From Euler to Langlands

Dihua Jiang
University of Minnesota

Ke Zhao Lecture, Sichuan University
December 20, 2019

Riemann Zeta Function

- One of the most mysterious and important functions in the modern Number Theory is

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- **converges absolutely** for $\mathrm{Re}(s) > 1$.
- **meromorphic continuation** to the complex plane $\mathbb{C}$.
- **functional equation:** $\zeta_\infty(s) := \Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}$

$$\Lambda(s) := \zeta_\infty(s)\zeta(s) = \zeta_\infty(1-s)\zeta(1-s) = \Lambda(1-s).$$

- **Euler product decomposition:** for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}.$$

Euler Product of Riemann Zeta Function

- **Euler Product Decomposition:**

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

for $\mathrm{Re}(s) > 1$ can be proved by using

Theorem (Foundamental Theorem of Arithmetic)

For any $r \in \mathbb{Q}$, there is prime numbers $p_1, p_2, \cdots, p_t$ and integers $e_1, e_2, \cdots, e_t$ such that

$$r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}.$$

This is unique up to permutation.

- $\zeta(s)$ has a simple pole at $s = 1$ if and only if there exists **infinitely many primes**.

Basic Structures of Numbers

- **Foundamental Theorem of Arithmetic** provides the fundamental **multiplicative structure** of numbers in terms of primes.
- and suggests the basic **local-global principle** in the modern Number Theory.
- From $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, to know r is equivalent to know all $p_i^{e_i}$, individually
- To measure r we use the usual absolute value; and to measure $p_i^{e_i}$ we use the so called p-adic absolute value.

p-adic Absolute Value

- Given a prime p, any $r \in \mathbb{Q}^\times$, we have $r = p^e \cdot \frac{a}{b}$, where $(p, a) = (p, b) = 1$.
- Define the p-adic absolute value

$$|r|_p := \begin{cases} p^{-e}, & \text{if } r \neq 0; \\ 0, & \text{if } r = 0. \end{cases}$$

- $|\cdot|_p$ defines a nontrivial metric on $\mathbb{Q}$.
- For $r \in \mathbb{Q}^\times$, the **Artin product formula**

$$\prod_v |r|_v = 1.$$

Locally Compact Topological Fields

- Over $\mathbb{Q}$, we have $|\cdot|_\infty$ and $|\cdot|_p$ for all p's.
- Take the completion, we have

$$\overline{(\mathbb{Q}, |\cdot|_\infty)} = \mathbb{R}; \quad \overline{(\mathbb{Q}, |\cdot|_p)} = \mathbb{Q}_p.$$

- They are only locally compact topological fields containing $\mathbb{Q}$ as a dense set.

$$\begin{array}{ccccccc}
\mathbb{R} & \mathbb{Q}_2 & \mathbb{Q}_3 & \mathbb{Q}_5 & \cdots & \mathbb{Q}_p & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
|\cdot|_\infty & |\cdot|_2 & |\cdot|_3 & |\cdot|_5 & \cdots & |\cdot|_p & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
\infty & 2 & 3 & 5 & \cdots & p & \cdots
\end{array}$$

Locally Compact Topological Fields

- The Euler product $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\prod_p \frac{1}{1-p^{-s}}$
- We may put it into the following diagram:

$$\begin{array}{ccccccc} \mathbb{R} & \mathbb{Q}_2 & \mathbb{Q}_3 & \mathbb{Q}_5 & \cdots & \mathbb{Q}_p & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \frac{1}{1-2^{-s}} & \frac{1}{1-3^{-s}} & \frac{1}{1-5^{-s}} & \cdots & \frac{1}{1-p^{-s}} & \cdots \end{array}$$

- One of the fundamental properties of locally compact groups is the existence of Haar measure, unique up to scalar.
- For $v = \infty$ or p, denote the Haar measure dx_v on $\mathbb{Q}_v$.
- The Harmonic Analysis on $(\mathbb{Q}_v, dx_v)$ provides an important interpretation of the local Euler factors $\zeta_p(s) = \frac{1}{1-p^{-s}}$ and $\zeta_\infty(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) = \pi^{-\frac{s}{2}}\int_0^\infty x^{\frac{s}{2}-1}e^{-x}dx$, following the famous thesis of **J. Tate**.

