多复变：简介与进展

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四川大学, 2018年9月12日

门卫问题(犹如哲学终极问题)：

你是谁；
你从哪里来；
你到哪里去

学科发展的类似问题：

多复变是什么；
多复变从哪里来；
多复变到哪里去

Poincaré: 展望未来，最好的方式是研究过去与现在

钟家庆在为《中国大百科全书》数学卷撰写多复变函数论条目时，称多复变函数论是“数学中研究多个复变量的全纯函数的性质和结构的分支学科”。

用范畴（category）的语言来说

多复变：研究复解析范畴的学问。

这里，范畴的对象（objects）是复流形（乃至复空间），态射（morphisms）是复流形（乃至复空间）间的复解析映照。

另名：(complex) analytic geometry (Serre引入）

布尔巴基学派的观点：“解析几何是解析空间的理论，这是所有数学学科当中最深刻、最困难的理论之一”，in Dieudonné：“The Work of Nicholas Bourbaki”，The American Math. Monthly, 1970

Dieudonné: A panorama of pure mathematics,
列“解析几何”入Boubarki关注度（Bourbaki density)A类

ringed space: a pair $(X, \mathscr{A})$, X a topological space, $\mathscr{A}$ a sheaf of rings

complex space: ringed space, complex model space

Define differential manifold, scheme,...: based on the same way

Behnke-Stein & H. Cartan: normal complex spaces
Serre: reduced complex spaces
H. Grauert: complex space

关于复空间的历史，见Remmert 的综述文章： “From Riemann Surfaces to Complex Spaces”

解析集：locally zero set of holomorphic functions

Weierstrass预备定理:

$\mathcal{O}_0$ is a Noether ring

对多项式环的希尔伯特零点定理对环$\mathcal{O}_0$也成立

local behavior of analytic subvariety

衍生物：CR结构，亚纯映照，多次调和函数，currents（positive closed）等等。

integration on complex analytic subvariety

19世纪：函数论的世纪(Volterra)

出现多复变函数的例子：abel 函数，theta函数，超几何函数，带参数的复变函数

19世纪末：多复分析只是单复分析的简单推广

复可微
全纯
复解析
separately complex analytic

identity theorem, maximum modulus principle, openness of holomorphic functions,...

多复变的产生背景

源自于该学科若干有自身特点的本质发现：Poincaré、Hartogs 的发现，从而多复变作为一个独立研究方向获得发展。

20世纪初：

（1）由经典的 Riemann 映照定理：
$\mathbb{C}$ 中单连通域（非 $\mathbb{C}$）$\cong \Delta$。特别，任意有界凸域均双全纯等价。

Poincaré 发现，当 $n \geq 2$ 时，

$$\begin{array}{l} \text{单位球 } \{|z_1|^2 + \cdots + |z_n|^2 < 1\} \\ \text{多圆盘 } \{|z_1| < 1, \cdots, |z_n| < 1\} \end{array}$$

不双全纯等价。

解析延拓: “analytic continuation (holomorphic extension) is a fundamental phenomena in complex analysis”

从区域到更大的区域（形）的延拓;
从子流形到母流形的延拓

Cauchy, Riemann, Weierstrass’s pioneer works

单复变：解析延拓的思想导致Riemann面的引入、黎曼 ζ-函数的研究、reflection principle、单值化定理的证明等

Techniques for analytic continuation: Cauchy’s integral formula, Weierstrass’s power series method, origin of the concept analytic sheaf

（2）Hartogs 现象：

当 $n \geq 2$ 有$\mathbb{C}^n$中的区域，其上所有全纯函数都能解析延拓至更大的区域。

例如，当 $n \geq 2$，穿孔单位球 $B_n \backslash \{0\}$ 上的所有全纯函数一定在单位球上全纯。

Adolf Hurwitz在1897年首届国际数学家大会的演讲中给出.

