An unguided tour started from chirality

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We will report an unguided mathematical tour started from research on chirality since 2000 and, attracted by questions around attractors, led to a zigzag path across topology and dynamics, often switched dimensions.

People who joined this tour at various stages include F. Ding, B. Jiang, Y. Liu, Y. Ni, J. Pan, H. Sun, C. Wang, S. Wang, J. Yao, Y. Zhang, H. Zheng, Q. Zhou and B. Zimmermann. Conversations with R. Edwards, L. Wen, C. Bonatti, J. Hillman, S. Kamada and others, added to the twists and turns that made the trip more fun.

References

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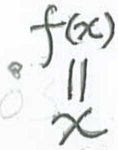
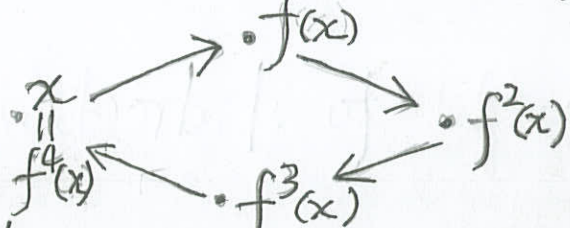
For a manifold M and an autormorphism f (discrete case), or a flow $\tau(\cdot, t)$ (continuous case).

Topology concern

- fixed pt set, Periodic pt set $P(f)$





- zeros and closed orbits of $\tau(\cdot, t)$

Dynamics mostly concern more: the asymptotic properties

$$f^n \qquad\qquad \tau(\cdot, t)$$

$$n \longrightarrow \infty, \qquad t \longrightarrow \infty$$

Basic Notions in Dynamics
(Only stated for discrete case)

1. Say $x \in M$ is non-wandering pt of f, if for any open set $U \ni x$, $\{f^n(x), n>0\} \cap U \neq \phi$.

i.e. the f-orbit back close to x again and again.

非游荡集
Non-Wandering set $\Omega(f)$ is the Union of Non-Wandering pts of f.

2. $\underline{\Delta} \subset M$ is + attractor (吸引子)
If $\exists$ a closed neighborhood U of Δ s.t. $f(U) \subset \text{int}\, U$ and

$$\Delta = \cap_{n=1}^{\infty} f^n(U) = \Omega(f|U)$$

Also call attractors of f^{-1} a negative attractor or repeller of f.

结构稳定 P3 Ω-稳定

3) Call f stuructural stable (Ω-stabe):

If h is C^1-close to f, then f and h are <u>conjugated</u> on M (on $\Omega(f)$)

$$(h = gfg^{-1};\ g: M^2)$$

Hence "stable" $\Longrightarrow$ Small pertubation preserving dynamics (since it preserving $\Omega(f) \supset P(f)$, attractors, closed orbits)

Periods

Ex 1 (a) Your watch is dynamics

(b) Our solar system is a dynamics

We expect the watch and the solar system are stable.

This reflect the motivation of the above definition "stabe"

When f stable?

A necessary condition is Axiom A (Smale):

f is hyperbolic in $\Omega(f)$ and $\overline{P(f)} = \Omega(f)$.