Adele Ring of $\mathbb{Q}$

- One might consider $\prod_v \mathbb{Q}_v$, but it is not locally compact.
- For each $r = \pm p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ involves finitely many primes.
- The ring of adeles is defined to be

$$\mathbb{A} := \{(x_v) \in \prod_v \mathbb{Q}_v \ : \ |x_p|_p \leq 1, \text{ for almost all } p\}.$$

- $\mathbb{A}$ is a locally compact ring containing all $\mathbb{Q}_v$; and $\mathbb{Q}$ is discrete in $\mathbb{A}$ such that $\mathbb{A}/\mathbb{Q}$ is compact.
- $(\mathbb{A}, \mathbb{Q})$ is a **modern analogy** of the classical pair $(\mathbb{R}, \mathbb{Z})$.

Tate's Thesis

- For each v, $\exists$ a Schwartz function ϕ_v, s.t.

$$\int_{\mathbb{Q}_v^\times} \phi_v(x)|x|_v^s d^\times x_v = \begin{cases} \frac{1}{1-p^{-s}} & \text{if } v = p, \\ \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } v = \infty. \end{cases}$$

- $\exists$ a Schwartz function $\phi = \otimes_v \phi_v$ on $\mathbb{A}$, s.t.

$$\int_{\mathbb{A}^\times} \phi(x)|x|_{\mathbb{A}}^s d^\times x = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \cdot \prod_p \frac{1}{1-p^{-s}} = \zeta_\infty(s)\zeta(s).$$

- **Functional Equation:** $\zeta_\infty(s)\zeta(s) = \zeta_\infty(1-s)\zeta(1-s)$ follows from the **Poisson Summation Formula** for the **Fourier Transform** on $\mathbb{A}/\mathbb{Q}$.
- **Local-Global** relation in *Harmonic Analysis* approaches the **Local-Global** relation in *Arithmetic*!

Solutions of Polynormial Equations

- Find solutions for

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

  with $a_i \in \mathbb{Q}$.
- It is well-known that for $n \leq 4$, one has a formula to express all the solutions in complex numbers $\mathbb{C}$.
- **Classical Problem:** Why is there no formula for the roots of $P(x) = 0$ with $n \geq 5$ in terms of a_i, using only *addition, subtraction, multiplication, division, square roots, cube roots, etc*?
- The **Abel Impossibility Theorem (1824)** confirms no formula in general for $n \geq 5$.
- However, a more conceptual understanding of this classical problem is not known until the work of **E. Galois** in 1830.

Galois Groups and Algebraic Number Theory

- According to Galois, the symmetric group of the roots of $P(x) = 0$, which is called its **Galois Group**, is much more fundamental than the explicit expression of the roots.
- This leads to the **Galois Theory** of $P(x) = 0$, which is to study the finite field extension of $\mathbb{Q}$.
- **Algebraic Number Theory** is to understand how the primes p behaves under a finite field extension F over $\mathbb{Q}$.
- Example:

$$\begin{array}{ccc} \mathbb{Q}(\sqrt{-1}) & (1-\sqrt{-1})(1+\sqrt{-1}) & 3 \\ \vdots & \vdots & \vdots \\ \mathbb{Q} & 2 & 3 \end{array}$$

Hilbert Ramification Theory

- F is a Galois field extension of $\mathbb{Q}$ of degree m.
- $\mathcal{O}$ is the ring of algebraic integers in F.
- Take a prime p:

$$\begin{array}{ccc} F & & p \cdot \mathcal{O} = \mathfrak{p}_1^e \cdot \mathfrak{p}_2^e \cdot \cdots \cdot \mathfrak{p}_r^e \\ \vdots & & \vdots \\ \mathbb{Q} & & p \end{array}$$

- The Galois group $\Gamma_{F/\mathbb{Q}}$ acts transitively on $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots \mathfrak{p}_r\}$.
- The residue field $\kappa(\mathfrak{p}_i) = \mathcal{O}/\mathfrak{p}_i \cong \mathcal{O}/\mathfrak{p}_j = \kappa(\mathfrak{p}_j)$ for any i, j.
- Let $[\kappa(\mathfrak{p}_i) : \mathbb{F}_p] = f$ with $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Then $m = e \cdot r \cdot f$.
- **Decomposition group:** $\Gamma(\mathfrak{p}_i) := \{x \in \Gamma_{F/\mathbb{Q}} \mid x \cdot \mathfrak{p}_i = \mathfrak{p}_i\}$. Then one has $\Gamma(\mathfrak{p}_i) \cong \Gamma(\mathfrak{p}_j)$.
- $1 \to \mathrm{I}_{\mathfrak{p}} \to \Gamma(\mathfrak{p}) \to \mathrm{Gal}(\kappa(\mathfrak{p})/\mathbb{F}_p) \to 1$.

