Special Kähler Geometry of the Hitchin System

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Joint work with David Baraglia

Hitchin Integrable System

The notions of Hitchin systems were introduced by N. Hitchin in 1987. They rapidly formed a subject lying on the crossroads of representation theory, symplectic geometry, and algebraic geometry.

• A stable Higgs Bundle is a solution to Yang-Mill's equation.
• Applications in Super Symmetric Field Theory.
• A conjecture about the Hyper-Kähler metric on $\mathcal{M}$ (Gaiotto, Moore and Neitzke).

Moduli space of Higgs bundles

In this talk, let

1. Σ be a compact Riemann surface with genus g ($g \geq 2$);
2. E be a rank n degree d holomorphic vector bundle;
3. A holomorphic section $\Phi \in \mathrm{H}^0(\Sigma, \mathrm{End}(E) \otimes K)$ is called a **Higgs field** of E, where K is the canonical line bundle with total space $T^*\Sigma$.

A pair (E, Φ) is called a **Higgs bundle**. A Higgs bundle is stable if for any sub-bundle F of E such that $\Phi F \subseteq F \otimes K$, we have

$$\frac{deg(F)}{rank(F)} < \frac{deg(E)}{rank(E)}.$$

Moduli space of Higgs bundles

We say (E_1, Φ_1) and (E_2, Φ_2) are isomorphic if there exists a vector bundle isomorphism $g : E_1 \to E_2$ such that the following diagram commutes.

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi_1} & E_1 \otimes K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g \otimes id_K} \\ E_2 & \xrightarrow{\Phi_2} & E_2 \otimes K \end{array}$$

The moduli space $\mathcal{M}$ of stable rank n degree d Higgs bundles is the quotient of the set of stable Higgs bundles by the group of isomorphisms.

Moduli space of Higgs bundles

A Dolbeault operator on a vector bundle $E \to \Sigma$ is a $\mathbb{C}$-linear map

$$\bar{\partial}_E : \Omega^0(\Sigma; E) \to \Omega^{0,1}(\Sigma; E),$$

satisfying the following rule: $\bar{\partial}_E(fs) = (\bar{\partial} f)s + f(\bar{\partial}_E s)$, for any $f \in C^\infty(\Sigma; \mathbb{C})$ and $s \in \Omega^0(\Sigma; E)$.

Let $\mathrm{Dol}(E)$ be the space of all Dolbeault operators on E, and $\mathcal{G}_E$ be the group of automorphisms of E, $g \in \mathcal{G}_E$ acts on $\bar{\partial}_E$ by $(g \cdot \bar{\partial}_E)(s) := g(\bar{\partial}_E(g^{-1}s))$.

Theorem

Let $Hol(E)$ be the space of holomorphic structures on E, then there are bijections:

$$Hol(E) \longleftrightarrow Dol(E),$$
$$Hol(E)/isomorphism \longleftrightarrow Dol(E)/\mathcal{G}_E.$$

The mappings send a holomorphic structure $\mathcal{E}$ on E to a Dolbeault operator $\bar{\partial}_E$ such that $\bar{\partial}_E(s) = 0$ for all holomorphic sections $s \in \mathcal{E}$.

A symplectic form of $\mathcal{M}$

The moduli space $\mathcal{M}$ is a symplectic manifold. The tangent space at a point (E, Φ) is given by the following

$$\mathrm{T}_{(E,\Phi)}\mathcal{M} = \frac{H}{\{(\alpha, \varphi) \mid \alpha = -\bar{\partial}_E(\lambda), \quad \varphi = [\lambda, \Phi] \ \}}.$$

where $H := \{(\alpha, \varphi) \mid [\alpha, \Phi] + \bar{\partial}_E \varphi = 0 \quad \}$,
$\alpha \in \Omega^{0,1}(\Sigma; \mathrm{End}(E))$,
$\varphi \in \Omega^{1,0}(\Sigma; \mathrm{End}(E))$,
$\lambda \in \Omega^{0}(\Sigma; \mathrm{End}(E))$.