导致多复变的基本概念

全纯域（domain of holomorphy）的定义：

区域$D \subset \mathbb{C}^n$ 不发生 Hartogs 现象，即存在一个全纯函数 f，使得 D 是 f 的自然定义（或存在）域（D 的边界称为自然边界）。

定理(H. Cartan-Thullen): 弱全纯域$\Leftrightarrow$ 全纯凸域$\Leftrightarrow$ 全纯域

单复变：每一开集都是全纯域

多复变：并非每一开集都如此，

对于 $D \subset \mathbb{C}^1$，存在全纯函数 f，使得 f 不能解析定义在更大的域上。

例如，

$\mathbb{C} \backslash \{1\}$ is a domain of holomorphy of Riemann ζ function;

$\sum z^{n!}$ 的自然定义（或存在）域是单位圆盘 Δ。

一般域情形参看 Rudin 的书《Real and Complex Analysis》

例如：$B_n \backslash \{0\}$ $(n \geq 2)$不是全纯域（由Hartogs现象）

凸域、有界齐性域、典型域、Teichmüller空间是全纯域（L. Bers）

关于全纯域与全纯包方向的一些著名问题如 Levi问题，Cousin I、II问题（单复变Mittag-Leffler、Weierstrass定理的相应推广）等的研究，

在Hartogs、Oka、Bochner、H. Cartan等多复变先驱在上世纪上半叶的重要工作基础上自然产生

构成多复变基本与核心内容，形成一条主线

概念：多次调和函数（40年代Lelong，Oka引入）
singularity, positive closed (1,1)-current

拟凸域（pseudo － convex domain）：存在域上多次调和函数，在边界趋于无穷大。

Levi问题: 拟凸域是全纯域

Oka, Norguet, Bremermann五十年代初解决

Levi问题就是把全纯域的刻画跳出了构造复（解析）函数的框框，而用构造实函数（多次调和函数）来刻画。

针对Levi问题，Cousin I、II问题（单复变Mittag-Leffler定理、Weierstrass零点定理在高维的类似）等

产生了与代数、几何、分析紧密相关的重要方法:

层及其上同调论(由J.勒雷引进)方法；
$\bar{\partial}$ 方程的L^2 方法；
积分表示方法；...

均可用来解决上述Levi问题，Cousin I、II问题等

关键角色：凝聚解析层(coherent analytic sheaf)

coherence: a local principle of analytic continuation

例子：

结构层，解析集的理想层，

normalization of coherent analytic sheaves,

Grauert’s direct image theorem (corollary: Remmert’s proper mapping theorem)

全纯域在复流形上的类似：Stein 流形（全纯凸，全纯分离）

H. Grauert (ICM 1962 plenary speaker) 解决复流形上的Levi 问题:
Kähler exhaustion implies Steinness

Stein 流形是仿射复子流形(embedding theorem due to Bishop, Narasimhan, Remmert;
Gromov-Eliashberg, Schürmann for minimal dimension $[3n/2]+1$ of the target complex euclidian space)

Stein 流形上的Cartan定理A、B：

Given a coherent analytic sheaf $\mathscr{F}$ on a Stein manifold M,

✓ Theorem A. the stalk $\mathscr{F}_x$ is generated by $\Gamma(M, \mathscr{F})$;

✓ Theorem B. $H^q(M, \mathscr{F}) = 0$ for $q \geq 1$

extension theorem (interpolation theorem): Given a closed complex subvariety S in the Stein manifold M, then any holomorphic section f of a holomorphic vector bundle restricting on S can be holomorphically extended to a holomorphic section F on the Stein manifold M.

used by Serre to establish GAGA principle

Stein 流形上的Oka-Grauert principle（Gromov等推广）

Grauert's h-principle: a continuous map from a Stein manifold to a complex Lie group is homotopic to a holomorphic map

复n维Stein 流形是n维CW复形(Stein结构对空间形状的影响)

