Pseudodifferential operators and complex powers of elliptic operators on noncommutative tori

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   - Complex Powers of an Elliptic ΨDO

Part 1. Pseudodifferential Calculus on Noncommutative Tori

Noncommutative Tori

- $\theta = (\theta_{jl})$, anti-symmetric real $n \times n$ matrix ($n \geq 2$).

Definition

$A_\theta = C^*$-algebra generated by the unitaries $U_1, \ldots U_n$ obeying the relation:

$$U_k U_j = e^{2i\pi\theta_{jk}} U_j U_k, \quad j, k = 1, \ldots, n.$$

Notation: $U^k := U_1^{k_1} \cdots U_n^{k_n}$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$.

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- There is a tracial state $\tau : A_\theta \to \mathbb{C}$ such that $\tau(1) = 1$ and $\tau(U^k) = 0$ for $0 \neq k \in \mathbb{Z}^n$.
- $\mathscr{H}_\theta :=$ the completion of A_θ w.r.t. the inner product,

$$(u|v) := \tau(uv^*).$$

- $u = \sum_{k \in \mathbb{Z}^n} u_k U^k$, $u_k = \left(u \middle| U^k\right)$, the Fourier series of $u \in \mathscr{H}_\theta$.

Noncommutative Tori

- There is a continuous action of $\mathbb{R}^n$ on A_θ such that

$$\alpha_s(U^k) = e^{is\cdot k} U^k, \qquad s \in \mathbb{R}^n,\ k \in \mathbb{Z}^n.$$

Smooth noncommutative torus

$$\begin{aligned}\mathscr{A}_\theta &:= \{u \in A_\theta;\ s \to \alpha_s(u) \text{ is a smooth map from } \mathbb{R}^n \text{ to } A_\theta\} \\ &= \{u = \textstyle\sum u_k U^k \in A_\theta;\ (u_k)_{k\in\mathbb{Z}^n} \text{ decays rapidly}\}.\end{aligned}$$

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- For $j = 1, \ldots, n$ define $\delta_j : \mathscr{A}_\theta \to \mathscr{A}_\theta$ by

$$\delta_j(U^k) = -i\partial_{s_j}\alpha_s(U^k)|_{s=0} = k_j U^k, \quad k = (k_1, \ldots, k_n) \in \mathbb{Z}^n.$$

Then δ_j is a derivation on $\mathscr{A}_\theta$, i.e., $\delta_j(uv) = \delta_j(u)v + u\delta_j(v)$ for all $u, v \in \mathscr{A}_\theta$.

Noncommutative Tori

- We equip $\mathscr{A}_\theta$ with the locally convex topology generated by the semi-norms,

$$u \longrightarrow \|\delta^\alpha u\|, \qquad \alpha \in \mathbb{N}_0^n.$$

$\mathscr{A}_\theta$ is a Fréchet $*$-algebra with respect to this topology.

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NC Torus	Ordinary Torus ($\theta = 0$)
A_θ	$C(\mathbb{T}^n)$
$\mathscr{A}_\theta$	$C^\infty(\mathbb{T}^n)$
U_j $(j = 1, \ldots, n)$	the function $x \to e^{ix_j}$
U^k $(k \in \mathbb{Z}^n)$	the function $x \to e^{ix\cdot k}$
δ_j $(j = 1, \ldots, n)$	$D_{x_j} := -i\partial_{x_j}$
$\delta^\alpha := \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}$ $(\alpha \in \mathbb{N}_0^n)$	$D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$
the trace τ	the integral on $\mathbb{T}^n$
$\mathscr{H}_\theta$	$L^2(\mathbb{T}^n)$

Differential Operators

Definition (Connes)

A differential operator of order m on $\mathscr{A}_\theta$ is a linear operator $P : \mathscr{A}_\theta \to \mathscr{A}_\theta$ of the form,

$$Pu = \sum_{|\alpha| \leq m} a_\alpha \delta^\alpha u, \qquad a_\alpha \in \mathscr{A}_\theta.$$

The symbol of P is the map $\rho : \mathbb{R}^n \to \mathscr{A}_\theta$ defined by $\rho(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$, $\xi \in \mathbb{R}^n$.