The symplectic form of $\mathcal{M}$ is defined by[Hit87]

$$\Omega((\alpha_1, \varphi_1), (\alpha_2, \varphi_2)) = \int_\Sigma \mathrm{tr}(\alpha_1 \wedge \varphi_2) - \mathrm{tr}(\alpha_2 \wedge \varphi_1).$$

Hitchin system

The characteristic polynomial:

$$p_a(x) = \det(xI - \Phi) = x^n + a_1 x^{n-1} + ... + a_n$$

where $a_i \in \mathrm{H}^0(\Sigma; K^{\otimes i})$, $a = (a_1, ..., a_n) \in B := \bigoplus_{i=1}^{n} \mathrm{H}^0(\Sigma; K^{\otimes i})$, $x \in T^*\Sigma$. Note that $p_a : T^*\Sigma \to (T^*\Sigma)^n$. The Hitchin map is defined to be:

$$h : \mathcal{M} \to B$$
$$(E, \Phi) \mapsto a$$

Theorem

[Hit87] $(\mathcal{M}, h)$ is an integrable system, known as Hitchin system. We call B the base of the Hitchin system.

The fibers of the Hitchin system

Let B^{reg} be the set of regular values of h, note that $p_a : T^*\Sigma \to (T^*\Sigma)^n$ is a section of $\pi^*(K^n)$ where $\pi : T^*\Sigma \to \Sigma$. The spectral curve S_a of a point a is defined to be $p_a^{-1}(\mathbf{0})$, where $\mathbf{0}$ is the zero section.

- S_a is a R.S. and $S_a \xrightarrow{\pi} \Sigma$ is an n-fold branched covering.
- The regular fiber of the Hitchin system is the Jacobian variety $\mathrm{Jac}(S_a)$ of S_a [Hitchin, 1987].
- B^{reg} has a Kähler metric called special Kähler metric.

The Kähler metric is given in the following way: Let $\{a_1, ..., a_{gS}, b_1, ..., b_{gS}\}$ be a symplectic basis of $\mathrm{H}_1(S_a; \mathbb{Z})$, $\{\omega_1, ..., \omega_{g_s}\}$ be a basis of holomorphic 1-forms of S_a such that $\int_{a_i} \omega_j = \delta_{ij}$, then the b-periods of S_a are

$$\tau_{ij} = \int_{b_j} \omega_i.$$

Special Kähler metric of B

Write $\omega = \Omega_1 + i\Omega_2$, where Ω_1, Ω_2 are real symplectic forms, and $\{x_1, ..., x_{2g_s}, y_1, ..., y_{2g_S}\}$ be real local coordinates of $\mathcal{M}$ with $\{y_i\}$ the coordinates of B^{reg}, $\{x_i\}$ the coordinates of the fibers . Then

$$\Omega_1 = \sum a_{ij} dx_i \wedge dy_j + \sum b_{ij} dy_i \wedge dy_j.$$

Choose a homology class $A \in \mathrm{H}_1(\mathrm{Jac}(S), \mathbb{Z}) \cong \mathrm{H}_1(S, \mathbb{Z})$, we obtain a one-form α_1 on B^{reg}

$$\alpha_1 = \sum (\int_A a_{ij} dx_i) \wedge dy_j.$$

Since Ω_1 is closed, α_1 is also closed, hence locally there is a function $u_A : B^{reg} \to \mathbb{R}$ such that $du_A = \alpha_1$. Hence we have a map

$$u : B^{reg} \to \mathrm{H}^1(S, \mathbb{R}).$$

By using Ω_2, we have another map $v : B^{reg} \to \mathrm{H}^1(S, \mathbb{R})$.

Special Kähler metric

[Hit99] u, v are local diffeomorphisms. The image of the map

$$(u, v) : B^{reg} \to \mathrm{H}^1(S, \mathbb{R}) \times \mathrm{H}^1(S, \mathbb{R})$$

is a Lagrangian submanifold with respect to the following symplectic forms on $\mathrm{H}^1(S, \mathbb{R}) \times \mathrm{H}^1(S, \mathbb{R})$:

$$\widetilde{\omega}_1((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \int_S \alpha_1 \wedge \beta_2 + \int_S \alpha_2 \wedge \beta_1$$

$$\widetilde{\omega}_2((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \int_S \alpha_1 \wedge \beta_1 - \int_S \alpha_2 \wedge \beta_2.$$

Theorem

The restriction of the metric

$$g((\alpha, \beta), (\alpha, \beta)) = \frac{1}{2} \int_S \alpha \wedge \beta$$

to B^{reg} is a special Kähler metric of B^{reg}.