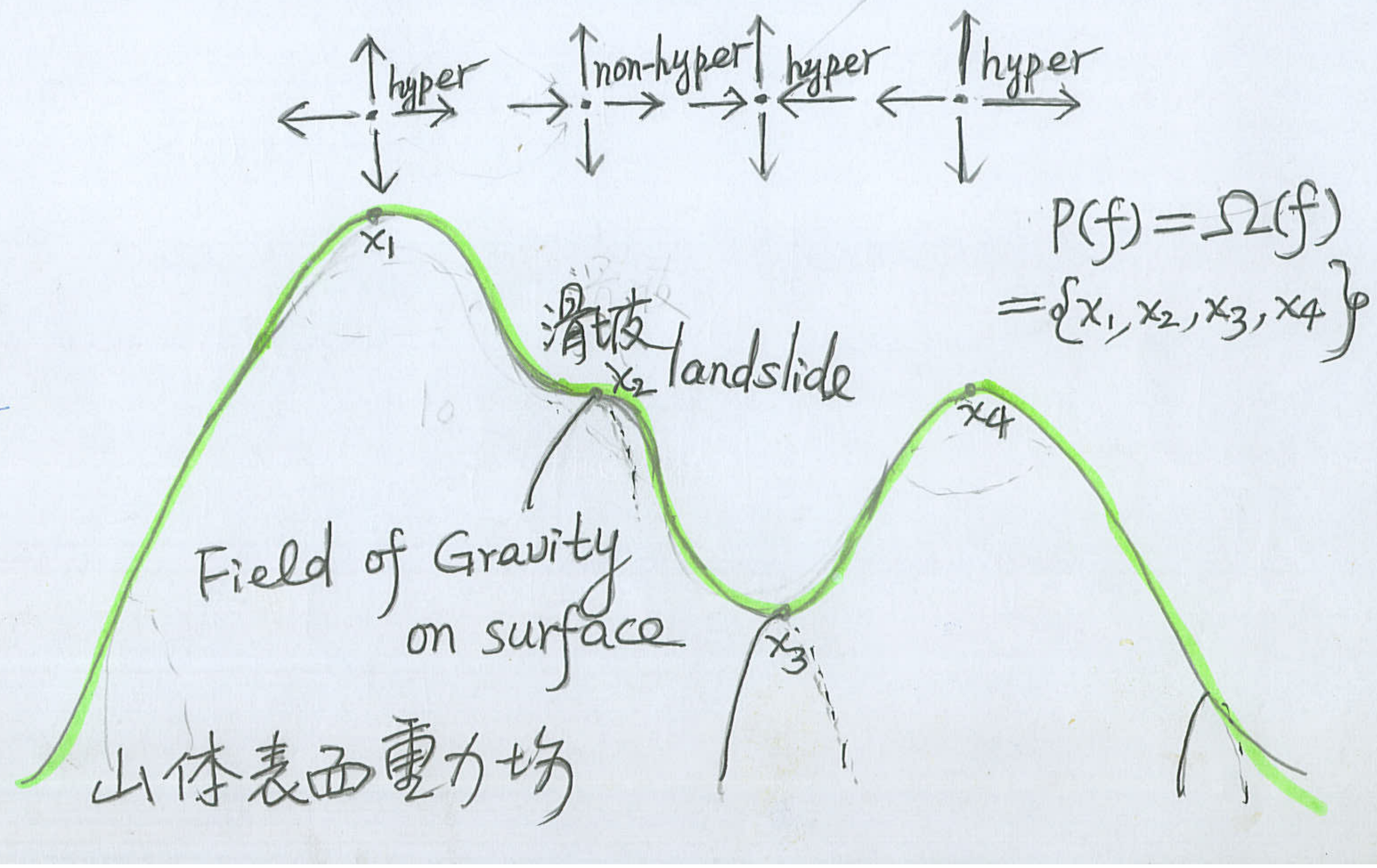
Call f is hyperbolic on $\Lambda \subset M$,

if $\exists$ a splitting $\underset{\uparrow \text{ tangent bundle}}{TM}\big|_{\Lambda} = E^s_\Lambda \oplus E^u_\Lambda$ s.t

f_* retracts & expends on E^s_Λ & E^u_Λ

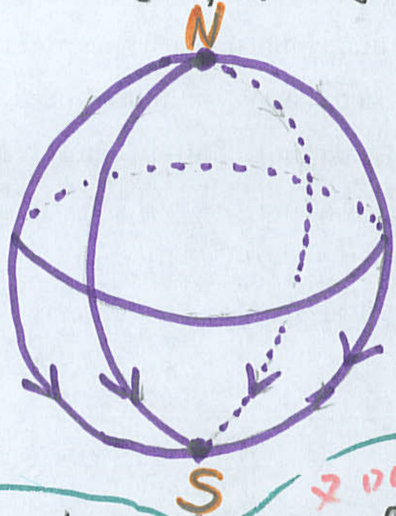
resp. and "obviously".

Ex 2. Hyperbolicity & stability



Examples meet Axiom A

Ex 2 北京麻球 (N-S dynamics)

