Resistance Growth of Branching Random Networks

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Outline

Introduction

Markov chains: recurrent vs transient

Transience $\iff$ non-degenerated bounded harmonic function exists.

Recurrence $\iff R(o \longleftrightarrow \infty) = \lim_{N\to\infty} R(o \longleftrightarrow N) = \infty$.

NASH-WILLIAMS, C.ST.J.A., Random Walks and Electric Currents in Networks, Proc. of the Cambridge Philosophical Soc., 65(1959), 181-194.

GRIFFEATH, D. & LIGGETT, T.M., Critical phenomena for Spizer's reversible nearest-particle systems. Ann. Probab. 10 (1982), 881-895.

LYONS, T., A Simple criterion for transience of a reversible markov chain. Ann. Probab. 11 (1983), 393-402.

DOYLE, P. G. & SNELL, J. L., Random Walks and Electrical Networks, Mathematical Association of America, 1984.

An easy example

T_d = regular tree, each vertex has d children (degree $=d+1$)
$|x|$ = the distance from vertex x to the root o.
$|e|$ = the distance from edge e to the root o.
$\lambda^{|e|}$ = the resistance of edge e
R_n = the resistance between root o and the level n.

$$R_n = \sum_{k=1}^{n} (\frac{\lambda}{d})^k.$$

$$\lim_n R_n = \infty \Longleftrightarrow \lambda \geq d.$$

Introduction

Glaton-Watson process, $\{p_k, k \geq 0\}$,
X_n the number of descendants of generation n.
$m = \sum_k kp_k =$ the mean of children.

subcritical, $m < 1, \lim_n X_n = 0$ a.s., ;
critical, $m = 1$, $EX_n = 1$, $\lim_n X_n = 0$ a.s., the most delicate case.
supercritcal, $m > 1$, $\lim_n X_n = \infty$ a.s., $\lim_n X_n/m^n$ exists.

K. Athreya, P. Ney, *Branching Processes*. Die Grundlehren der mathematischen Wissenschaften, Band 196, Springer–Verlag, New York–Heidelberg, 1972.

Introduction

A supercritcal GW process $\Longleftrightarrow$ Random Trees

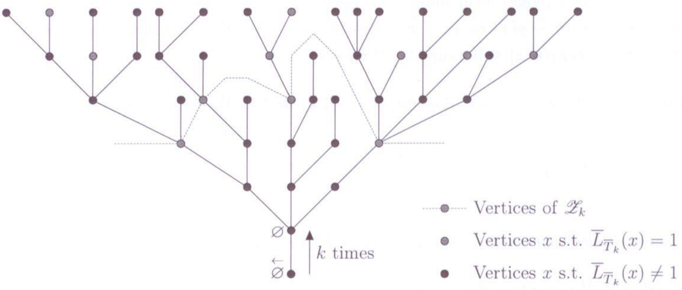


Figure 1: The set $\mathscr{X}_k$.

Introduction

Galton-Watson tree

$\lambda^{|e|}$ = resistance of edge e

R_n = resistance between root o and the level n.

$$\lim_n R_n = \infty \iff \lambda \geq m.$$

R. Lyons. Random walks and percolation on trees. *Ann. Probab.* **18** (1990), 931–958.

R. Lyons. Random walks, capacity and percolation on trees. *Ann. Probab.* **20** (1992), 2043–2088.

Introduction

Regular Tree with random resistance.
$\xi(e)d^{|e|}$ = the resistance of edge e.

$$\mathbf{E}[R_n] = \mathbf{E}[\xi]\, n - \frac{\mathrm{Var}[\xi]}{\mathbf{E}[\xi]} \log n + O(1)$$

$$\mathbf{E}[C_n] = \frac{1}{\mathbf{E}[\xi]} \frac{1}{n} + \frac{\mathrm{Var}[\xi]}{\mathbf{E}[\xi]^3} \frac{\log n}{n^2} + O(n^{-2})$$

where $C_n = 1/R_n$ = the conductance,

$$\mathrm{Var}[R_n] = O(1) \quad \text{and} \quad \mathrm{Var}[C_n] = O(n^{-4}).$$

A sub-Gaussian tail bound

$$\mathbf{E}|R_n - \mathbf{E}[R_n]|^k = O(1) \qquad \text{for all } k \geq 1.$$

L. Addario-Berry, N. Broutin and G. Lugosi. Effective resistance of random trees. *Ann. Appl. Probab.* **19** (2009), 1092–1107.