Two sets of local coordinates of B^{reg}

Let θ be the canonical one-form on S_a, then there is a pair of local holomorphic coordinate systems $\{z^1, ..., z^{gS}\}$ and $\{w_1, ..., w_{gS}\}$ for B^{reg} given by:

$$z^i = \int_{a_i} \theta, \quad w_i = \int_{b_i} \theta.$$

Properties of the affine coordinates [Fre99]:

- The special Kähler connection ∇ on B^{reg} is trivial in the real coordinate system $\{Re(z^1), ..., Re(z^{g_s}), Re(w_1), ..., Re(w_{g_s})\}$;
- The Kähler form: $\omega = \frac{\sqrt{-1}}{2} Im(\tau_{ij}) dz^i \wedge d\bar{z}^j = d(Rez_i) \wedge dRe(y_i)$;
- $\tau_{ij} = \frac{\partial w_j}{\partial z^i}$;

The Donagi-Markmann cubic $c_{ijk} := \frac{\partial \tau_{ij}}{\partial z^k}$ is symmetric in i, j, k.

Special Kähler geometry of B^{reg}

In these coordinates, the special Kähler metric on B^{reg} is:

$$g_{sk} = \frac{1}{2}\sum_{i,j}^{g_s} \mathrm{Im}(\tau_{ij})(ds_1^i \otimes ds_1^j + ds_2^i \otimes ds_2^j),$$

where $z_j = s_1^j + is_2^j$.
We can see that the Kähler form and special Kähler metric depend on the periods of the spectral curves.

Theorem

[BH17]For any covering $S_a \to \Sigma$ with only simple branch points, the Donagi-Markman cubic can be computed by

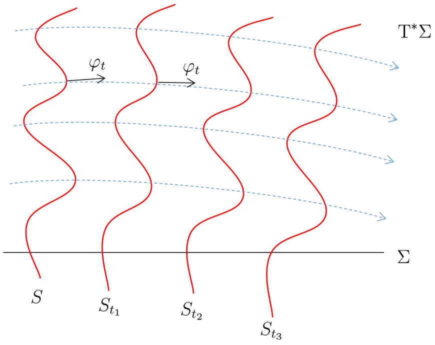
$$\frac{\partial \tau_{jk}}{\partial z_i} = -2\pi i \sum_{p_l} Res(\frac{\omega_i\omega_j\omega_k\xi}{\theta};\ p_l)$$

where ξ is the vector field of the $\mathbb{C}^*$-action on $T^*\Sigma$ given by multiplication in the fibres, p_l are the ramification points of $\pi : S \to \Sigma$.

In other words if x is the local coordinate of Σ, (x, y) be the local coordinates of $T^*\Sigma$ corresponding to the point (x, ydx), then $\theta = ydx$ and $\xi = y\frac{\partial}{\partial y}$. We have

$$\frac{\partial \tau_{jk}}{\partial z_i} = -2\pi i \sum_{p_l} \mathrm{Res}(\frac{\omega_i\omega_j\omega_k}{dxdy}).$$

STEP 1. Let $\partial_i = \dfrac{\partial}{\partial z^i}$ be a tangent vector of B^{reg} at a, then ∂_i determines a normal vector field V_i on S_a

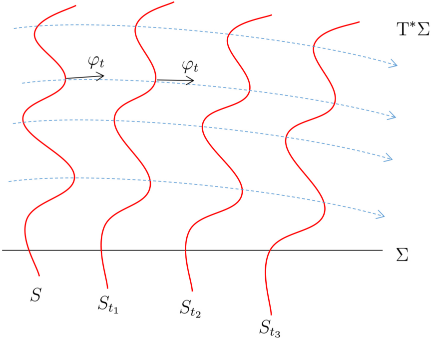


STEP 1

Recall that $S_a \subset T^*\Sigma$, let K_S be the canonical bundle of S_a, N_S be the normal bundle. Then the deformation of S_a is described by a map:

$$\chi : T_a B^{reg} \cong B \to H^0(S_a, N_{S_a}),$$

called the characteristic map.



A sketch of proof

Also, use the short exact sequence,

$$0 \to \mathrm{T}S \to \mathrm{T}(\mathrm{T}^*\Sigma) \to N_i \to 0,$$

We have the Kodaira-Spencer map $\kappa = \delta \circ \chi$ for variation of complex structures given by

$$\mathrm{T}_a B^{reg} \xrightarrow{\chi} \mathrm{H}^0(S; N_i) \xrightarrow{\delta} \mathrm{H}^1(S; \mathrm{T}S).$$

A sketch of proof

STEP 1. Let $\partial_i = \dfrac{\partial}{\partial z^i}$ be a tangent vector of B^{reg} at a, then ∂_i determines a normal vector field V_i on S_a

STEP 2. The integral curves of the normal vector field generates a one-parameter group of diffeomorphisms φ_{t^i} of S_a

STEP 3. Let θ_{t^i} be the pull back of θ by φ_{t^i}, define

$$\partial_i \theta := \frac{d\theta_{t^i}}{dt^i} = \mathcal{L}_{V_i}(\theta_{t^i}).$$

and prove the following lemma.