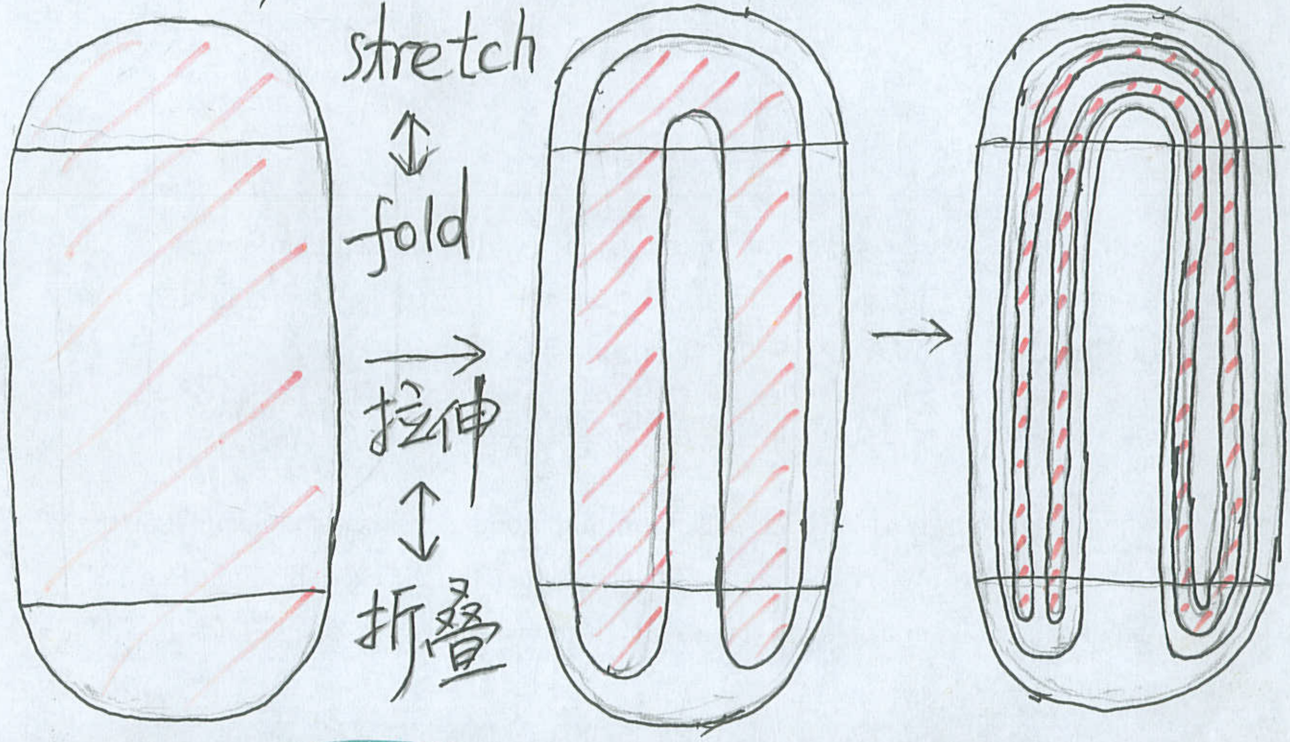


$\Omega(f) = P(f)$ $\quad f|_{\Omega(f)}$ hyper

$\| \{N, S\}$ ± attractor

对称性 symmetry

Ex 3 兰州拉面 (Smale horseshoe)