Main results

Assign a random resistance $\xi(e)m^{|e|}$ to edge e for a supercritical Galton-Watson tree,

Theorem

$$\lim_{n\to\infty} n\,\mathbf{E}\,C_n = \frac{1}{c_1}.$$

If additionally $p_1\, m < 1$, *then*

$$\lim_{n\to\infty} \frac{\mathbf{E}\,R_n}{n} = c_1\,\mathbf{E}\,\frac{1}{W}.$$

D. Chen, Y Hu & S. Lin, Resistance growth of branching random networks, Electronic Journal of Probability, Volume 23 (2018), paper no. 52, 17 pp. https://arxiv.org/abs/1801.05043, https://projecteuclid.org/euclid.ejp/1527818430

Main results

Theorem

Assuming that $\mathbf{E}[\xi + \xi^{-1} + \nu^2] < \infty$*, we have the almost convergence*

$$\lim_{n\to\infty} \frac{C_n}{\mathbf{E} C_n} = W. \tag{1}$$

Assume that $p_0 = 0$. $\sum p_k k log k < \infty$.

$$\frac{X_n}{m^n} \to W > 0 \ a.s., \quad \mathbf{E} W = 1. \qquad \mathbf{E} W^2 = \frac{\sum k^2 p_k - m}{m(m-1)}.$$

$W^{(x)} = \lim_{n\to\infty} m^{|x|-n} \# \mathbb{T}_n[x]$ has the same distribution as W.

$$W = m^{-n} \sum_{|x|=n} W^{(x)}.$$

$\xi(e) m^{|e|}$ = the resistance of edge e.

Main results

$$
\begin{aligned}
a_1 &:= m^{-2}\,\mathbf{E}[\nu(\nu-1)], && (2)\\
b_1 &:= \mathbf{E}[\xi], \\
c_1 &:= \frac{a_1 b_1}{1-m^{-1}}. && (3)\\
a_2 &:= m^{-3}\,\mathbf{E}\left[\nu(\nu-1)(\nu-2)\mathbf{1}_{\{\nu\geq 2\}}\right], && (4)\\
b_2 &:= \mathbf{E}\left[\xi^2\right], \\
c_2 &:= (1-m^{-2})^{-1}\left(\frac{3a_1^2}{m-1}+a_2\right), && (5)\\
c_3 &:= \frac{2a_1c_1}{m-1}-\frac{2b_1c_2}{m}, && (6)\\
c_4 &:= \frac{b_1}{1-m^{-1}}\left(\frac{c_3}{c_1}+a_1\right)-b_2\frac{c_2}{c_1}. && (7)
\end{aligned}
$$

Main results

Theorem

Assume that $\mathbf{E}[\xi^3 + \xi^{-1} + \nu^4] < \infty$. *Then there exists a constant* $c_0 \in \mathbb{R}$ *such that, as* $n \to \infty$,

$$\mathbf{E}\, C_n = \frac{1}{c_1 n} - \frac{c_4}{c_1^2}\frac{\log n}{n^2} - \frac{c_0}{c_1^2}\frac{1}{n^2} + O(\frac{(\log n)^2}{n^3}).$$

The constant c_0 will be defined later.

Main results

Theorem

Assuming that $\mathbf{E}[\xi^3 + \xi^{-1} + \nu^4] < \infty$, *we have, as* $n \to \infty$,

$$n\Big(\frac{C_n}{\mathbf{E}\,C_n} - W\Big) \longrightarrow \sum_{\ell=1}^{\infty} \frac{1}{m^\ell} \sum_{|x|=\ell} W^{(x)} \left(1 - \frac{\xi_x}{c_1} W^{(x)}\right),$$

in prob. P, *and,* $R_n - A_n/W$ *converges to 0 in prob.* P *as* $n \to \infty$, *where*

$$A_n = c_1 n + c_4 \log n + \left(c_0 - \frac{1}{W} \sum_{\ell=1}^{\infty} \frac{1}{m^\ell} \sum_{|x|=\ell} W^{(x)} \left(c_1 - \xi_x W^{(x)}\right)\right).$$

with the same constant c_0 *in Theorem 9.*

Example, the infinite cluster of bond percolation on $\mathbb{Z}^d$ can be seen as a random electric network in which each open edge has unit resistance and each closed edge has infinite resistance.

Grimmett, Kesten and Zhang proved that when $d \geq 3$, the effective resistance of this network between a fixed point and infinity is a.s. finite,

Thus the simple random walk on this infinite percolation cluster is a.s. transient.

