A simple and efficient finite volume WENO method for hyperbolic conservation laws

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Outline

- **Introduction**
- **Description of Simple WENO schemes**
- **Numerical results**
- **Conclusions**

1 Introduction

- We consider hyperbolic conservation laws:

$$\begin{cases} u_t + \nabla \cdot f(u) = 0, \\ u(x,0) = u_0(x). \end{cases}$$

- Hyperbolic conservation laws and convection dominated PDEs play an important role arise in applications, such as gas dynamics, modeling of shallow waters,...

- There are special difficulties associated with solving these systems both on mathematical and numerical methods, for discontinuous may appear in the solutions for nonlinear equations, even though the initial conditions are smooth enough.

Introduction

- This is why devising robust, accurate and efficient methods for numerically solving these problems is of considerable importance and as expected, has attracted the interest of many researchers and practitioners.
- Many numerical methods have been developed to solve these problems.
  - Monotone schemes (the first order schemes): Godunov scheme, Lax-Friedrichs scheme, EO(Engquist-Osher) scheme.
  - TVD (Total-Variation-Diminishing） scheme
  - ENO (Essentially Non-Oscillatory) scheme
  - WENO (Weighted Essentially Non-Oscillatory) scheme
  - Discontinuous Galerkin finite element method
- The ENO and WENO methods are the popular numerical methods for hyperbolic conservation laws.

- ENO schemes
  - Finite volume ENO schemes: Harten, Engquist, Osher and Chakravarthy, JCP 1987.
  - Finite difference ENO schemes: Shu and Osher, JCP 1988, 1989.
  - Uniform high order polynomial interpolation or reconstruction for the numerical fluxes (or using other basis functions).
  - The stencil is locally adaptive: among several candidate stencils, *one* is chosen according to local smoothness.

- WENO schemes
  - The first WENO scheme is constructed by Liu, Osher and Chan for a third order finite volume version, in 1994 JCP.
  - In 1996, third and fifth order finite difference WENO schemes in multi space dimensions are constructed by Jiang and Shu, with a general framework for the design of *smoothness indicators and nonlinear weights.*
  - Instead of using just one candidate stencil, a linear combination of all candidate stencils is used.
  - A key idea in WENO schemes is a linear combination of lower order fluxes or reconstruction to obtain a higher order approximation.

- WENO schemes
  - The choice of the weight to each candidate stencil, which is a *nonlinear* function of the grid values, is a key to the success of WENO.
  - Both ENO and WENO schemes use the idea of adaptive stencils to automatically achieve high order accuracy and non-oscillatory property near discontinuities.
  - For the system case, WENO schemes based on local characteristic decompositions and flux splitting to avoid spurious oscillatory.

- Advantages of ENO and WENO schemes
  - Uniform high order accuracy in smooth regions including at smooth extrema, unlike second order TVD schemes which degenerate to first order accuracy at smooth extrema.
  - Sharp and essentially non-oscillatory (to the eyes) shock transition.
  - Robust for many physical systems with strong shocks.
  - Especially suitable for simulating solutions containing both discontinuities and complicated smooth solution structure, such as shock interaction with vortices.

- **Application of WENO**
  - Computational fluid dynamics
  - Astronomy and astrophysics
  - Semiconductor device simulation
  - Traffic flows
  - Computational biology
  - Hamilton Jacobi equation

Chi-Wang Shu, *"High Order Weighted Essentially Nonoscillatory Schemes for Convection Dominated Problems"* SIAM Review, 51 (2009), pp. 82-126;
"High order WENO and DG methods for time-dependent convection-dominated PDEs: A brief survey of several recent developments", J. Comput. Phys., 316 (2016), pp. 598-613.

Introduction

- Some drawbacks of current WENO schemes
  - The computation cost is very high.
  - The optimal (linear) weights are depended on geometry of mesh.
  - The drawback is more evident with the increase of the space dimension.