Runge 逼近定理（Weil, Oka）

Stein 流形上存在无临界点的全纯函数（Forstnerič，Acta Math. 2003) (related to Hodge conjecture) on Stein manifold every class of even-dimensional cohomology with rational coefficients can be realized as a closed complex submanifold (Griffiths)

holomorphic version of de Rham theorem on Stein manifolds

Serre's problem on fibre bundle over Stein manifolds

covering of a Stein space is Stein (Stein)

Stein流形由其上全纯函数代数的谱刻画

Solving $\bar{\partial}$-equation:

Dolbeault isomorphism theorem

$H^q(M, \Omega^p) = 0$ iff $\bar{\partial}$-equation is solvable

e.g. M is a Stein manifold, then $H^q(M, \Omega^p) = 0$ for $q > 0$ (by Cartan theorem A, B)

Question: the converse is true?

Prove the Hartogs phenomena via solving $\bar{\partial}$-equation (Ehrenpreiss, 1961)

L^2 Hodge theory: Weyl, Hodge, Kodaira, Morrey, Spencer, ...

L^2 method for solving $\bar{\partial}$-equation: Hörmander, Andreotti-Vesentini, Kohn,...

in the setting of unbounded (closed, densely defined) operator on Hilbert space

L^2 existence theorem

Applications: extension theorem (Analytic continuation of holomorphic functions on analytic subvarieties to the Stein manifolds), Levi problem, Cousin problems I、II,...

什么样的复流形是Stein流形？（Levi问题）
什么样的复流形是射影代数流形？(Kodaira embedding)

Philosophy of Levi problem, Kodaira embedding theorem, Hömander's L^2 estimate with singular weights:

► complex differential geometric conditions imply complex analytic conclusion

► existence or construction of specified holomorphic objects (functions, sections) is reduced to the construction of specified plurisubharmonic functions (real-valued, may take $-\infty$)

• "hard" objects (rigid): holomorphic functions, sections

• "soft" objects (flexible): psh functions, currents

新近方法：

乘子理想层(multiplier ideal sheaf)
（层及其上同调论，$\bar{\partial}$ 方程的L^2方法之综合）

1950’s: Cartan, Serre, Grauert,
1960’s: Hörmander, Kohn, Andreotti-Vesentini,
1970’s: Bombieri, Skoda,
1990’s: Y.-T. Siu, Nadel, Demailly（法国科学院院士，ICM 2006 大会报告人）
……

multiplier ideal sheaf:

Definition (Nadel 1990 Ann. of Math.): to a plurisubharmonic function φ, is associated an ideal subsheaf $\mathcal{I}(\varphi)$ of $\mathcal{O}$: germs of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 e^{-2\varphi}$ is locally integrable.

Nadel theorem: $\mathcal{I}(\varphi)$ is coherent

Theorem: multiplier ideal sheaf is integrally closed, i.e., the integral closure of $\mathcal{I}(\varphi)$ is itself

Nadel vanishing theorem: Let $(L, e^{-\varphi})$ be a big line bundle on a compact Kähler manifold (X, ω) (i.e., the curvature current Θ of the singular Hermitian metric is a Kähler current: $\exists \epsilon > 0$, s.t., $\Theta \geq \epsilon\omega$). Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $q \geq 1$.

corollaries:

Kodaira vanishing theorem, Kodaira embedding theorem,

Kawamata-Viehweg vanishing theorem,

Grauert's solution of Levi problem on complex manifolds multiplier ideal sheaves give a unified treatment to the solution of Levi problem and proof of Kodaira embedding theorem

philosophy:

- singularities of psh functions play a key role:

create singularities、use singularities

Green (Green function), Riemann, H. Weyl, Dirac, ...

singular integral, Newtonian potential, fundamental (weak) solution, generalized function (distribution), currents, Poincaré-Lelong equation, ...