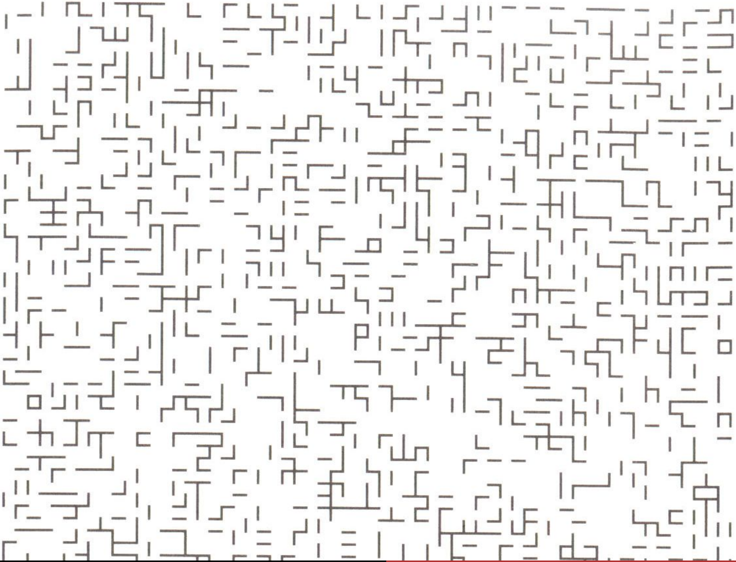
G. Grimmett, H. Kesten and Y. Zhang. Random walk on the infinite cluster of the percolation model. *Probab. Theory Relat. Fields*, **96** (1993), 33–44.

D. Chen. On the infinite cluster of the Bernoulli bond percolation in the Scherk's graph. *J. Applied Probab.*,

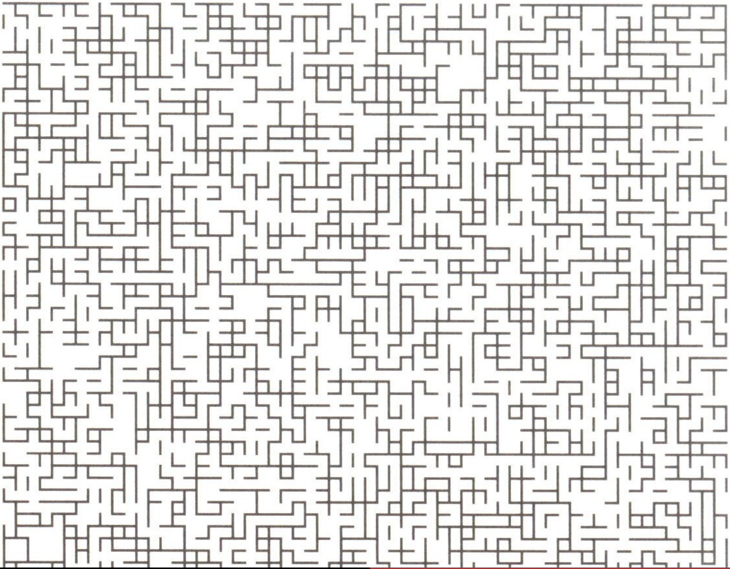
Vol.38, No.4, (2001), 828–840

Related Results

What is Percolation? [1.2



Related Results



Related Results

Benjamini and Rossignol showed that point-to-point effective resistance has submean variance in $\mathbb{Z}^2$, whereas the mean and the variance are of the same order when $d \geq 3$.

I. Benjamini and R. Rossignol. Submean variance bound for effective resistance on random electric networks. *Commun. Math. Phys.* **280** (2008), 445–462.

Related Results

complete graph on n vertices.

For a particular class of resistance distribution on the edges, as $n \to \infty$, the limit distribution of the random effective resistance between two specified vertices was identified as the sum of two i.i.d. random variables, each with the distribution of the effective resistance between the root and infinity in a Galton–Watson tree with a supercritical Poisson offspring distribution.

G. Grimmett and H. Kesten. Random electrical networks on complete graphs. *J. London Math. Soc. (2)* **30** (1984), 171–192.