- Our motivation
  - Develop more efficient and simpler scheme
  - The optimal (linear) weights are independed on geometry of mesh.

2 Description of WENO schemes

- We consider one dimensional nonlinear hyperbolic conservation laws: $u_t + f(u)_x = 0$. The mesh is distributed into some cells $I_i = [x_{i-1/2}, x_{i+1/2}]$, with the cell size is denoted as $x_{i+1/2} - x_{i-1/2} = \Delta x_i$ and associated cell centers are defined as $x_i = \frac{1}{2}(x_{i+1/2} + x_{i-1/2})$

- Integrate $u_t + f(u)_x = 0$ on cell I_i, we have:

$$\frac{d\bar{u}_i(t)}{dt} + \frac{1}{\Delta x_i}(f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t))) = 0, \tag{1}$$

where $\bar{u}_i(t) = \frac{1}{\Delta x_i} \int_{I_i} u(x,t)dx$ is the cell average. Replace $f(u(x_{i+1/2}, t))$ by the numerical flux $\hat{f}_{i+1/2} = \hat{f}(u^-_{i+1/2}, u^+_{i+1/2})$, if $u^\pm_{i+1/2}$ are the $(2r+1) - th$ order approximation to $u(x_{i+1/2}, t)$, then

$$\frac{d\bar{u}_i(t)}{dt} = L(u_i) = -\frac{1}{\Delta x_i}(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}), \tag{2}$$

is the $(2r+1) - th$ order approximation to (1).

Description of WENO schemes

- We take the Lax-Friedrichs flux:

$$\hat{f}(a,b) = \frac{1}{2}[f(a) + f(b) - \alpha(b-a)], \tag{3}$$

where $\alpha = max_u|f'(u)|$ is a constant and the maximum is taken over the whole range of u.

- Time discretization

Use explicit, nonlinear stable high order Runge-Kutta method to discretize (2), for example the third order version:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \tfrac{3}{4}u^n + \tfrac{1}{4}u^{(1)} + \tfrac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \tfrac{1}{3}u^n + \tfrac{2}{3}u^{(2)} + \tfrac{2}{3}\Delta t L(u^{(2)}). \end{aligned}$$

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- The reconstruction of $u^{\pm}_{i+1/2}$

We present the procedure for the reconstruction of $u^{-}_{i+1/2}$ by the $2r+1$ order WENO approximation. Here we only show the case $r=2$. The procedure for the reconstruction of $u^{+}_{i+1/2}$ is mirror symmetric of that for $u^{-}_{i+1/2}$ with respect to $x_{i+1/2}$.

- Given the bigger stencil $\mathcal{T}_1 = (I_{i-2}, \ldots, I_{i+2})$, two the small stencils $\mathcal{T}_2 = (I_{i-1}, I_i)$ and $\mathcal{T}_3 = (I_i, I_{i+1})$.
- For classical WENO: Given three the small stencils $\mathcal{S}_1 = (I_{i-2}, I_{i-1}, I_i)$, $\mathcal{S}_2 = (I_{i-1}, I_i, I_{i+1})$ and $\mathcal{S}_3 = (I_i, I_{i+1}, I_{i+2})$,

Description of WENO schemes

- We construct polynomials $p_k(x)$ on stencil $\mathcal{T}_k$ such that:

$$\frac{1}{\Delta x_{i+l}}\int_{I_{i+l}} p_1(\xi)d\xi = \bar{u}_{i+l},\ \ l=-2,\ldots,2;$$

and

$$\frac{1}{\Delta x_{i+l}}\int_{I_{i+l}} p_2(\xi)d\xi = \bar{u}_{i+l},\ \ l=-1,0;\ \ \frac{1}{\Delta x_{i+l}}\int_{I_{i+l}} p_3(\xi)d\xi = \bar{u}_{i+l},\ \ l=0,1$$