- singularity of a psh φ: $\varphi(z) = -\infty$

  e.g., for $\varphi = c\, log(|f_1|^2 + \cdots + |f_k|^2)$ is psh, where $c > 0$

- relation between an analytic subset and a pluripolar set:

  $f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) = \varphi^{-1}(-\infty)$

- psh with analytic singularities

- singular hermitian metric on a holomorphic line bundle:

  locally $e^{-\varphi}, \varphi \in L^1_{loc}$

  curvature $\Theta = i\partial\bar{\partial}\varphi$ in the sense of currents

  pseudoeffective line bundle: $\Theta \geq 0$, i.e., φ is psh.

invariants:

- Lelong number: $v(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{ln|z-x|}$
- complex singularity exponent (log canonical threshold) $c_x(\varphi) = sup\{c \geq 0 : \exp^{-2c\varphi}$ is L^1 w.r.t. the Lebesgue measure on $\mathbb{C}^n$ on a neighborhood of $x\}$
- multiplier ideal sheaf: $\{f = 0| \int |f|^2 e^{-\varphi} < \infty\}$ = $\{e^{-\varphi}$ not locally integrable $\}$ $\subset \{\varphi = -\infty\}$

Oka-Cartan extension theorem:

Given a closed complex subvariety S in the Stein manifold M, then any holomorphic section f of a holomorphic vector bundle restricting on S can be holomorphically extended to a holomorphic section F on the Stein manifold M.

A natural question:

in the setting of the above theorem, if the holomorphic function or section is of a special property (say, invariant w.r.t. a group action, L^p, bounded or L^2), could the holomorphic extension be still of the same special property?

L^2 extension problem:

for a suitable pair (M, S), in the above Oka-Cartan extension theorem, if the function or section f is further L^2 on S, find an L^2 extension F on M together with a good or even optimal estimate?

see Demailly: Analytic methods in algebraic geometry, Higher Education Press, Beijing, 2010.

L^2 extension theorem: Analytic continuation of L^2 holomorphic functions on analytic subvarieties to the Stein manifolds (Ohsawa-Takegoshi, Ohsawa)

L^2 division theorem (Skoda, ICM1978 speaker)

Deformation invariance of plurigenera for projective algebraic manifolds (Siu)

• Y.-T. Siu, Some recent transcendental techniques in algebraic and complex geometry, Proc. of ICM. 2002.

• J.P. Demailly, Kähler manifolds and transcendental techniques in algebraic geometry. Proc. of ICM. 2006.

多复变在代数几何的作用:

transcendental techniques in algebraic geometry:
Abel, Jacobi, Riemann, Weyl, Hodge, Kodaira, Hirzebruch, Grauert,...

compact Riemann surface is projective algebraic,
noncompact Riemann surface is Stein

GAGA principle (W.L. Chow, Serre)
for the projective category:
complex analytic geometry = complex algebraic geometry

positive line bundle in the sense of Grauert: Kodaira embedding theorem generalized to complex spaces

多复变在代数几何的作用:

Unified treatment based on L^2-estimates: Kodaira-Akizuki-Nakano (precise) vanishing theorem, Nakano vanishing theorem, Cartan theorem B (for holomorphic vector bundles), Kodaira-Serre (unprecise) vanishing theorem, Levi problem on complex manifolds,...

Relation between Stein manifolds and projective algebraic manifolds：projective manifold $\setminus$ the zero set of a nontrivial holomorphic section of the positive line bundle is a Stein manifold

- L^2 extension plays a connection role

多复变在量子场论的作用:

全纯包的构造，量子场论中色散关系（第一个实际应用，discovered the edge of the wedge theorem 劈边定理(high dimensional analogue of reflection principle), N.N. Bogolyubov, V. Vladimirov (both are ICM1958 plenary speaker)）

the edge of the wedge theorem stimulated M. Sato to obtain the theory of hyperfunctions

the extended future tube conjecture: the extended future is a domain of holomorphy
solved by Xiangyu Zhou in 1997

习近平总书记在2014年6月及2016年5月两次公开讲话中列举：

多复变函数论

为新中国的科学成就之一

华罗庚先生：中国多复变研究的开创者

华罗庚(1910-1985)