Galois Group: Local-Global Structure

- The Galois group $\Gamma_{F/\mathbb{Q}}$ produces $\Gamma(\mathfrak{p})$ for each prime p.
- **E. Artin** studies the complex representation of the Galois group $\Gamma_{F/\mathbb{Q}}$, in order to understand its structure and the associated field extension $F/\mathbb{Q}$.
- $\rho \ : \ \Gamma_{F/\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$ is a goup homomorphism.
- ρ also produces a family of representations

$$\rho_p \ : \ \Gamma(\mathfrak{p}) \to \mathrm{GL}_n(\mathbb{C})$$

- The well-known **Chebotarev Density Theorem:** The **Local Symmetries** carried by $\rho_p(\Gamma(\mathfrak{p}))$ for all primes p recover the **Global Symmetries** of $\rho(\Gamma_{F/\mathbb{Q}})$.

Artin L-functions

- For almost of primes p, the local represemtation ρ_p is trivial on the inertia group $\mathrm{I}_{\mathfrak{p}}$.
- Hence $\rho_p(\Gamma(\mathfrak{p})) =< \rho(\mathrm{Frob}_{\mathfrak{p}}) >$ is a cyclic group generated by the Frobenius $\mathrm{Frob}_{\mathfrak{p}}$ at p.
- It is known that the Galois representation ρ is of Frobenius semi-simple, i.e. $\rho(\mathrm{Frob}_{\mathfrak{p}})$ is conjugate to a diagonal element in $\mathrm{GL}_n(\mathbb{C})$.
- The characteristic polynomial

$$\det(\mathrm{I}_n - \rho_p(\mathrm{Frob}_{\mathfrak{p}}))$$

determines the semi-simple conjugate class $\rho(\mathrm{Frob}_{\mathfrak{p}})$, which is also called the **Frobenius-Hecke class** of ρ.

Artin L-functions

- Since there are infinitely manu primes p, it is better to consider the Euler product

$$\Pi_p \det(\mathrm{I}_n - \rho_p(\mathrm{Frob}_{\mathfrak{p}})).$$

- The **Bad News** is this Euler product will never be convergent!
- It is **Artin** who introduces the following object, called the **Artin** L-function of ρ:

$$L(s, \rho, F/\mathbb{Q}) := \Pi_p \det(\mathrm{I}_n - \rho_p(\mathrm{Frob}_{\mathfrak{p}})p^{-s})^{-1}.$$

- $L(s, \rho, F/\mathbb{Q})$ converges absolutely for real part of s large, and has meromorphic conitnuation to $s \in \mathbb{C}$.
- **Artin Conjecture:** If ρ is irreducible, then $L(s, \rho, F/\mathbb{Q})$ has analytic continuation to an entire function in $s \in \mathbb{C}$.

Local Langlands Recirocity Conjecture

$\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the absolute Galois group and $\rho : \Gamma_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$

$$\begin{array}{cccccc}
\rho_\infty(Frob_\infty) & \rho_2(\mathrm{Frob}_2) & \rho_3(\mathrm{Frob}_3) & \cdots & \rho_p(\mathrm{Frob}_p) & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \\
\mathbb{R} & \mathbb{Q}_2 & \mathbb{Q}_3 & \cdots & \mathbb{Q}_p & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \\
\pi_\infty & \pi_2 & \pi_3 & \cdots & \pi_p & \cdots
\end{array}$$

- π_p is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$.
- The **correspondence** $\rho_p \leftrightarrow \rho_p(\mathrm{Frob}_{\mathfrak{p}}) \leftrightarrow \pi_p$ is the **local Langlands reciprocity conjecture** for GL_n.
- It was proved by R. Langlands for $\mathrm{GL}_n(\mathbb{R})$ and by M. Harris and R. Taylor, by G. Henniart, and by P. Scholze for $\mathrm{GL}_n(\mathbb{Q}_p)$.