- For classical WENO, we construct polynomials $q_l(x)$ on stencil $\mathcal{S}_l$ such that:

$$\frac{1}{\Delta x_{i-3+j+l}}\int_{I_{i-3+j+l}} q_l(\xi)d\xi = \bar{u}_{i-3+j+l},\ \ j=0,1,2;\ \ l=1,2,3;$$

Description of WENO schemes

- With the similar idea by Levy, Puppo and Russo for central WENO methods, we rewrite $p_1(x)$ as:

$$p_1(x) = \gamma_1(\frac{1}{\gamma_1}p_1(x) - \frac{\gamma_2}{\gamma_1}p_2(x) - \frac{\gamma_3}{\gamma_1}p_3(x)) + \gamma_2 p_2(x) + \gamma_3 p_3(x) \quad (4)$$

  It is obvious that for any positive linear weights $\gamma_1, \gamma_2, \gamma_3$ with $\gamma_1 + \gamma_2 + \gamma_3 = 1$, the (4) is the fifth order approximation to $u(x,t)$.

- For classical WENO, we compute linear weights $\gamma_1^c, \gamma_2^c, \gamma_3^c$ such that:

$$p_1(x_{i+1/2}) = \sum_{l=1}^{3} \gamma_l^c q_l(x_{i+1/2})$$

  The linear weights are depended on geometry of mesh, and where you reconstruct.

Description of WENO schemes

- We compute the smoothness indicator, denoted as β_k for each stencil $\mathcal{T}_k$, which measures how smooth the function $p_k(x)$,

$$\beta_k = \sum_{l=1}^{r_k} \int_{I_i} (\Delta x_i)^{2l-1} (p_k^{(l)})^2 dx,$$

where $p_k^{(l)}$ is the lth-derivative of $p_k(x)$, and $r_1 = 4,\ r_2 = r_3 = 1$.

- For classical WENO,

$$\beta_k^c = \sum_{l=1}^{2} \int_{I_i} (\Delta x_i)^{2l-1} (q_k^{(l)})^2 dx,$$

where $q_k^{(l)}$ is the lth-derivative of $q_k(x)$.

Description of WENO schemes

Compute the nonlinear weights based on the linear weights and the smoothness indicators. We define:

$$\tau = (\frac{|\beta_1 - \beta_2| + |\beta_1 - \beta_3|}{2})^2 = O(\Delta x_i^6). \tag{5}$$

Then the non-linear weights are defined as

$$\omega_\ell = \frac{\bar{\omega}_\ell}{\sum_{\ell\ell=1}^{3} \bar{\omega}_{\ell\ell}}, \qquad \bar{\omega}_\ell = \gamma_\ell (1 + \frac{\tau}{\varepsilon + \beta_\ell}), \quad \ell = 1, 2, 3. \tag{6}$$

Here ε is a small positive number to avoid the denominator of (6) to become zero.

For classical WENO:

$$\omega_\ell^c = \frac{\bar{\omega}_\ell}{\sum_{\ell\ell=1}^{3} \bar{\omega}_{\ell\ell}}, \qquad \bar{\omega}_\ell = \frac{\gamma_\ell}{(\varepsilon + \beta_\ell^c)^2}, \quad \ell = 1, 2, 3. \tag{7}$$

Description of WENO schemes

By the usage of Taylor expression in the smooth region for β_ℓ , we can obtain:

$$\frac{\tau}{\varepsilon + \beta_\ell} = O(\Delta x_i^4), \ell = 1, 2, 3,$$

on condition that $\varepsilon \ll \beta_\ell$. Therefore the nonlinear weights ω_ℓ, $\ell = 1, 2, 3$, satisfy the order accuracy condition for the fifth order accuracy to the WENO scheme.