早期：多复变自守函数，与C.L. Siegel同期工作相比肩

回国后：典型域上的多复变函数论

open problem：construction of Cauchy-Szegö kernel on the unit ball；
Dirichlet problem on the classical domains

构造出4 类典型域的各类核：Bergman 核，Cauchy-Szegö 核，Poisson 核;
与陆启铿解决了对应Bergman度量的Laplace-Beltrami方程的狄利克雷问题；
特征流形（Shilov 边界）：最先由华先生发现

1950年回国后完成

1956年获首届国家自然科学一等奖

被评价为“领先西方至少十几年”

“Hua operator”，“Hua system”，“Hua equation”，“Hua measure”

“在中国人民的地位，有如爱因斯坦之在美国”（Science）

大陆首位当选为美国科学院外籍院士

华老的学生

中国科学院院士

陆启铿先生(1927-2015)

陆启铿, 1957：50年代在国际上较早地得到了多复变函数的 Schwarz 引理，对有界齐性域上的 Bergman 度量 $f^*ds^2 \leq kds^2$；

引入了典型域上一些由酉曲率确定的全系解析不变量(在国际上被称为“Lu constant”）

对有界全纯映照，发现可用 Bergman 度量张量来控制映照的 Jacobian。作为推论，Caratheodory 度量一定不超过 Bergman 度量，这是多复变中的基本结果。

另一推论， 在国外(Krantz’s book)被称为”Cartan-Caratheodory-Kaup-Wu Theorem” (Kaup, Wu’s papers appeared ten years later)

1966年提出了“陆启铿定理”：具常西曲率的完备 Bergman 度量的有界域解析等价于单位球，

提出了 Bergman 核有无零点的问题，在国际上被称为“陆启铿猜想”，该猜想成立的区域被称为“陆启铿域”（有界齐性域是“陆启铿域”），

“陆启铿猜想（问题）”已载入多复变的著名教科书，列为综述文献的题目，至今一直有人研究。

Boas(Bergman Prize winner): Lu Qikeng problem.

70年代用联络论及平行移动来诠释规范场，率先指出物理上规范场与数学上的主纤维丛的联络的关系，证明杨振宁的规范场的积分定义等价于沿一曲线的平行移动。

Look, K. H.： Yang-Mills fields, and connections on principal fibre bundles. (Chinese. English summary) Acta Phys. Sinica 23 (1974), no. 4, 249 – 263.
The relation between the theory of Yang-Mills fields and that of connections of principal bundles is established.

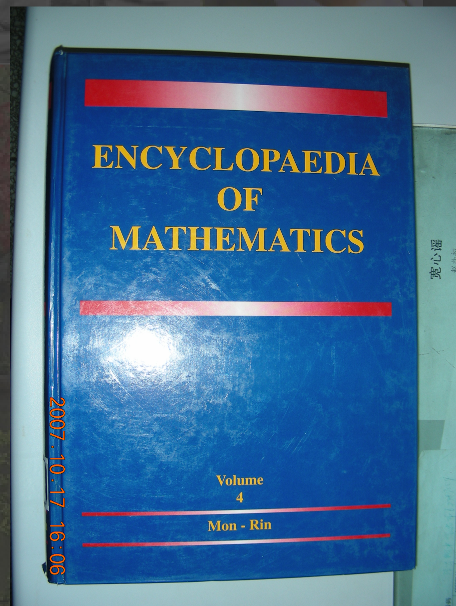
Donaldson： ”The realization that the gauge fields of particle physics and the connections of differential geometry are one and the same has had wide-ranging consequences” (Encyclopaedia of Math. Physics”

“从华罗庚获奖的书中知道，4维 Siegel 上半平面的 Bergman 核函数可以写成 1/det ImZ 的若干次方的形式，是一个多次调和函数。他的学生的学生周向宇在解决“扩充未来光锥管域猜想”的证明中，重要的一步就是要构造一个在扩充未来光锥管域的多次调和函数，要从上述函数出发。这一着如果不是华学派的弟子是难以想到的。”