Global Langlands Reciprocity Conjecture

- Take $\pi := \pi_\infty \otimes (\otimes_p \pi_p)$, which is an irreducible admissible unitary representation of $\mathrm{GL}_n(\mathbb{A})$.
- **Langlands Conjecture:** There is a non-trivial $\mathrm{GL}_n(\mathbb{A})$-equivariant embedding of the space V_π of π into the following space of L^2-automorphic functions

$$L^2(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$$

- It consists of functions $f : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$, such that
  1. $f(zg) = \omega(z)f(g)$ for $z \in Z(\mathbb{A})$ the center of $\mathrm{GL}_n(\mathbb{A})$;
  2. $f(\gamma g) = f(g)$ for any $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ and $g \in \mathrm{GL}_n(\mathbb{A})$;
  3. $\int_{\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})} |f(g)|^2 dg < \infty$.

Langlands Conjecture implies Artin Conjecture

- It can be checked that if π corresponding to ρ occurs in the space $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}),\omega)$, then if ρ is irreducible, the L-function $L(s,\pi)$ is entire according the work of R. Godement and H. Jacquet.
- Hence the **Artin conjecture** is true.
- It is **highly non-trivial** to prove that $\pi := \pi_\infty \otimes (\otimes_p \pi_p)$ do occur in the space $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}),\omega)$.
- For the moment, we only know some special cases for GL_2 case.
- If someone knows this for $n \geq 3$ in general, he or she should win a **Fields Medal**, if the person is younger than 40 years old.

Langlands Automorphic L-functions

- An irreducible unitary representation (V_π, π) of $\mathrm{GL}_n(\mathbb{A})$ is called **automorphic** if

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})}(V_\pi, L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega)) \neq 0.$$

- For each automorphic representation $\pi = \otimes_p \pi_p$, there exists a **Frobenius-Hecke class** $c(\pi_p)$ in $\mathrm{GL}_n(\mathbb{C})$.
- **Strong Multiplicity One Theorem** of Jacquet-Shalika, the famiy $\{c(\pi_p)\}$ determines π uniquely.
- For any $r \; : \; \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{d_r}(\mathbb{C})$ with $d_r = \dim V_r$, one defines the general Langlands L-function:

$$L(s, \pi, r) = \Pi_p L(s, \pi_p, r)$$

Langlands Conjecture

- **Langlands Theorem:** For an irreducible admissible unitary representation $\pi = \otimes_p \pi_p$ of $\mathrm{GL}_n(\mathbb{A})$, the Euler product

$$L(s, \pi, r) = \Pi_p L(s, \pi_p, r)$$

converges absolutely for $\mathrm{Re}(s)$ sufficiently large.

- **Langlands Conjecture:** If an irreducible admissible unitary representation $\pi = \otimes_p \pi_p$ of $\mathrm{GL}_n(\mathbb{A})$ occurs in

$$L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}), \omega),$$

then the L-function $L(s, \pi, r)$ enjoys the following analytic properties:

1. $L(s, \pi, r)$ has meromorphic continuation to $s \in \mathbb{C}$;
2. Functional Equation: $L(s, \pi, r) = \epsilon(s, \pi, r)L(1 - s, \pi^\vee, r)$;
3. $L(s, \pi, r)$ has finitely many poles at $s \geq \frac{1}{2}$.

Langlands Conjecture: Analytic Properties

The **Langlands Conjecture** on *analytic properties of automorphic L-functions* has been verified for many families of cases.

- **Rankin-Selberg method**, which traces back to the work of Euler and Riemann.
- **Langlands-Shahidi method**, which is based on the Langlands theory of Eisenstein series.
- **Endoscopy Classification method**, which is based on the Arthur-Selberg trace formula.
- **Braverman-Kazhdan proposal**, which is based on theory of invariant distributions and harmonic analysis.
- **Beyond Endoscopy proposal**, which is based on the Langlands beyond endoscopy proposal for more general types of functoriality.
- The Langlands conjecture can be stated for general reductive algebraic groups defined over any number fields.