$$\omega_\ell = \gamma_\ell + O(\Delta x_i^4)$$

For classical WENO:

$$\omega_\ell^c = \gamma_\ell^c + O(\Delta x_i^2)$$

Description of WENO schemes

The new final reconstructions of the u at $x = x_{i+1/2}$ is given by:

$$u^-_{i+1/2} = \omega_1(\frac{1}{\gamma_1}p_1(x_{i+\frac{1}{2}}) - \frac{\gamma_2}{\gamma_1}p_2(x_{i+\frac{1}{2}}) - \frac{\gamma_3}{\gamma_1}p_3(x_{i+\frac{1}{2}})) + \omega_2 p_2(x_{i+\frac{1}{2}}) + \omega_3 p_3(x_{i+\frac{1}{2}}). \quad (8)$$

For classical WENO:

$$u^-_{i+1/2} = \omega_1 q_1(x_{i+\frac{1}{2}}) + \omega_2 q_2(x_{i+\frac{1}{2}}) + \omega_3 q_3(x_{i+\frac{1}{2}}). \quad (9)$$

The new final reconstructions of the u at $x = x_{i+1/2}$ is given by:

$$u^-_{i+1/2} = \omega_1\left(\frac{1}{\gamma_1}p_1(x_{i+\frac{1}{2}}) - \frac{\gamma_2}{\gamma_1}p_2(x_{i+\frac{1}{2}}) - \frac{\gamma_3}{\gamma_1}p_3(x_{i+\frac{1}{2}})\right) + \omega_2 p_2(x_{i+\frac{1}{2}}) + \omega_3 p_3(x_{i+\frac{1}{2}}). \quad (10)$$

For classical WENO:

$$u^-_{i+1/2} = \omega_1 q_1(x_{i+\frac{1}{2}}) + \omega_2 q_2(x_{i+\frac{1}{2}}) + \omega_3 q_3(x_{i+\frac{1}{2}}). \quad (11)$$

Description of WENO schemes

- We consider two dimensional nonlinear hyperbolic conservation laws:

$$u_t + f(u)_x + g(u)_y = 0,$$

and integrate it over the target cell $\triangle_0$ to obtain the semi-discrete finite volume formula as

$$\frac{d\bar{u}_0(t)}{dt} = -\frac{1}{|\triangle_0|}\int_{\partial\triangle_0} F\cdot\vec{n}ds = L(u), \tag{12}$$

where $\bar{u}_0(t) = \frac{1}{|\triangle_0|}\int_{\triangle_0} u(x,y,t)dxdy$, $F = (f,g)^T$, $\partial\triangle_0$ is the boundary of the target cell $\triangle_0$ which is composed of three edges (line segments), $|\triangle_0|$ is the area of the target cell $\triangle_0$ and $\vec{n}$ denotes the outward unit normal to the edge of the target cell.

- The line integrals in (12) are discretized by a two-point Gaussian integration formula:

$$\int_{\partial\triangle_0} F\cdot\vec{n}ds \approx \sum_{\ell\ell=1}^{3}|\partial\triangle_{0_{\ell\ell}}|\sum_{\ell=1}^{2}\sigma_\ell F(u(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t))\cdot\vec{n}_{\ell\ell}. \tag{13}$$

Description of WENO schemes

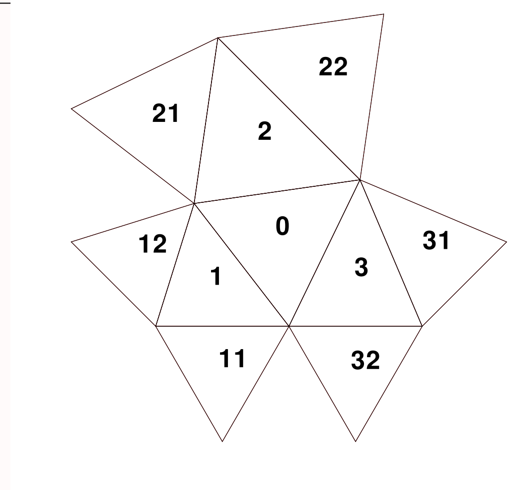
- And $F(u(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t)) \cdot \vec{n}_{\ell\ell}$, $\ell = 1, 2$, $\ell\ell = 1, 2, 3$, are reformulated by numerical fluxes such as the Lax-Friedrichs flux