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- Example 1: The (flat) Laplacian $\Delta := \delta_1^2 + \cdots + \delta_n^2$.
- Example 2: (Connes-Tretkoff) Conformally deformed Laplacian $\Delta_k := k^{-1} \Delta k^{-1}$ on a noncommutative 2-torus $\mathscr{A}_\theta$. Here $k \in \mathscr{A}_\theta$ is positive and invertible.
- Using the Fourier inversion formula we see that

$$Pu = \iint e^{is \cdot \xi} \rho(\xi) \alpha_{-s}(u) ds đ\xi \qquad \text{for all } u \in \mathscr{A}_\theta.$$

Motivation

- We can get much geometric information about a compact Riemannian manifold (M, g) from the zeta function $\zeta(s) := \mathrm{Tr}(\Delta_g^{-s})$, where Δ_g is the Laplacian associated with g.
- e.g., $M =$ compact Riemann surface $\Rightarrow \zeta(0) = \frac{1}{6}\chi(M) - \dim\ker\Delta_g$.

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- In 2009, Connes-Tretkoff proved the following version of Gauss-Bonnet theorem for a noncommutative 2-torus $\mathscr{A}_\theta$: the value of the zeta function $\zeta(s; \Delta_k) := \mathrm{Tr}(\Delta_k^{-s})$ at $s = 0$ is independent of the choice of positive and invertible element $k \in \mathscr{A}_\theta$.

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- The following tools play a central role in Connes-Tretkoff's paper and its subsequent results.
  - Pseudodifferential operators (ΨDOs) on noncommutative tori.
  - Parametric ΨDOs on noncommutative tori.
  - Holomorphic families of ΨDOs.
  - Complex powers of an elliptic ΨDO.

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  - Parametric ΨDOs on noncommutative tori.
  - Holomorphic families of ΨDOs.
  - Complex powers of an elliptic ΨDO.
- The aim of this mini-course is to give detailed accounts on these notions as an aid in researches initiated by Connes-Tretkoff.

Symbols on Noncommutative Tori

- (Standard symbols, Baaj, Connes) $\mathbb{S}^m(\mathbb{R}^n; \mathscr{A}_\theta)$, $m \in \mathbb{R}$, consists of maps $\rho(\xi) \in C^\infty(\mathbb{R}^n; \mathscr{A}_\theta)$ such that, $\forall \alpha, \beta \in \mathbb{N}_0^n$, $\exists C_{\alpha\beta} > 0$ such that

$$\|\delta^\alpha \partial_\xi^\beta \rho(\xi)\| \leq C_{\alpha\beta}(1 + |\xi|)^{m-|\beta|} \qquad \forall \xi \in \mathbb{R}^n.$$

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- (Homogeneous symbols) $S_q(\mathbb{R}^n; \mathscr{A}_\theta)$, $q \in \mathbb{C}$, consists of smooth maps $\rho : \mathbb{R}^n \setminus 0 \to \mathscr{A}_\theta$ such that $\rho(\lambda\xi) = \lambda^q \rho(\xi)$ $\forall \xi \in \mathbb{R}^n \setminus 0$ $\forall \lambda > 0$.