$\Omega(f) = (\text{Cantor set})^2$

Ex 4 Anosov map

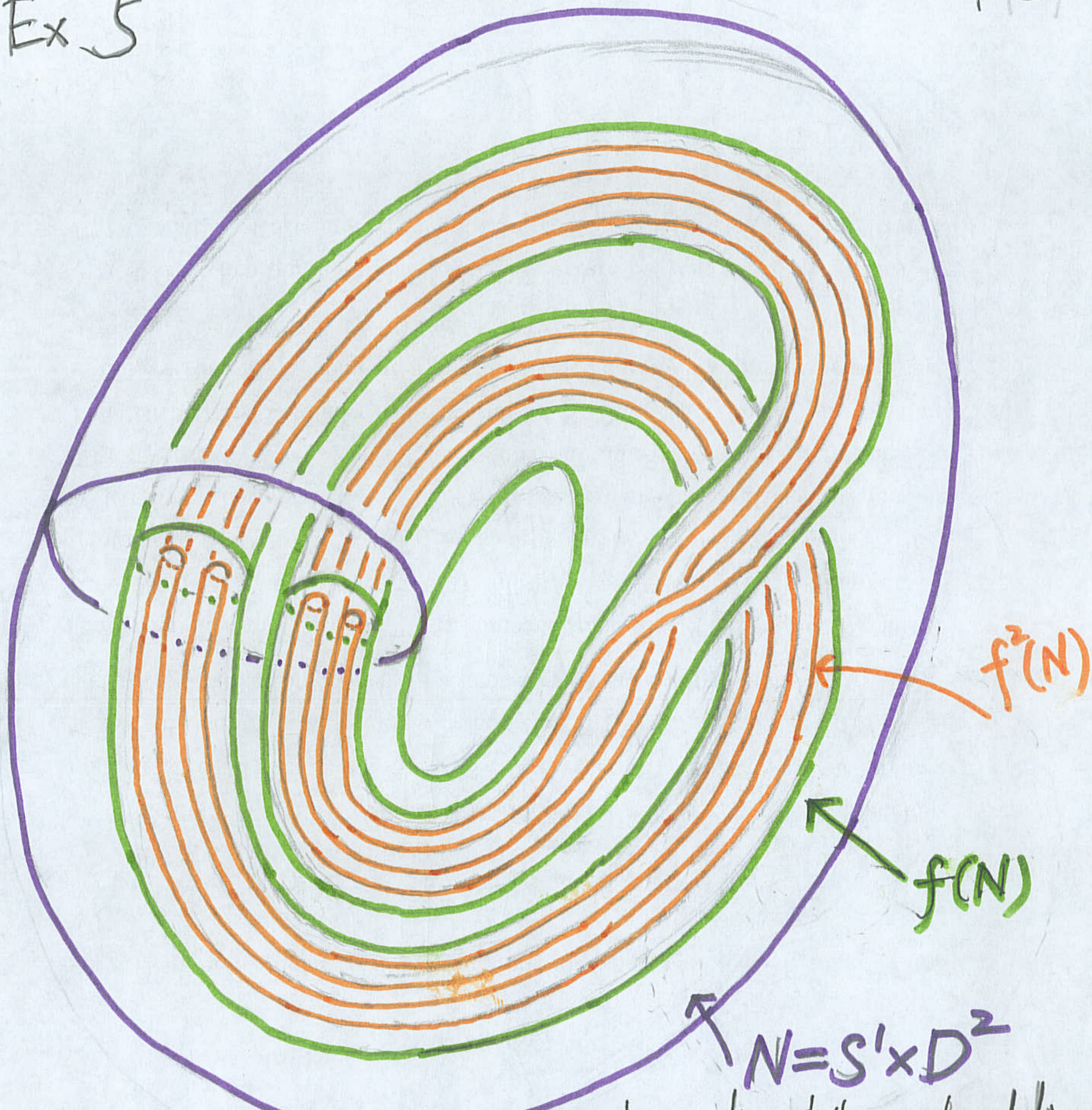
$f:$  $\circlearrowleft$

$\tilde{f}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ induced by

$$\tilde{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\overline{P(f)} = \Omega(f) = M$$

Ex 5



$f: N \xrightarrow[\text{winding \# 2}]{\text{linearly} \; \text{embedding}} N$

(p7) 2-adic solenoid

$$S = \bigcap_1^\infty f^n(N)$$ compact connected

Suspansion on cantor set, uncountably many path-connected cpts

Topo. dim 1   fractal dim $1 + \log$

In 1967, Smale thought that $\Omega(f)$ meets Axiom A should be constructed from some "basic sets"

His samples are:

(i) 0-dimensional (Including isolated pt & horseshoe)

(ii) Anosov type (Including DA)

(iii) Solenoid type (Including GS)

Recall in classical Morse thy:

$\Omega(g)$ is finite & hyperbolic for the gradient field g of a Morse function on M

Study Topology of M via $\Omega(g)$ is Important aspect of Morse thy.

Classical samples:

(1) $|\Omega(g)| = 2$ implies $M = S^n$ (Reeb, 1952)

(2) $|\Omega(g)| = 3$ implies M is projective-like spaces (Eells-Kuiper 1961)

What happens if $\Omega(f)$ is hyperbolic but singular (infinite). In otherwords How small is local models are realized globly by dynamics on closed manifolds such researches are places topology and dynamics reacts.

Below is a sample.

Thm 1 (2004. Jiang-Ni-W, 姜伯駒-倪忆-王)

Suppose M is a closed orientable 3-mfd.

$$\exists f \text{ on } M \text{ s.t } \Omega(f) = \left\{ \begin{array}{c} \text{finitely many} \\ \text{solenoids} \end{array} \right\}$$

$\Longleftrightarrow$ M is Lens space $L(p,q)$, $p \neq 0$.

Moreover in "$\Longleftarrow$", $\Omega(f)$ consists of two $(p+1)$-adic solenoids which are $\pm$ attractors of f;

f is Ω-stable, but not structurally stable.

Definition of Lens spaces

(1) $S^3 \overset{\text{unit sphere}}{\subset} \mathbb{C}^2$

Period p action $\tau : (\mathbb{C}^2, S^3) \circlearrowleft$

given by $\tau(z_1, z_2) = (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2q\pi i}{p}} z_2)$