Sketch of Proof (1)

Theorem

For supercritical Galton-Watson tree,

$$\lim_{n\to\infty} n\,\mathbf{E}[C_n] = \frac{1}{c_1}.$$

If additionally $p_1\, m < 1$, *then*

$$\lim_{n\to\infty} \frac{\mathbf{E}[R_n]}{n} = c_1\,\mathbf{E}\Big[\frac{1}{W}\Big].$$

Bounds on the expected conductance

The effective conductance C_n between the root and the level set $\{x \in \mathbb{T} \colon |x| = n\}$ satisfies

$$C_n := C(\{\varnothing\} \leftrightarrow \mathbb{T}_n) = \pi(\varnothing) P_{\varnothing,\omega}\left(\tau_n < T_{\varnothing}^{+}\right),$$

where

$$\tau_n := \inf\{k \geq 0 \colon |X_k| = n\}, \qquad T_{\varnothing}^{+} := \inf\{k \geq 1 \colon X_k = \varnothing\}.$$

Immediately, $C_n \geq C_{n+1}$.

$$C_{n+1} = \frac{1}{m} \sum_{i=1}^{\nu} \frac{C_n^{(i)}}{1 + \xi_i C_n^{(i)}}. \tag{8}$$

Sketch of Proof

Lemma

If $\mathbf{E}\xi^{-1} < \infty$*, then* $\mathbf{E}C_n \leq \frac{\mathbf{E}\xi^{-1}}{n}$ *for all* $n \geq 1$*.*

Lemma

Assume that $\mathbf{E}\xi^{-1} < \infty$*. For* $2 \leq k \leq 4$*, if* $\mathbf{E}\nu^k < \infty$*, then*

$$\mathbf{E}(C_n)^k = O(n^{-k}) \qquad \text{as } n \to \infty.$$

Lemma

If $\mathbf{E}\xi \in (0, \infty)$ *and* $\mathbf{E}\nu^2 < \infty$*, then there exists a constant* $c > 0$ *such that* $\mathbf{E}C_n \geq \frac{c}{n}$ *for all* $n \geq 1$*.*

Flow, minimum energy, the Dirichlet principle.

Let θ be the a flow from A to Z with strength $\|\theta\|$, i.e., it satisfies Kirchhoff's node law that $\operatorname{div}\theta(x) = 0$ for all $x \notin A \cup Z$, and that

$$\|\theta\| = \sum_{a\in A} \sum_{y\sim a, y\notin A} \theta(\overrightarrow{ay}) = \sum_{z\in Z} \sum_{y\sim z, y\notin Z} \theta(\overrightarrow{yz}).$$

then

$$R(A \leftrightarrow Z) := \inf_{\|\theta\|=1} \sum_{e\in E} r(e)\theta(e)^2. \tag{9}$$

Sketch of Proof (2)

Theorem

Assume that $\mathbf{E}[\xi^3 + \xi^{-1} + \nu^4] < \infty$. *Then there exists a constant* $c_0 \in \mathbb{R}$ *such that, as* $n \to \infty$,

$$\mathbf{E} C_n = \frac{1}{c_1 n} - \frac{c_4}{c_1^2} \frac{\log n}{n^2} - \frac{c_0}{c_1^2} \frac{1}{n^2} + O\left(\frac{(\log n)^2}{n^3}\right).$$

Sketch of Proof

Asymptotic expansion of the expected conductance

For every integer $n \geq 1$, we write

$$x_n := \mathbf{E}C_n, \qquad y_n := \mathbf{E}C_n^2, \qquad z_n := \mathbf{E}C_n^3.$$

By Lemma 7, $x_n = O(n^{-1}), y_n = O(n^{-2})$ and $z_n = O(n^{-3})$.

$$\begin{aligned}
x_{n+1} &= x_n - b_1 y_n + b_2 z_n + O(n^{-4}), && (10)\\
y_{n+1} &= \frac{y_n}{m} + a_1\, x_{n+1}^2 - \frac{2b_1}{m} z_n + O(n^{-4}) \\
&= \frac{y_n}{m} + a_1\, x_n^2 - \left(2a_1 b_1 x_n y_n + \frac{2b_1}{m} z_n\right) + O(n^{-4}), && (11)\\
z_{n+1} &= \frac{z_n}{m^2} + \frac{3a_1}{m} x_{n+1} y_n + a_2 x_{n+1}^3 + O(n^{-4}) \\
&= \frac{z_n}{m^2} + \frac{3a_1}{m} x_n y_n + a_2 x_n^3 + O(n^{-4}). && (12)
\end{aligned}$$