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- (Classical Symbols, Baaj) $S^q(\mathbb{R}^n; \mathscr{A}_\theta)$, $q \in \mathbb{C}$, consists of maps $\rho(\xi) \in C^\infty(\mathbb{R}^n; \mathscr{A}_\theta)$ that admit an expansion $\rho(\xi) \sim \sum_{j \geq 0} \rho_{q-j}(\xi)$, $\rho_{q-j}(\xi) \in S_{q-j}(\mathbb{R}^n; \mathscr{A}_\theta)$. Here $\sim$ means that, $\forall N \geq 1$ $\forall \alpha, \beta \in \mathbb{N}_0^n$, $\exists C_{N\alpha\beta} > 0$ such that
$$\left\| \delta^\alpha \partial_\xi^\beta \Big(\rho - \sum_{j<N} \rho_{q-j} \Big)(\xi) \right\| \leq C_{N\alpha\beta} |\xi|^{\Re q - N - |\beta|} \quad \forall \xi \in \mathbb{R}^n \text{ with } |\xi| \geq 1.$$
In this case $\rho_q(\xi)$ is called the principal symbol of $\rho(\xi)$.
- We have an inclusion $S^q(\mathbb{R}^n; \mathscr{A}_\theta) \subset \mathbb{S}^{\Re q}(\mathbb{R}^n; \mathscr{A}_\theta)$.

Amplitudes and Oscillating Integrals

Let $m \in \mathbb{R}$.

$\mathscr{A}_\theta$-valued Amplitudes

$A^m = A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathscr{A}_\theta)$ consists of maps $a(s,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathscr{A}_\theta)$ such that, for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, there is $C_{\alpha\beta\gamma} > 0$ such that

$$\|\delta^\alpha \partial_s^\beta \partial_\xi^\gamma a(s,\xi)\| \leq C_{\alpha\beta\gamma}(1 + |s| + |\xi|)^m \qquad \forall (s,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

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- Let $\chi(s,\xi) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\chi(s,\xi) = 1$ near $(0,0)$. Set

$$L = \chi(s,\xi) + \frac{1 - \chi(s,\xi)}{|s|^2 + |\xi|^2} \sum_{1 \leq j \leq n} (\xi_j D_{s_j} + s_j D_{\xi_j}),$$

  where $D_{x_j} = \frac{1}{i}\partial_{x_j}$.
- Observe that $L(e^{is\cdot\xi}) = e^{is\cdot\xi}$.
- Define the transpose of L by $\iint L(f)g = \iint f\ L^t(g)$.
  $\Rightarrow L^t$ gives rise to a continuous linear map from A^m to A^{m-1}.

Amplitudes and Oscillating Integrals

Let $a(s,\xi) \in A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathscr{A}_\theta)$.

- If $m < -2n$, the map $(s,\xi) \to e^{is\cdot\xi}a(s,\xi)$ is integrable on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, for all $N \geq 0$, we have

$$\iint e^{is\cdot\xi}a(s,\xi)ds\,đ\xi = \iint L^N[e^{is\cdot\xi}]a(s,\xi)ds\,đ\xi$$
$$= \iint e^{is\cdot\xi}(L^t)^N[a(s,\xi)]ds\,đ\xi.$$

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- (Oscillating integrals) For general $m \in \mathbb{R}$, we define

$$J(a) = \iint e^{is\cdot\xi} (L^t)^N [a(s,\xi)] ds đ\xi,$$

  where N is any non-negative integer such that $m - N < -2n$. Here the choice of N is irrelevant. When $m < -2n$ we may take $N = 0$. In this case we have $J(a) = \iint e^{is\cdot\xi} a(s,\xi) ds đ\xi$.

ΨDOs on NC Tori

If $\rho(\xi) \in \mathbb{S}^m(\mathbb{R}^n; \mathscr{A}_\theta)$ and $u \in \mathscr{A}_\theta$, then $\rho(\xi)\alpha_{-s}(u) \in A^{m_+}(\mathbb{R}^n \times \mathbb{R}^n; \mathscr{A}_\theta)$, where $m_+ := \max(m, 0)$.