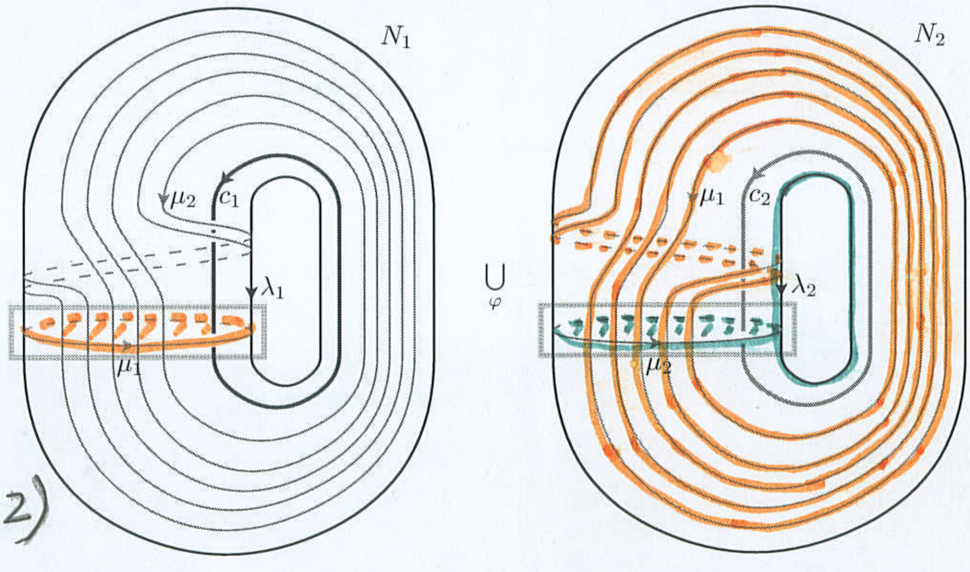
$(p, q) = 1$

Then $L(p,q) = S^3/\tau$.

Note $\pi_1(L(p,q)) = \mathbb{Z}_p$, $L(1,q) = S^3$

(2)

$L(p,q) =$



p times ↑

q times →

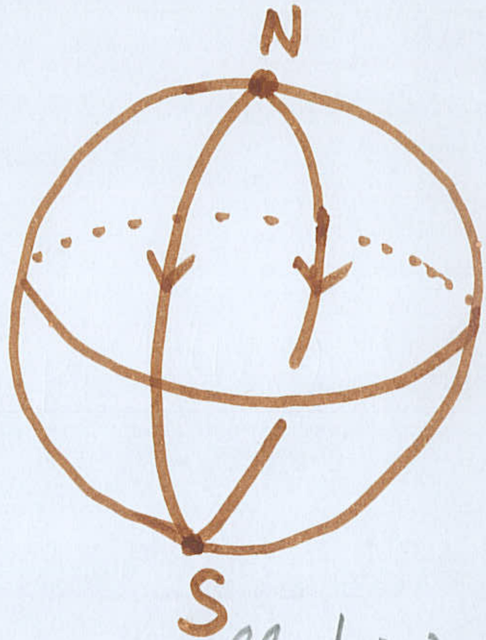
$(p, q) = (5, 2)$

FIGURE 1. Lens space $L(p,q)$ as union of solid tori $N_1 \underset{\varphi}{\cup} N_2$

M admits a dynamics f s.t $\Omega(f) =$ {two hyperbolic $\pm$ attractors} presents a symmetry of M with stability

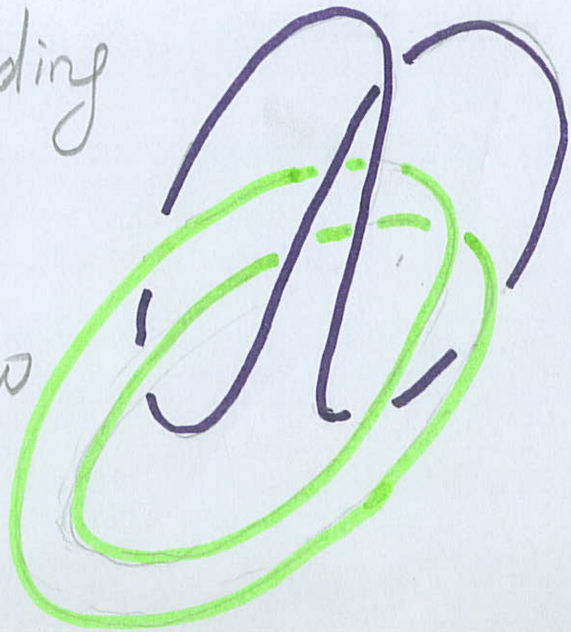
从源到汇 From Sourse to sink.

Reeb's Thm claim:
When $\pm$ attractors are isolated pts, $M = S^n$,
Which is simplest since:



1. topology of attractors
2. Embedding of attractors
3. Dynamics on attractor are all trivial

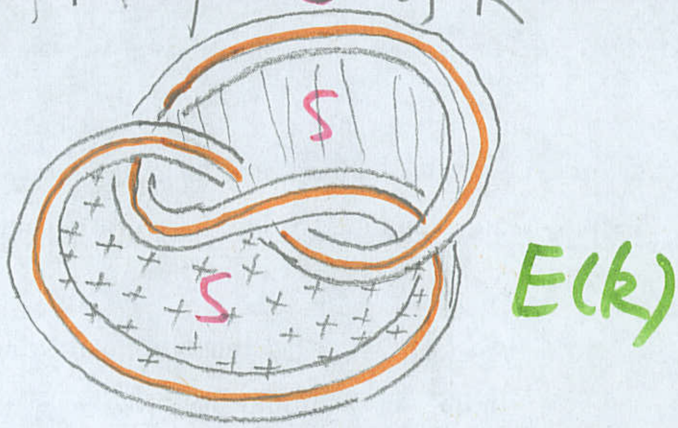
In our result, $\pm$ attractors are solenoids
its Topology, its embedding into mfds, its dynamics are non-trivial
Those non-triviality allow ∞'ly 3-mfds having Symmetries of hyperbolic $\pm$ attractors.