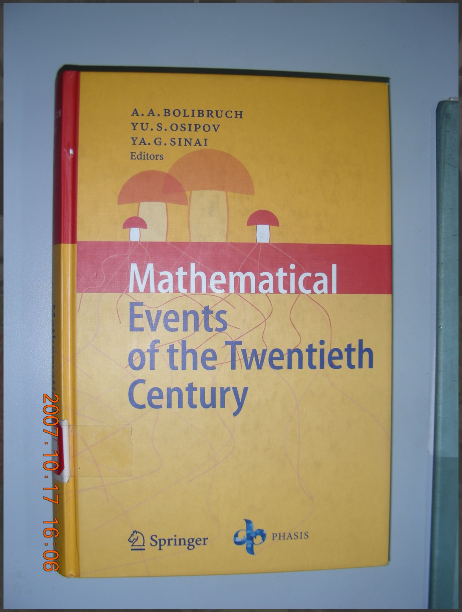
（陆启铿：华罗庚—- 敏锐的直觉、惊人的技巧。中科院创新案例汇编。)

关键：基于华-陆学派的工作: E. Cartan-Weyl-Siegle等关于对称域及典型群的工作的深刻发展

反映了华先生、陆先生工作的长期影响



在苏联《数学百科全书》大辞典中被列为未解决问题

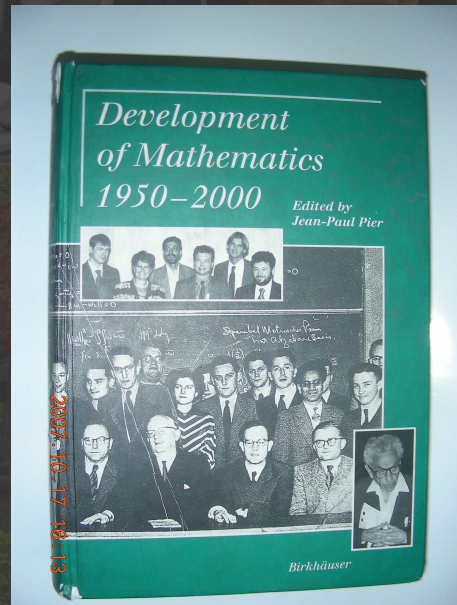


被俄国科学院院士Vladimirov写入

《二十世纪的数学大事》（2006年，Springer）

“this difficult problem was solved only in 1997 by the Chinese mathematician Zhou”

“show how mathematics helps physics to obtain new knowledge which is hidden in the axioms!”



被瑞典皇家科学院院士Kiselman写入

《数学的发展：1950-2000》（2000年，Birkhäuser）

该工作被认为是“数学发展的亮点之一”

L^2 extension and multiplier ideal sheaf

natural development of

✓ Hartogs, Levi, Cousin: foundational works on analytic extensions and domains of holomorphy

✓ Oka, Grauert's solutions of Levi problem

✓ Bergman kernel and metric: Bergman, 华罗庚, 陆启铿

✓ Oka - Cartan's global theory on Stein manifolds, analytic extensions, problems of Cousin I and II

✓ Hörmander's L^2 method for solving $\bar\partial$ equation

✓ Kodaira-Grauert embedding theory (Kodaira-Grauert vanishing, embedding theorems, positivity in the sense of Grauert)

► many mathematicians have contributions to L^2 extension theorem
Ohsawa-Takegoshi,
Ohsawa (ICM 1990 邀请报告人),
Siu(美国国家科学院院士，ICM 2002 大会报告人),
Demailly(法国科学院院士，ICM 2006 大会报告人),
Berndtsson(瑞典皇家科学院院士，ICM 2018 邀请报告人)
also obtained explicit good estimates

► natural open problem – optimal L^2 extension problem

► very few connection between optimal L^2 extension and other problems (except Suita conjecture)

Guan, Zhou solved the optimal L^2 extension problem, as applications, solving several long-standing problems and finding some new connections

published in Ann. of Math. 2015, based on a series of works in Liouville's J. 2012, Comptes Rendus 2012, Science China Math. 2015