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Definition

1. A pseudodifferential operator (ΨDO) associated with $\rho(\xi) \in \mathbb{S}^m(\mathbb{R}^n; \mathscr{A}_\theta)$, $m \in \mathbb{R}$, is a linear map $P_\rho : \mathscr{A}_\theta \to \mathscr{A}_\theta$ defined by
$$P_\rho u = J(\rho(\xi)\alpha_{-s}(u)).$$
2. $\Psi^q(\mathscr{A}_\theta)$, $q \in \mathbb{C}$, consists of all linear operators $P : \mathscr{A}_\theta \to \mathscr{A}_\theta$ that are of the form $P = P_\rho$ for some symbol $\rho(\xi) \in S^q(\mathbb{R}^n; \mathscr{A}_\theta)$.

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- (Connes-Tretkoff) Let $\rho(\xi) \in \mathbb{S}^m(\mathbb{R}^n; \mathscr{A}_\theta)$, $m \in \mathbb{R}$. Then for every $u = \sum_{k \in \mathbb{Z}^n} u_k U^k \in \mathscr{A}_\theta$, we have
$$P_\rho u = \sum_{k \in \mathbb{Z}^n} u_k \rho(k) U^k.$$

Smoothing Operators

$\mathscr{A}'_\theta := \{v : \mathscr{A}_\theta \to \mathbb{C} : \ v \text{ is continuous and linear}\}$.
We equip $\mathscr{A}'_\theta$ with its strong topology, i.e., the LCS topology generated by the semi-norms,

$$v \longrightarrow \sup_{u \in B} |\langle v, u \rangle|, \qquad B \subset \mathscr{A}_\theta \text{ bounded.}$$

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We have continuous inclusions $\mathscr{A}_\theta \subset \mathscr{H}_\theta \subset \mathscr{A}'_\theta$.

- A linear operator $R : \mathscr{A}_\theta \to \mathscr{A}'_\theta$ is called smoothing when it extends to a continuous linear operator $R : \mathscr{A}'_\theta \to \mathscr{A}_\theta$.
- $\Psi^{-\infty}(\mathscr{A}_\theta) :=$ the space of smoothing operators.

Proposition (Ha-L.-Ponge)

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Proposition (Ha-L.-Ponge)

- *A linear operator $R : \mathscr{A}_\theta \to \mathscr{A}_\theta'$ is smoothing if and only if there is a symbol $\rho(\xi) \in \mathscr{S}(\mathbb{R}^n; \mathscr{A}_\theta)$ such that $R = P_\rho$.*
- *Every ΨDO $P : \mathscr{A}_\theta \to \mathscr{A}_\theta$ uniquely extends to a continuous linear map $P : \mathscr{A}_\theta' \to \mathscr{A}_\theta'$.*

Composition of ΨDOs

Let $\rho_j(\xi) \in \mathbb{S}^{m_j}(\mathbb{R}^n; \mathscr{A}_\theta)$, $m_j \in \mathbb{R}$, $j = 1, 2$. Set

$$\rho_1 \sharp \rho_2(\xi) = \iint e^{it\cdot\eta} \rho_1(\xi + \eta) \alpha_{-t}[\rho_2(\xi)] dt đ\eta.$$

The integral on the right-hand side makes sense as an oscillating integral. It can be shown that $\rho_1 \sharp \rho_2(\xi) \in \mathbb{S}^{m_1+m_2}(\mathbb{R}^n; \mathscr{A}_\theta)$ and $P_{\rho_1} P_{\rho_2} = P_{\rho_1 \sharp \rho_2}$.

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Proposition

Let $P_j \in \Psi^{q_j}(\mathscr{A}_\theta)$, $q_j \in \mathbb{C}$, $j = 1, 2$. In addition, let $\rho(\xi)$ and $\sigma(\xi)$ be the respective principal symbols of P_1 and P_2. Then

1. $P_1 P_2 \in \Psi^{q_1+q_2}(\mathscr{A}_\theta)$.
2. $\rho(\xi)\sigma(\xi)$ *is the principal symbol of* $P_1 P_2$.