Unexpected application of Our Solenoid result

$k \subset S^3$, Exterior of k: $E(k) = S^3 - \mathrm{int}\, N(k)$

Seifert surface S of k



$E(k)$ has ∞ cyclic covering $\widehat{E(k)}$ (unique)

Call k is fibered, if $E(k)$ is a surface bundle

Question (J. Stallings)

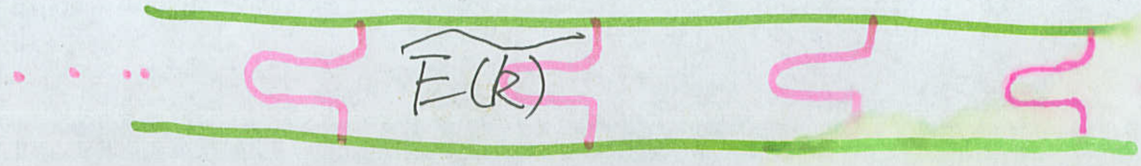
Suppose $k \subset S^3$ is not fibered, when $\widehat{E(k)} \subset R^3$?

Property IE

Note fiber knots always IE

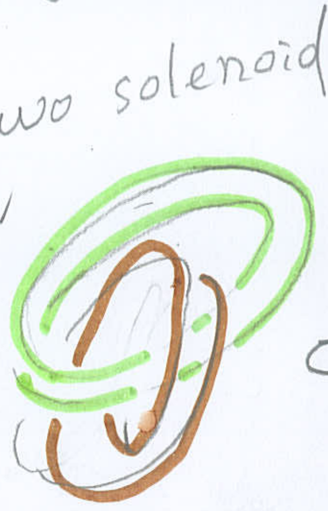
Neither positive nor negative Examples

in last 20 Years



In result of Jiang-Ni-W, $\exists\, f: S^3 \circlearrowleft$

s.t $\Omega(f) = \mathcal{S}_1 \cup \mathcal{S}_2$ two solenoid



then $f|_{S^3 \setminus \mathcal{S}_1 \cup \mathcal{S}_2}$ is a good free action and $\subset S^3$

the quotient $S^3 \setminus \mathcal{S}_1 \cup \mathcal{S}_2$

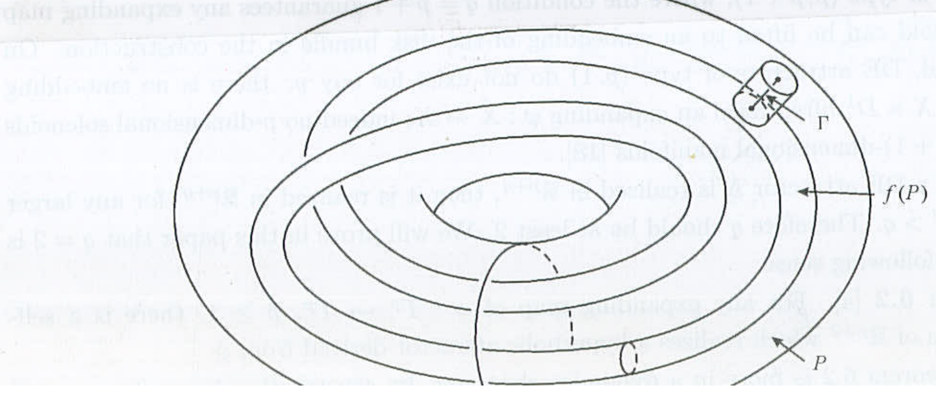
is a compact 3-mfd M whose infinite cyclic covering $\widetilde{M} = S^3 \setminus \mathcal{S}_1 \cup \mathcal{S}_2 \subset S^3$.