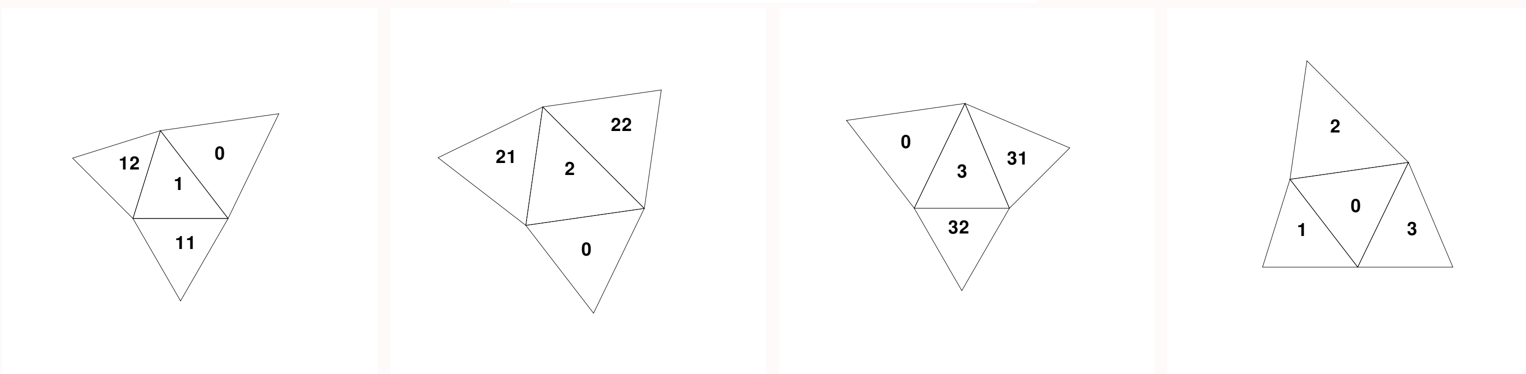
$$F(u(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t)) \cdot \vec{n}_{\ell\ell} \approx \frac{1}{2}[(F(u^+(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t)) + F(u^-(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t))) \cdot \vec{n}_{\ell\ell} - \alpha(u^+(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t) - u^-(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t))], \ell = 1, 2, \ell\ell = 1, 2, 3, \quad (14)$$

  in which α is taken as an upper bound for the eigenvalues of the Jacobian in the $\vec{n}_{\ell\ell}$ direction, and u_*^+ and u_*^- are the conservative values of u inside and outside of the boundaries of the target triangular cell (inside of the neighboring triangular cell) at different Gaussian points and $|\partial\triangle_{0_{\ell\ell}}|$, $\ell\ell = 1, 2, 3$, are the length of the line segments.

- The reconstruction of function $u(x, y, t)$ at different Gaussian quadrature points $(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}})$, $\ell = 1, 2$, $\ell\ell = 1, 2, 3$, on the boundaries of target cell $\triangle_0$ is narrated as follows.

Description of WENO scheme





The typical stencils. From top to bottom and left to right: T_1, T_2, T_3, T_4, T_5.

Description of WENO schemes

- For the third order scheme, we construct the quadratic polynomial $p_1(x,y)$ has the same cell average of u on the target cell $\triangle_0$, and matches the cell averages of u on the other triangles in the set $T_1 \setminus \{\triangle_0\}$ in a least square sense.
- For the fourth order scheme, we construct a cubic polynomial $p_1(x,y)$ on T_1 to obtain a fourth order approximation of conservative variable u by requiring that it has the same cell averages of u on the target cell $\triangle_0$ and the other triangles:

$$\frac{1}{|\triangle_\ell|}\int_{\triangle_\ell} p_1(x,y)dxdy = \bar{u}_\ell, \ell = 0,1,2,3,11,12,21,22,31,32. \tag{15}$$

- *Remark: If some triangles are merged and the cells in the big stencil are less than six for the third order or ten for the fourth order scheme, we should use triangular cells in the next layer to guarantee information enough to reconstruct the polynomial in the least squares.*

Description of WENO schemes

- Construct four linear polynomials $p_i(x,y), i = 2,3,4,5$, which satisfy the cell average of the conservative variable u on the target cell $\triangle_0$ and match the cell averages of u on the other triangles in a least square sense.