Sobolev Spaces on NC Tori

Let $s \in \mathbb{R}$. Then $\Lambda^s := (1+\Delta)^{\frac{s}{2}} \in \Psi^s(\mathscr{A}_\theta)$ and Λ^s has symbol $(1+|\xi|^2)^{\frac{s}{2}}$.

Definition (Sobolev Spaces)

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- Given any $s' > s$, the inclusion $\mathscr{H}_\theta^{(s')} \subset \mathscr{H}_\theta^{(s)}$ is compact.

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- Given any $s' > s$, the inclusion $\mathscr{H}_\theta^{(s')} \subset \mathscr{H}_\theta^{(s)}$ is compact.
- Let $\rho(\xi) \in \mathbb{S}^m(\mathbb{R}^n; \mathscr{A}_\theta)$, $m \in \mathbb{R}$. For every $s \in \mathbb{R}$, P_ρ uniquely extends to a continuous linear map $P_\rho : \mathscr{H}_\theta^{(s+m)} \to \mathscr{H}_\theta^{(s)}$. In particular, if $m = 0$ then P_ρ gives rise to a bounded operator on $\mathscr{H}_\theta$.
- (Baaj, Connes) If $m < 0$, then P_ρ gives rise to a compact operator $P_\rho : \mathscr{H}_\theta \to \mathscr{H}_\theta$.

Trace-Class Property of ΨDOs

1. $\Delta = \delta_1^2 + \cdots + \delta_n^2$ is isospectral to the flat Laplacian on $\mathbb{T}^n$.
2. Weyl's law: $\lambda_k(\Delta)(ck^{-1})^{\frac{2}{n}} \to 1$ as $k \to \infty$. Here $c := \pi^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)^{-1}$.

Trace-Class Property of ΨDOs

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Proposition (Ha-L.-Ponge)

Let $m < -n$. Then for every $\rho(\xi) \in \mathbb{S}^m(\mathbb{R}^n; \mathscr{A}_\theta)$, P_ρ is trace-class, and its trace is given by

$$\mathrm{Tr}(P_\rho) = \sum_{k \in \mathbb{Z}^n} \tau[\rho(k)].$$

Part 2. Resolvents and Complex Powers of Elliptic ΨDOs on NC Tori

Elliptic ΨDOs on NC Tori

Definition

$P \in \Psi^q(\mathscr{A}_\theta)$, $q \in \mathbb{C}$, is called elliptic when its principal symbol $\rho_q(\xi)$ is invertible for all $\xi \in \mathbb{R}^n \setminus 0$.

- Example 1: The (flat) Laplacian $\Delta := \delta_1^2 + \cdots + \delta_n^2$.
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$P \in \Psi^q(\mathscr{A}_\theta)$, $q \in \mathbb{C}$, is elliptic if and only if it admits a parametrix, i.e., an operator $Q \in \Psi^{-q}(\mathscr{A}_\theta)$ such that $PQ = QP = 1 \bmod \Psi^{-\infty}(\mathscr{A}_\theta)$.

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Consequences

1. Let $s \in \mathbb{R}$. Then, for any $u \in \mathscr{A}'_\theta$, $Pu \in \mathscr{H}_\theta^{(s)} \Leftrightarrow u \in \mathscr{H}_\theta^{(s+\Re q)}$.
2. For every $s \in \mathbb{R}$, $P : \mathscr{H}_\theta^{(s+m)} \to \mathscr{H}_\theta^{(s)}$ is a Fredholm operator.

Spectra and Partial Inverses of Elliptic ΨDOs

Let $P \in \Psi^q(\mathscr{A}_\theta)$ be an elliptic ΨDO with $m := \Re q > 0$.

- The resolvent set of $P := \{\lambda \in \mathbb{C};\ P - \lambda : \mathscr{H}_\theta^{(m)} \to \mathscr{H}_\theta \text{ is bijective}\}$.
- Sp P := the complement of the resolvent set.