$M \overset{\text{almost}}{=\!=} 9_{46} =$ 

First positive answer to stalling Q.. On the negative side

Thm 2 (Jiang-Ni-Zhou-W)

If a non-fibered knot k of genus 1 has property IE, then $\Delta(k) = \begin{cases} 1, \text{ or} \\ 2t^2 - 5t + 2, \end{cases}$



Planarity and Achirality

可平面性和手性

Suppose $A \subset \mathbb{R}^3$

Call A is achiral / marked achiral

if $\exists$ motion τ of R^3 s.t. $\tau(A)=A^*$

$\tau(A)=A^*$ and $\tau(x)=x^*$ $\forall x \in A$

A^*, x^* 是 A, x 的镜像

$\Longleftrightarrow$ A is invariant under / pointwise fixed under

an Orientation Reversing homeo of R^3

ORH

$\Longleftrightarrow$ setwise version / pointwise version

姜伯驹 / defined by Jiang-Wang in 1999

The achirality in our real word is between the two mathematical abstractions above

Call a space P is (abstract) planar if
$\exists$ embedding $\subset \mathbb{R}^2 \subset S^2$

Thm (Jiang-Wang) 2000

Let K be a finite graph
(or finite polyhedron)

There exits an marked
achiral embedding $i: K \hookrightarrow \mathbb{R}^3$

iff K is (abstract) planar.

It is natrural to wondon. if the condition
"finite polyhedron" can be replaced by
contiua (compact. conneted 度量空间)

Q^*: Suppose $P \xrightarrow{S.A.E} S^3$ for 连续流 P.
Does P 可平面化?

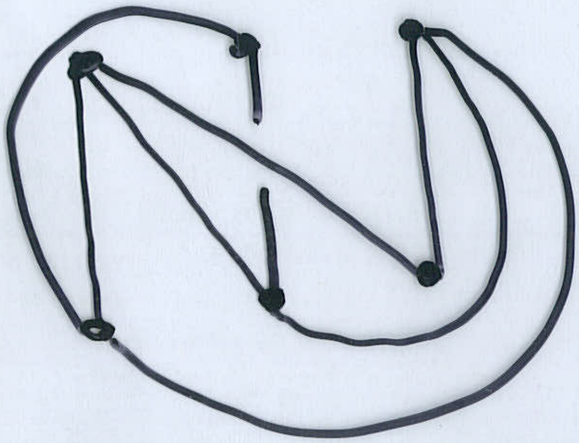
To avoid misunderstanding of the above Thm:

S^1

abstract planar

marked achiral

chiral



$\mathbb{R}^3$

任-embedding chiral

在寻找 Q^* 反例时, Solenoid 成为首选:

① Solenoid 不可平面化 (Bing 1962)
(quick proof and generlized Result
Jiang-王-Zheng 2008)

② Solenoid 是可平面化集(圈周)的极限.

在构造 S^3 上 ORH 以 Solenoid $\mathcal{S} \subset S^3$ 为不动点集时发现了 Thm (JNW, 2004)

Q^* 的反例则是又过了12年才证明的.

Thm (Jiang-王-Zheng-Zhou, 2007, 2011)

$\exists$ solenoid $\mathcal{S} \xhookrightarrow{S.A.E} S^3$.

key figure:

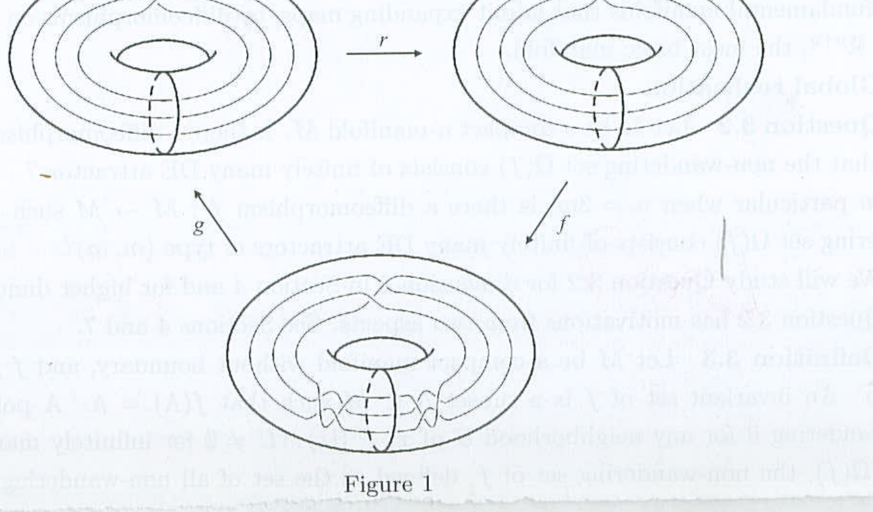


Figure 1

那么所谓动力系统的结果是否真的引起Dynamicists的兴趣呢？

Yes. 来自专家们(中.法.美.墨西哥)的问题.