  WENO by Hu and Shu: For the third order case, nine linear polynomials are constructed; For the fourth order case, six quadratic polynomials are constructed.

- Similar to one dimensional case, we rewrite $p_1(x,y)$ as:

$$p_1(x,y) = \gamma_1(\frac{1}{\gamma_1}p_1(x,y) - \sum_{l=2}^{5}\frac{\gamma_l}{\gamma_1}p_l(x,y)) + \sum_{l=2}^{5}\gamma_l p_l(x,y). \tag{16}$$

  It is obvious that for any positive linear weights $\gamma_1, \cdots, \gamma_5$ with $\sum_{l=1}^{5}\gamma_l = 1$, the (16) is the third order approximation to $u(x,y,t)$.

  WENO by Hu and Shu: Only the mostly near-uniform meshes can guarantee the linear weights are positive.

Description of WENO schemes

- Compute the smoothness indicators, denote by β_ℓ, $\ell = 1, \ldots, 5$, which measure how smooth the functions $p_\ell(x,y)$, $\ell = 1, \ldots, 5$, are in the target cell $\triangle_0$. The smaller these smoothness indicators, the smoother the functions are in the target cell.

$$\beta_\ell = \sum_{|l|=1}^{r} \int_{\triangle_0} |\triangle_0|^{|l|-1} \left(\frac{\partial^{|l|}}{\partial x^{l_1} \partial y^{l_2}} p_\ell(x,y) \right)^2 dxdy, \ \ell = 1, \ldots, 5, \tag{17}$$

where $l = (l_1, l_2)$, $|l| = l_1 + l_2$. And for $\ell = 1$, $r = 2$ or 3 for the third or fourth order, respectively; for $\ell = 2, 3, 4, 5$, $r = 1$.

Description of WENO schemes

- Compute the nonlinear weights based on the linear weights and the smoothness indicators.

$$\tau = \left(\frac{|\beta_1-\beta_2|+|\beta_1-\beta_3|+|\beta_1-\beta_4|+|\beta_1-\beta_5|}{4}\right)^2 = O(|\triangle_0|^3). \qquad (18)$$

Then the associate nonlinear weights are defined as

$$\omega_\ell = \frac{\bar{\omega}_\ell}{\sum_{\ell\ell=1}^{5}\bar{\omega}_{\ell\ell}}, \qquad \bar{\omega}_\ell = \gamma_\ell\Big(1+\frac{\tau}{\varepsilon+\beta_\ell}\Big), \quad \ell = 1,2,3,4,5. \qquad (19)$$

Here ε is a small positive number to avoid the denominator of (19) to become zero. By the implementation of (18) in the smooth region, it satisfies

$$\frac{\tau}{\varepsilon+\beta_\ell} = O(|\triangle_0|^2), \ell = 1,2,3,4,5, \qquad (20)$$

- The nonlinear weights satisfy: $\omega_\ell = \gamma_\ell + O(|\triangle_0|^2)$

Description of WENO schemes

- The final reconstruction polynomial to approximate to $u(x,y,t)$ is given:

$$Q(x,y) = \omega_1\left(\frac{1}{\gamma_1}p_1(x,y) - \sum_{j=2}^{5}\frac{\gamma_j}{\gamma_1}p_j(x,y)\right) + \sum_{j=2}^{5}\omega_j p_j(x,y), \tag{21}$$

and at different Gaussian quadrature points $(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}})$, $\ell = 1,2$, $\ell\ell = 1,2,3$, on different line segments of the boundaries of the target cell $\triangle_0$, the approximations are given by

$$u^-(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}, t) = Q(x_{G_{\ell\ell_\ell}}, y_{G_{\ell\ell_\ell}}),\ \ell = 1,2,\ \ell\ell = 1,2,3. \tag{22}$$

WENO by Hu and Shu: For the linear weights are depended on Gaussian quadrature points, we have to compute nonlinear weights six times.