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Suppose Sp $P \neq \mathbb{C}$. Then the map $\mathbb{C} \setminus \mathrm{Sp}\, P \ni \lambda \to (P - \lambda)^{-1} \in \mathscr{L}(\mathscr{H}_\theta, \mathscr{H}_\theta^{(m)})$ is holomorphic. The root space $E_\lambda(P)$ and Riesz projection $\Pi_\lambda(P)$ of $\lambda \in \mathrm{Sp}\, P$ are defined by

$$E_\lambda(P) = \bigcup_{\ell \geq 0} \ker(P - \lambda)^\ell, \qquad \Pi_\lambda(P) = \frac{1}{2i\pi} \int_{|\zeta - \lambda| = r} (\zeta - P)^{-1} d\zeta.$$

Here r is small enough so that $\{\zeta \in \mathbb{C};\ |\zeta - \lambda| \leq r\} \cap \mathrm{Sp}\, P = \{\lambda\}$.

Spectra and Partial Inverses of Elliptic ΨDOs

- $\Pi_\lambda(P)^2 = \Pi_\lambda(P)$ and $\Pi_\lambda(P)\Pi_\mu(P) = 0$ if $\lambda \neq \mu$.
- $E_\lambda(P)$ is a finite dimensional subspace of $\mathscr{A}_\theta$. In particular, there is $N \in \mathbb{N}$ such that $E_\lambda(P) = \ker(P - \lambda)^N$.

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- $\Pi_\lambda(P)$ is a projection onto $E_\lambda(P)$ with kernel $E_{\overline{\lambda}}(P^*)^\perp$. In particular, we have direct-sum decomposition $\mathscr{H}_\theta = E_\lambda(P) \dotplus E_{\overline{\lambda}}(P^*)^\perp$.
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- P induces a linear homeomorphism $P_1 : E_0(P^*)^\perp \cap \mathscr{H}_\theta^{(m)} \to E_0(P^*)^\perp$.

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Definition

The partial inverse of P is the operator $P^{-1} : \mathscr{H}_\theta \to \mathscr{H}_\theta^{(m)}$ defined by

- $P^{-1} = 0$ on $E_0(P)$.
- $P^{-1}u = P_1^{-1}u$ for all $u \in E_0(P^*)^\perp$.

We have $PP^{-1} = 1 - \Pi_0(P)$ on $\mathscr{H}_\theta$ and $P^{-1}P = 1 - \Pi_0(P)$ on $\mathscr{H}_\theta^{(m)}$.

Pseudo-cones and Vectors with Parameter

- In what follows, we let $\Lambda \subset \mathbb{C}$ be an open pseudo-cone, i.e., Λ is of the form $\Lambda = \Theta \setminus D$. Here Θ is an open cone in $\mathbb{C} \setminus 0$ about the negative real axis and D is the closed disk at the origin.
- For open pseudo-cones Λ_1 and Λ_2, we denote by $\Lambda_1 \subset\subset \Lambda_2$ if $\overline{\Lambda_1} \subset \Lambda_2$.

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Definition

Let E be a locally convex space. $\mathrm{Hol}^d(\Lambda, E)$, $d \in \mathbb{Z}$, consists of holomorphic families $(x(\lambda))_{\lambda\in\Lambda}$ with values in E such that, for all continuous semi-norm p on E and pseudo-cones $\Lambda' \subset\subset \Lambda$, there is $C_{p\Lambda'} > 0$ such that

$$p(x(\lambda)) \leq C_{p\Lambda'}(1 + |\lambda|)^d \qquad \forall \lambda \in \Lambda'.$$

ΨDOs with Parameter

In what follows, let $w > 0$, $q \in \mathbb{C}$, $d \in \mathbb{Z}$. Set $d_- = \sup(0, -d)$ and $m = \Re q + wd_-$.