Q**：能否在4维流形 M 上构造出

$f: M \longrightarrow M$ s.t. $\Omega(f)$ 由两个 $(2,2)$ 型的Solenoid组成？

Q***：High dimensional solenoids 作微分同胚吸引子的实现问题？

Positive answer of Q** $\Longrightarrow$ Negative

Negative answer of Smale's conjecture of Anosov map：If $f: M \longrightarrow M$ anosov, then $\Omega(f) = M$. (对流的类似猜测有反例 J. Franks)

So

Df：

$$\begin{tikzcd} X^p \times D^q \arrow[r, "e"] \arrow[d, "\pi"] & X^p \times D^q \arrow[d, "\pi"] \\ X^p \arrow[r, "\varphi"] & X^p \end{tikzcd}$$

φ expending

e embedding

and

$S = \bigcap_{1}^{\infty} e^n(N)$：DE solenoid attractor of type (p, q)

潘建中

Thm (DPWY 2010) 若 $f: M \longrightarrow M$ s.t

$\Omega(f) = \{\pm DE \text{ attractors}\}$. 则 M 是有理同调球，且 attractors are of type $(n-2, 2)$.

推论：No $f: M^n \longrightarrow M^n$ s.t. $\Omega(f) = \{\pm DE. \text{attra.}\}$

if either $n=4$, or $n=2k$, DE. attra type (k, k)

在证明这个定理时用到 Gromov 的定理：任一 expending map 共轭于幂零李群的 automorphism. 及 Nomizu 一个关于幂零流形上同调的定理.

Q: $\exists f: M^n \longrightarrow M^n$, $n > 3$, s.t. $\Omega(f) = \{\pm DE \text{ attra.}\}$?

Thm (DLWYI 2011)

(1) For any expending map $\phi: M^p \longrightarrow M^p$.

$\exists$ diffeo of R^{p+q} realizing DE attra. derived from ϕ, for $q > p$

(2) For any expending map $\phi: T^p \longrightarrow T^p$, p维环面.

$\exists$ diffeo of R^{p+2} realizing DE attra. derived from ϕ

RK: 2 是不能再降的

Thm (DLWYI 2011) 显然 closely related

对 $M \subset \mathbb{R}^{p+q}$ M上的一个 automor. 能扩張

到 $\mathbb{R}^{p+q}$ 上，这又回到拓扑问题上来.

(1) 的证明依赖于吴文俊关于 Whitney 定理

的扩張.

(2) 则用到在 DLWYI 中证明的一个定理

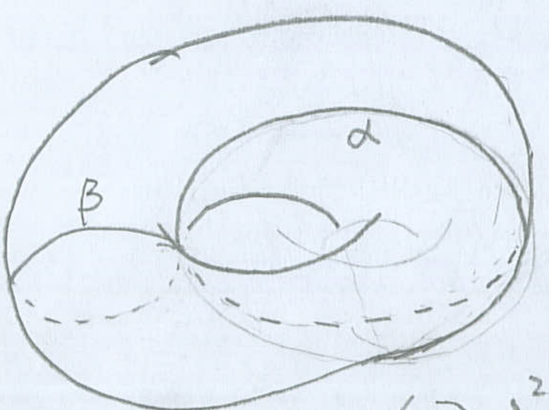
定义： MCG_M 记 M 的映射类群

对 $e: M^p \hookrightarrow \mathbb{R}^{p+2}$,

记 $E_e = \{\tau \in MCG_M \mid \tau \text{ extends over } \mathbb{R}^{p+2}\}$

定理：For any unknotted embedding

$e: T^p \longrightarrow \mathbb{R}^{p+2}$, $[MCG_{T^p} : E_e] \leq 2^p - 1$



$\subset \mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{R}^1 = \mathbb{R}^4$

$(D_\beta)^2$ 和 D_α extends

$\langle D_\beta^2, D_\alpha \rangle$ 生成 MCG_{T^2} 指数为 3

子君群

对 embeding $e: M^p \hookrightarrow R^{p+2}$

如何确定某个 $\tau: M \to M$ 不能扩張 [DLWYII]

定理：$\forall\, e: M^p \hookrightarrow R^{p+2}$

the induced spin structure $e^{\#}(\zeta)$

is invariant under each $\tau \in E_e$.

Applications:

定理：$\forall$ unknotted $T^p \xhookrightarrow{e} R^{p+2}$

$$[MCG_{T^p} : E_e] \geq 2^p - 1$$

....

定理：$\forall\, F_g \xhookrightarrow{e} R^4$

$$[MCG_{F_g} : E_e] \geq 2^{2g-1} + 2^{g-1}.$$

推论：$\exists\, \tau \in MCG_{F_g}$ s.t. $\tau \notin E_e$ for any $e: F_g \hookrightarrow R^4$

最近 In [LNSW]，类似经典纽结论中之格 Thurston Norm 的概念已被引入

$T^2 \hookrightarrow R^4$ 的研究，并和 E_e 有紧密的关系。