus $1-m \le d \le m-1$, for $m := u_M - w_M \ge 1$.

Lemma (1) $\prod_{\ell=0}^{m-1} (1 + v^{2\ell} z) = \sum_{b=0}^{m} v^{(m-1)b} \begin{bmatrix} m \\ b \end{bmatrix} z^b$, for $m \ge 1$

(2) $\sum_{b=0}^{m} (-1)^b v^{-db} \begin{bmatrix} m \\ b \end{bmatrix} = 0$, for $m \ge 1$, $|d| \le m-1$, $d \equiv m-1 \pmod 2$.

☺ (1) is standard; (2) follows by setting $z = -v^{1-m-d}$ in (1).

Theorem [Ringel, Green] • $\exists$ $\mathbb{Q}(\underline{v})$-algebra embedding

$$\Psi: \mathcal{U}^+|_{v=\underline{v}} \longrightarrow (\mathcal{H}(kQ), \diamond_R) \quad \text{or} \quad \mathcal{U}^+|_{v=\underline{v}} \longrightarrow (\mathcal{H}(kQ), \diamond_B)$$
$$E_i \longmapsto [S_i] \qquad\qquad E_i \longmapsto [S_i]/(q-1)$$

• For Q Dynkin (only), Ψ is an $\cong$.

☺ Sketch • The rank 2 computation above $\Rightarrow$ Ψ is a homomorphism

• $\exists$ $\{\,,\,\}$ on $\mathcal{U}^+$ compatible w/ the "Hopf Pairing" $\{\,,\,\}$ on $\mathcal{H}(kQ)$.

• $\{\,,\,\}$ on $\mathcal{U}^+$ is non-degenerate [Lusztig] $\Rightarrow$ Ψ is injective

• Q Dynkin $\mathcal{U}^+$ & $H(kQ)$ has same graded dimension $\Leftarrow$ [Gabriel]
$\Downarrow$
Ψ is surjective. //

Lecture 2 (1/7/21) : Hall algebra realization of the whole quantum group $\tilde{\mathcal{U}}$ or $\mathcal{U}$.
[Lu] same Beijing Time Thursday