- Notation: $C^{\infty,d}(\mathbb{R}^n \times \Lambda; \mathscr{A}_\theta) := \mathrm{Hol}^d(\Lambda, C^\infty(\mathbb{R}^n; \mathscr{A}_\theta))$.

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Definition (Classical Symbols with Parameter)

$S^{q,d}(\mathbb{R}^n \times \Lambda; \mathscr{A}_\theta)$ consists of maps $\rho(\xi; \lambda) \in C^{\infty,d}(\mathbb{R}^n \times \Lambda; \mathscr{A}_\theta)$ for which there are maps $\rho_{q-j} : (\mathbb{R}^n \setminus 0) \times \Theta \to \mathscr{A}_\theta$, $j \geq 0$, such that

- $\rho_{q-j}(t\xi; t^w \lambda) = t^{q-j} \rho_{q-j}(\xi; \lambda) \quad \forall t > 0 \quad \forall (\xi, \lambda) \in (\mathbb{R}^n \setminus 0) \times \Theta$.
- $\rho_{q-j}(\xi; \lambda)$ is smooth w.r.t. ξ and holomorphic w.r.t. λ.
- $\forall N \in \mathbb{N} \quad \forall \Lambda' \subset\subset \Lambda$ and $\forall \alpha, \beta \in \mathbb{N}_0^n$, $\exists C_{N\Lambda'\alpha\beta} > 0$ such that, for all $\lambda \in \Lambda'$ and $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$, we have

$$\left\| \delta^\alpha \partial_\xi^\beta \Big(\rho - \sum_{j<N} \rho_{q-j} \Big)(\xi; \lambda) \right\| \leq C_{N\Lambda'\alpha\beta} (1 + |\lambda|)^d |\xi|^{m-N-|\beta|}.$$

ΨDOs with Parameter and the Agmon Pseudo-cone

Definition (ΨDOs with Parameter)

$$\Psi^{q,d}(\mathscr{A}_\theta;\Lambda) := \{(P_{\rho(\cdot;\lambda)})_{\lambda\in\Lambda};\ \rho(\xi;\lambda) \in S^{q,d}(\mathbb{R}^n \times \Lambda;\mathscr{A}_\theta)\}.$$

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Proposition (L.-Ponge)

$$P_j(\lambda) \in \Psi^{q_j,d_j}(\mathscr{A}_\theta;\Lambda) \Rightarrow P_1(\lambda)P_2(\lambda) \in \Psi^{q_1+q_2,d_1+d_2}(\mathscr{A}_\theta;\Lambda).$$

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- $\breve{\Theta}(P) :=$ the complement of the closed cone described above.
- $r := \inf\{\lambda \in \mathbb{C};\ \lambda \in \operatorname{Sp} P,\ \lambda \neq 0\}$.
- $\Lambda(P) := \breve{\Theta}(P) \setminus \overline{D(0,R)}$, where $R := \frac{1}{2}r$.

The Resolvent of an Elliptic ΨDO

- Let P be an elliptic differential operator of order m with principal symbol $\rho_m(\xi)$.
- $P - \lambda \in \Psi^{m,1}(\mathscr{A}_\theta; \Lambda(P))$ and $P - \lambda$ has principal symbol $\rho_m(\xi) - \lambda$.

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Moreover, the following holds.

Proposition (L.-Ponge)

$(P - \lambda)^{-1} \in \Psi^{-m,-1}(\mathscr{A}_\theta; \Lambda(P))$. *Moreover* $(P - \lambda)^{-1}$ *has principal symbol* $(\rho_m(\xi) - \lambda)^{-1}$.

Holomorphic Families of ΨDOs

Let Ω be an open subset of $\mathbb{C}$.