- *Remark: For system cases, the reconstructions are performed in the local characteristic directions to avoid spurious oscillations.*

3 Numerical results

In this section we present the results of numerical tests to illustrate the good performance of the new scheme. The CFL number is set as 0.6. For the purpose of testing whether the random choice of the linear weights would pollute the optimal order accuracy of WENO-ZQ scheme or not, we set different type of linear weights in the numerical accuracy cases as.

For one dimensional cases

(1) $\gamma_1 = 0.98, \gamma_2 = \gamma_3 = 0.01$; (2) $\gamma_1 = \gamma_2 = \gamma_3 = 1.0/3.0$;

(3) $\gamma_1 = 0.01, \gamma_2 = \gamma_3 = 0.495$.

For two dimensional cases

(1) $\gamma_1 = 0.96, \gamma_l = 0.01, l = 2, 3, 4, 5$; (2) $\gamma_l = 0.2, l = 1, 2, 3, 4, 5$;

(3) $\gamma_1 = 0.01, \gamma_l = 0.2475, l = 2, 3, 4, 5$; (4) $\gamma_l = 0.01, l = 1, 2, 3, 4, \gamma_5 = 0.96$.

And in the test cases with shock, we set γ_1=0.98, $\gamma_2 = \gamma_3 = 0.01$ and $\gamma_1 = 0.96, \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0.01$ for 1D and 2D, respectively.

Numerical results

- One-dimensional Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mu\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho\mu\\ \rho\mu^2+p\\ \mu(E+p)\end{pmatrix}=0. \tag{23}$$

In which ρ is density, μ is the velocity in $x-$direction , E is total energy and p is pressure. The initial conditions are: $\rho(x,0) = 1 + 0.2\sin(\pi x)$, $\mu(x,0) = 1$, $p(x,0) = 1$, $\gamma = 1.4$. The computing domain is $x \in [0,2]$. Periodic boundary condition is applied in this test. $T = 2.0$