Sketch of Proof

$$\varepsilon_n := \frac{1}{x_{n+1}} - \frac{1}{x_n} - c_1 = \frac{c_4}{n} + O(n^{-2} long),$$

$$\frac{1}{x_n} - \frac{1}{x_1} = c_1(n-1) + \sum_{i=1}^{n-1} \varepsilon_i = c_1 n + c_4 \log n + o(\log n),$$

Finally

$$\mathbf{E} C_n = x_n = \frac{1}{c_1 n} - \frac{c_4}{c_1^2}\frac{\log n}{n^2} - \frac{c_0}{c_1^2}\frac{1}{n^2} + O(\frac{(\log n)^2}{n^3}).$$

Sketch of Proof (3)

Theorem

Assuming that $\mathbf{E}[\xi^3 + \xi^{-1} + \nu^4] < \infty$, *we have, as* $n \to \infty$,

$$n\Big(\frac{C_n}{\mathbf{E}\, C_n} - W\Big) \longrightarrow \sum_{\ell=1}^{\infty} \frac{1}{m^\ell} \sum_{|x|=\ell} W^{(x)} \left(1 - \frac{\xi_x}{c_1} W^{(x)}\right),$$

in prob. (P), *and,* $R_n - A_n/W$ *converges to 0 in probability* P *as* $n \to \infty$, *where*

$$A_n = c_1 n + c_4 \log n + \left(c_0 - \frac{1}{W} \sum_{\ell=1}^{\infty} \frac{1}{m^\ell} \sum_{|x|=\ell} W^{(x)} \left(c_1 - \xi_x W^{(x)}\right)\right).$$

with the same constant c_0 *in Theorem 9.*

Sketch of Proof

Almost sure convergence and rate of convergence

$$
\begin{aligned}
Y_n &:= \frac{C_n}{\mathbf{E}\, C_n} - W, \\
\Pi_n &:= C_n\Big(\frac{1}{x_{n+1}} - \frac{1}{x_n} - \frac{1}{x_{n+1}}\frac{\xi\, C_n}{1+\xi\, C_n}\Big).
\end{aligned}
$$

$$
Y_n = \frac{1}{m}\sum_{i=1}^{\nu} Y_{n-1}^{(i)} + \frac{1}{m}\sum_{i=1}^{\nu} \Pi_{n-1}^{(i)}.
$$

Since $W = m^{-k}\sum_{|x|=k} W^{(x)}$, by induction,

$$
Y_n = \frac{1}{m^k}\sum_{|x|=k} Y_{n-k}^{(x)} + \sum_{\ell=1}^{k}\frac{1}{m^\ell}\sum_{|y|=\ell}\Pi_{n-\ell}^{(y)} \quad \text{for any } 1 \le k < n.
$$

$$
\mathbf{E}\Big(\frac{1}{m^k}\sum_{|x|=k} Y_{n-k}^{(x)}\Big)^2 = m^{-k}\,\mathbf{E}(Y_{n-k})^2 \le C'\, m^{-k}.
$$

Sketch of Proof

Meanwhile,

$$\mathbf{E}\Big[\sum_{\ell=1}^{k}\frac{1}{m^{\ell}}\sum_{|y|=\ell}|\Pi_{n-\ell}^{(y)}|\Big] \le \sum_{\ell=1}^{k}\frac{C}{n-\ell} \le \frac{Ck}{n-k}.$$

It follows that

$$\mathbf{E}|Y_n| \le \sqrt{C'\,m^{-k}} + \frac{Ck}{n-k}.$$

By taking $k = C'' \log n$ for some constant C'' sufficiently large,

$$\mathbf{E}|Y_n| = O(\frac{\log n}{n}).$$

Questions

For Galton-Watson tree, resistance of edge $e = \xi(e)\lambda^{|e|}$.
If $\lambda < m$, then the effective resistance $R(o \longleftrightarrow \infty) < \infty$ a.s.
What is the distribution of $R(o \longleftrightarrow \infty)$?

Theorem

Fix $\lambda > m$. Assuming that $\mathbf{E}[\xi + \xi^{-1} + \nu^2] < \infty$, we have

$$\{C_n(\lambda)\} \longrightarrow W \qquad \text{a.s.} \qquad as\, n \to \infty$$

If $\mathbf{E}[\xi^2 + \xi^{-1} + \nu^3] < \infty$, then, as $n \to \infty$, the limit of

$$\left(\frac{\lambda}{m}\right)^n \mathbf{E} C_n(\lambda)$$

exists and is strictly positive.

can the limit be identified?

Acknowledgement

Thank You !

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