Definition (Fathi-Ghorbanpour-Khalkhali)

$(\rho(z)(\xi))_{z\in\Omega} \subset S^*(\mathbb{R}^n; \mathscr{A}_\theta)$ is said to be a holomorphic family when:

1. The order $w(z)$ of $\rho(z)(\xi)$ depends analytically on z.
2. $z \to \rho(z)(\xi)$ is a holomorphic map from Ω to $C^\infty(\mathbb{R}^n; \mathscr{A}_\theta)$.
3. $\rho(z)(\xi) \sim \sum_{j\geq 0} \rho(z)_{w(z)-j}(\xi)$, $\rho(z)_{w(z)-j}(\xi) \in S_{w(z)-j}(\mathbb{R}^n; \mathscr{A}_\theta)$.

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Definition (Holomorphic Families of ΨDOs)

$\mathrm{Hol}(\Omega, \Psi^*(\mathscr{A}_\theta)) := \{(P_{\rho(z)})_{z\in\Omega};\ (\rho(z)(\xi))_{z\in\Omega} \in \mathrm{Hol}(\Omega, S^*(\mathbb{R}^n; \mathscr{A}_\theta))\}$.

- $P(z), Q(z) \in \mathrm{Hol}(\Omega, \Psi^*(\mathscr{A}_\theta)) \Rightarrow P(z)Q(z) \in \mathrm{Hol}(\Omega, \Psi^*(\mathscr{A}_\theta))$.

Complex Powers of an Elliptic ΨDO

- Let Γ be the contour in $\Lambda(P)$ of the form $\Gamma = \Gamma_1^{(-)} \cup \Gamma_2 \cup \Gamma_1^{(+)}$, where

$$\Gamma_1^{(-)} = (\infty, r_1 e^{i(2\pi - \phi)}],$$
$$\Gamma_2 = \{\lambda \in \mathbb{C};\ |\lambda| = r_1,\ \phi \leq \arg \lambda \leq 2\pi - \phi\},$$
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- We orient Γ in clockwise direction.

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Definition (Complex Powers of an Elliptic ΨDO)

We define a family of operators $(P^s)_{\Re s<0}$ by

$$P^s = \frac{1}{2i\pi}\int_\Gamma \lambda^s (P-\lambda)^{-1} d\lambda, \qquad \Re s < 0.$$

Complex Powers of an Elliptic ΨDO

$$P^s = \frac{1}{2i\pi}\int_\Gamma \lambda^s (P-\lambda)^{-1} d\lambda, \qquad \Re s < 0.$$

Recall that $(P-\lambda)^{-1} \in \Psi^{-m,-1}(\mathscr{A}_\theta; \Lambda(P))$.

- Using the result that $(P-\lambda)^{-1} \in \Psi^{-m,-1}(\mathscr{A}_\theta; \Lambda(P))$ we can compute the symbol of P^s, and prove that $(P^s)_{\Re s<0}$ gives rise to a holomorphic family of ΨDOs of order ms.
- For general $s \in \mathbb{C}$, we define P^s as the ΨDO such that $P^s = P^k P^{s-k}$, where k is any positive integer $> \Re s$. Here the choice of k is irrelevant.

Complex Powers of an Elliptic ΨDO

Theorem (L.-Ponge)

$(P^s)_{s\in\mathbb{C}} \in \mathrm{Hol}(\mathbb{C}, \Psi^*(\mathscr{A}_\theta))$ *and* $\mathrm{ord} P^s = ms$. *Moreover, we have*

- $P^{s_1+s_2} = P^{s_1}P^{s_2}$ $\forall s_j \in \mathbb{C}$, $j = 1, 2$.
- P^{-k} = *the* k*th power of the partial inverse of* P $\forall k \in \mathbb{N}$.
- P^ℓ *(in the sense of complex powers)* $= (1 - \Pi_0(P))P^\ell$ $\forall \ell \in \mathbb{N}$.
- P^0 *(in the sense of complex powers)* $= 1 - \Pi_0(P)$.

Thank you for your attention!