A sketch of proof

STEP 3.: Prove the following lemma

Lemma

[BH17] In Dolbeault cohomology,

$$\partial_i[\theta] = [\partial_i\theta] = [\omega_i]. \tag{1}$$

$$0 = \int_S \theta \wedge \partial_i\partial_j\theta, \tag{2}$$

$$\frac{\partial\tau_{ij}}{\partial z_k} = -\int_S \theta \wedge \partial_i\partial_j\partial_k\theta. \tag{3}$$

Apply ∂_k to (2) and combined with (3), we have

$$\begin{aligned}\frac{\partial\tau_{ij}}{\partial z_k} &= -\int_S \partial_j\partial_k\theta \wedge \partial_i\theta \\ &= -\int_S \kappa(\partial_j)\omega_k \wedge \omega_i.\end{aligned} \tag{4}$$

A sketch of proof

STEP 1. Let $\partial_i = \dfrac{\partial}{\partial z^i}$ be a tangent vector of B^{reg} at a, then ∂_i determines a normal vector field V_i of S_a

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and prove the following lemma.

STEP 4. Use Griffiths transversality theorem, and the Kodaira-Spencer map to compute the RHS of (4) and apply (1) to get the result.

Further developments

The RHS of $\partial_i \tau_{jk} = -2\pi i \sum_{p_l} \text{Res} \left(\frac{\omega_i \omega_j \omega_k}{dxdy} \right)$ relates to the Eynard-Orantin invariants $W_m^{(0)}$, i.e.

$$-2\pi i \sum_{p_l} \text{Res} \left(\frac{\omega_i \omega_j \omega_k}{dxdy} \right) = -\left(\frac{1}{2\pi i} \right)^2 \int_{p \in b_i} \int_{p_1 \in b_j} \int_{p_2 \in b_k} W_3^{(0)}(p, p_1, p_2).$$

By using Eynard-Orantin topological recursion [Eynard, Orantin. 2007], and the variation formulas they developed we are able to compute [BH17]:

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-2}} \tau_{i_{m-1} i_m} = -\left(\frac{i}{2\pi} \right)^{m-1} \int_{p_1 \in b_{i_1}} \cdots \int_{p_m \in b_{i_m}} W_m^{(0)}(p_1, ..., p_m).$$

where $W_m^{(0)}$ is the Eynard-Orantin invariants for a spectral curve S_a. Therefore the power series expansion of the period τ_{ij} about a point $a \in B^{reg}$ is obtained.

Relation to Hurwitz Space

Let $\mathcal{H}_{n,m}$ be the set of all n-fold coverings with m simple branch points. Let $z_i = x_i = x(a_i)$ be the coordiante of $\mathcal{H}_{n,m}$ where x_i be a local coordinate of an open disc on Σ containing the branch point p_i. Then

Theorem

$$-\frac{1}{2\pi i}\int_{p\in b_i} W_1^{(1)}(p) = \delta_i F^{(1)},$$

where $F^{(1)}$ is called the free energy.

Further developments

Theorem

Let $B^{sim} \subset B^{reg}$ be the dense subset consisting of all points whose spectral curves only have simple ramification points. Let $\lambda_i = x_i = x(a_i)$ be the coordiante of $\mathcal{H}_{n,m}$ where x_i be a local coordinate of an open disc on Σ containing the branch point p_i. Then there is a natural map

$$J : B^{sim} \to \mathcal{H}_{n,m},$$

where $m = deg(K^{n(n-1)})$. And

$$\frac{\partial \lambda_j}{\partial z_i} = -\frac{\omega_i(a_j)}{y'(a_j)},$$

$$\frac{\partial^2 \lambda_j}{\partial z_i \partial z_k} = \sum_{a \neq a_j} \frac{\omega_k(a)\omega_i(a)B(a,q)}{2y'(a_j)y'(a)dr} + \frac{\omega_k(a_j)\omega_i(a_j)S_B(a_j)}{12y'(a_j)^2} + \frac{\omega_i(a_j)}{4y'(a_j)^2}\left(\omega_k''(a_j) - \frac{\omega_k(a_j)y'''(a_j)}{y'(a_j)}\right).$$

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Thank You!