Langlands Conjecture: Poles and Functoriality

- If an irreducible automorphic representation $\pi = \pi_\infty \otimes (\otimes_p \pi_p)$ of $\mathrm{GL}_n(\mathbb{A})$ is of Ramanujan type, i.e. π_p are tempered, then there exists an algebraic subgroup ${}^\lambda H_\pi$ of $\mathrm{GL}_n(\mathbb{C})$, such that for any finite dimensional representation $r \ : \ \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{d_r}(\mathbb{C})$, one has

$$-\mathrm{ord}_{s=1} L(s, \pi, r) = \dim Hom_{{}^\lambda H_\pi}(r, 1).$$

- **Theorem:** For an irreducible cuspidal automorphic representations τ of $\mathrm{GL}_{2n}(\mathbb{A})$, the exterior square L-function $L(s, \tau, \wedge^2)$ is holomorphic for $\mathrm{Re}(s) > 1$, and TFAE:
  1. $L(s, \tau, \wedge^2)$ has a simple pole at $s = 1$.
  2. τ is the Langlands transfer from $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{SO}_{2n+1})$.
- This theorem is the final statement accomplished through the work of Jacquet-Shalika, of Cogdell-Kim-PS-Shahidi, of Ginzburg-Rallis-Soudry, and finally of Jiang-Soudry.

Langlands Conjecture: Poles and Functoriality

- As representations of $\mathrm{Sp}_{2n}(\mathbb{C})$:

$$\Lambda^2(\mathbb{C}^{2n}) = \rho_2 \oplus \mathbf{1}_{\mathrm{Sp}_{2n}}$$

where ρ_2 is the second fundamental complex representation of $\mathrm{Sp}_{2n}(\mathbb{C})$, which is irreducible and has dimension $2n^2 - n - 1$, and $\mathbf{1}_{\mathrm{Sp}_{2n}}$ is the trivial representation of $\mathrm{Sp}_{2n}(\mathbb{C})$.

- **Theorem [J.]:** $L(s, \pi, \rho_2) = \frac{L(s,\tau,\Lambda^2)}{\zeta(s)}$ converges absolutely for $\mathrm{Re}(s) > 1$, has meromorphic continuation to the whole complex plane, satisfies the functional equation

$$L(s, \pi, \rho_2) = \epsilon(s, \tau, \rho_2) L(1 - s, \pi, \rho_2),$$

has possible poles at $s = 0, 1$, besides other possible poles in the open interval $(0, 1)$ (involving **GRH** and **Siegel zeros**).

Langlands Conjecture: Poles and Functoriality

- **Elliptic Endoscopy:** For $n = n_1 + n_2 + \cdots + n_r$ with $n_i > 0$,

  $$H_{[n_1 \cdots n_r]} := \mathrm{SO}_{2n_1+1} \times \mathrm{SO}_{2n_2+1} \times \cdots \times \mathrm{SO}_{2n_r+1}$$

  is an elliptic endoscopy group of SO_{2n+1} with the dual embedding:

  $$\mathrm{Sp}_{2n_1}(\mathbb{C}) \times \mathrm{Sp}_{2n_2}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{2n_r}(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C}).$$

- **Theorem [J.]:** $L(s, \pi, \rho_2)$ has a pole at $s = 1$ of **order** $r - 1$ if and only if $\exists$ a partition $n = \sum_{j=1}^{r} n_j$ with $n_j > 0$ s.t. π is an **endoscopy transfer** from an irred. cusp. autom. rep'n

  $$\pi_1 \otimes \cdots \otimes \pi_r$$

  of $H_{[n_1 \cdots n_r]}(\mathbb{A})$. ($\pi$ has a generic global Arthur parameter.)

Langlands Conjecture: Poles and Functoriality

Invariant Theory: For an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$, there exists a closed subgroup H_π^λ of $G^\vee(\mathbb{C})$, such that for $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}(V_\rho)$, the following holds:

$$\dim \mathrm{Hom}_{H_\pi^\lambda}(V_\rho, 1) = -\mathrm{ord}_{s=1} L(s, \pi, \rho).$$

Functorial Source: There exists a reductive algebraic group H defined over F with properties:

1. $\iota : H^\vee(\mathbb{C}) \subset H_\pi^\lambda \to G^\vee(\mathbb{C})$,
2. there exists an irreducible cuspidal automorphic representation σ of $H(\mathbb{A})$, which is primitive, such that π is the Langlands functorial transfer of σ.
3. $L^S(s, \pi, \rho) = L^S(s, \sigma, \rho \circ \iota)$.

Remark: My Theorem proves the **Langlands Conjecture** for $G = \mathrm{SO}_{2n+1}$ and $\rho = \rho_2$. This is one of the very few known cases.