Ohsawa 2015年专著"L^2 Approaches in Several Complex Variables"(Springer)，在前言中指出："Q. Guan and X.-Y. Zhou proved generalized variants and characterized those surfaces on which the inequality is strict. Their work gave the author a decisive impetus to start writing a survey to cover these remarkable achievements"

a solution of Ohsawa's question in
T. Ohsawa, On the extension of L^2 holomorphic functions VIII - a remark on a theorem of Guan and Zhou, IJM 2017

Guan and Zhou found in Ann. of Math. 2015 that their optimal L^2 extension theorem could imply

Berndtsson: $\log B_t(z)$ is a plurisubharmonic function with respect to (z,t).

implies Griffiths positivity of the relative canonical bundle (Guan, Zhou, Sci. China Math. 2017)

在Ohsawa的专著中称为"Guan-Zhou method"

Guan, Zhou solved Demailly's strong openness conjecture about the multiplier ideal sheaves in 2013

$$\mathcal{I}(\varphi) = \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi).$$

- published in Ann. of Math. 2015
- also conjectured by Y.T. Siu
- special case - openness conjecture was solved by Berndtsson in 2013

美国数学评论： "The proofs of both the openness and the strong openness conjectures are among the greatest achievements 'in the intersection' of complex analysis and algebraic geometry in recent years."

The meaning of the conjecture:

✓ $|f|^2e^{-\varphi}$ is locally integrable, then there exists an $\varepsilon_0 > 0$ s.t. $|f|^2e^{-(1+\varepsilon_0)\varphi}$ is also locally integrable

✓ $\{p \in \mathbb{R} : |f|^2e^{-p\varphi}$ is locally integrable$\}$ is open

origin from calculus:

$\{p \in \mathbb{R} : 1/|x|^{pc} = e^{-p\varphi}$ is locally integrable at the origin$\}$ is open, where $\varphi = c\, log|x|, c > 0$

$\{p \in \mathbb{R} : f(x)/|x|^{pc} = f(x)e^{-p\varphi}$ is locally integrable at the origin$\}$ is open

Corollaries:

- Demailly-Ein-Lazarsfeld conjecture (均为ICM 邀请报告人), also stated as an open problem in Lazarsfeld's book: Positivity in algebraic geometry
- Vanishing theorem for pseudoeffective line bundles（蕴含Kodaira、Kawamata-Viehweg、Nadel 消没定理）
- Boucksom (2018 ICM邀请报告人)-Farve-Jonson conjecture

Guan, Zhou solved

Demailly-Kollár's conjecture related to openness conj.,

Jonsson-Mustata's conjecture related to strong openness conj.

based on ideas of our work on optimal L^2 estimate

published in Invent. Math. 2015

Using Guan-Zhou’s solution of Demailly’s strong openness conjecture:

Meng, Zhou (to appear in JAG):
Nadel vanishing theorem without Kähler condition

vanishing theorem, finiteness theorem and surjectivity valued in multiplier ideal sheaves

Zhou, Zhu (JDG, 2018), establish:

optimal L^2 theorem on weakly pseudoconvex Kähler manifolds

positivity of twisted relative canonical bundles for Kähler fibration

by using and developing the following

Zhou, Zhu (MRL 2017)： generalized Siu's lemma

Zhou, Zhu (2017): positivity of twisted relative pluricanonical bundles and their direct images for Kähler fibration

case of algebraic fibre space： obtained by Berndtsson-Paun(Duke 2008), Paun-Takayama (JAG 2018)

Some authors are using

- our optimal L^2 extension theorem,
- ”Guan-Zhou method” and
- the solution of Demailly’s strong openness conjecture

to study some important problems in several complex variables and algebraic geometry

Demailly,
Ohsawa,
Berndtsson-Lempert,
Paun-Takayama,
Paun,
Hacon-Popa-Schnell


谢谢！