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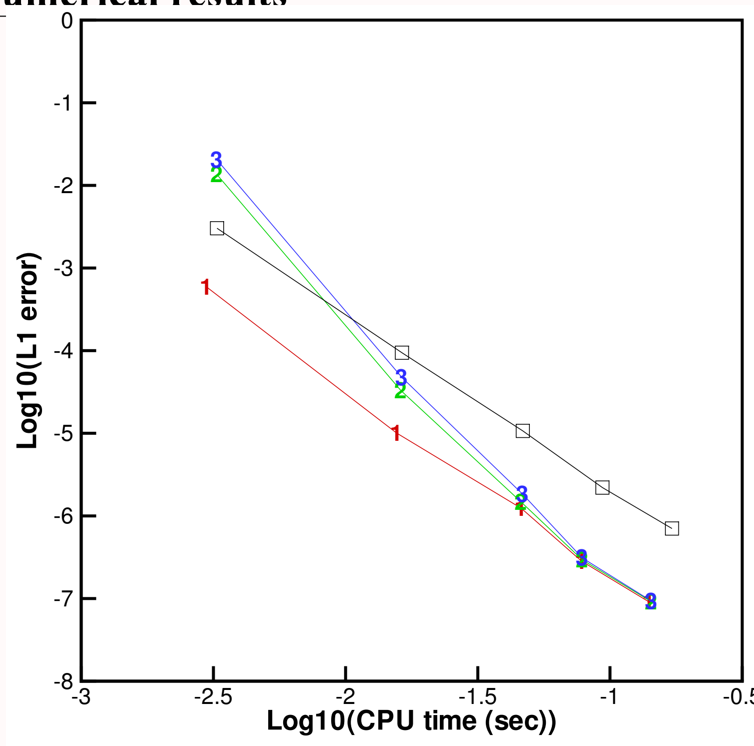
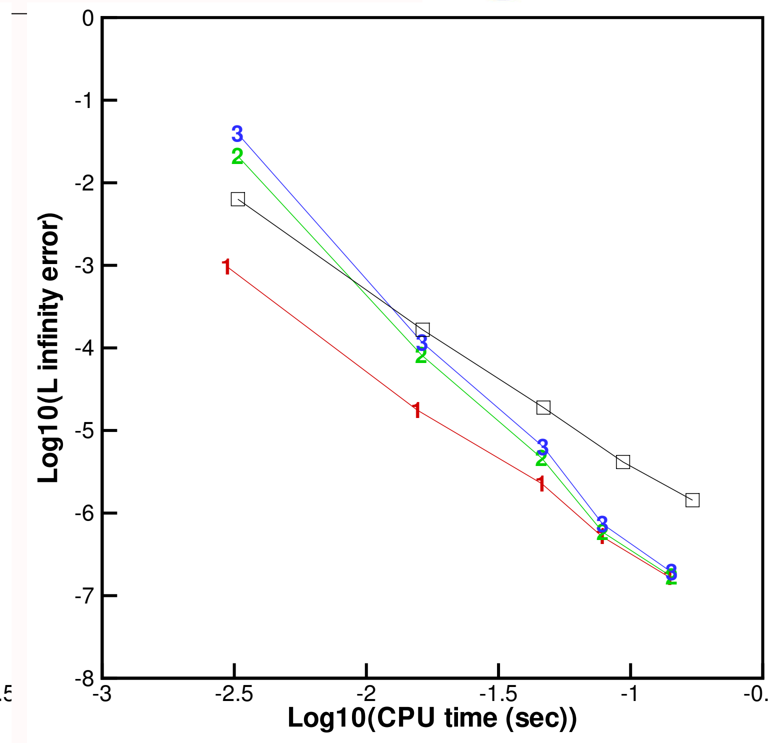
Numerical results

	WENO-ZQ (1) scheme				WENO-JS scheme			
grid cells	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
10	5.91E-4		9.57E-4		3.03E-3		6.32E-3	
20	1.00E-5	5.88	1.76E-5	5.76	9.44E-5	5.00	1.65E-4	5.25
30	1.23E-6	5.16	2.26E-6	5.06	1.06E-5	5.37	1.89E-5	5.35
40	2.83E-7	5.12	5.18E-7	5.12	2.20E-6	5.49	4.14E-6	5.27
50	9.10E-8	5.10	1.66E-7	5.09	7.03E-7	5.11	1.43E-6	4.76
	WENO-ZQ (2) scheme				WENO-ZQ (3) scheme			
grid cells	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
10	1.35E-2		2.09E-2		2.03E-2		3.90E-2	
20	3.29E-5	8.68	8.17E-5	8.00	4.74E-5	8.74	1.14E-4	8.41
30	1.47E-6	7.66	4.60E-6	7.09	1.82E-6	8.03	6.29E-6	7.16
40	2.94E-7	5.59	5.84E-7	7.17	3.09E-7	6.16	7.30E-7	7.48
50	9.17E-8	5.22	1.68E-7	5.57	9.29E-8	5.40	1.90E-7	6.03

Numerical results





1D-Euler equations. Computing time and error. Number signs and a solid red line denote the results of WENO-ZQ scheme with different linear weights (1), (2) and (3); squares and a solid line denote the results of WENO-JS scheme.

- Two-dimensional Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho\mu \\ \rho\nu \\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho\mu \\ \rho\mu^2+p \\ \rho\mu\nu \\ \mu(E+p)\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho\nu \\ \rho\mu\nu \\ \rho\nu^2+p \\ \nu(E+p)\end{pmatrix}=0. \tag{24}$$

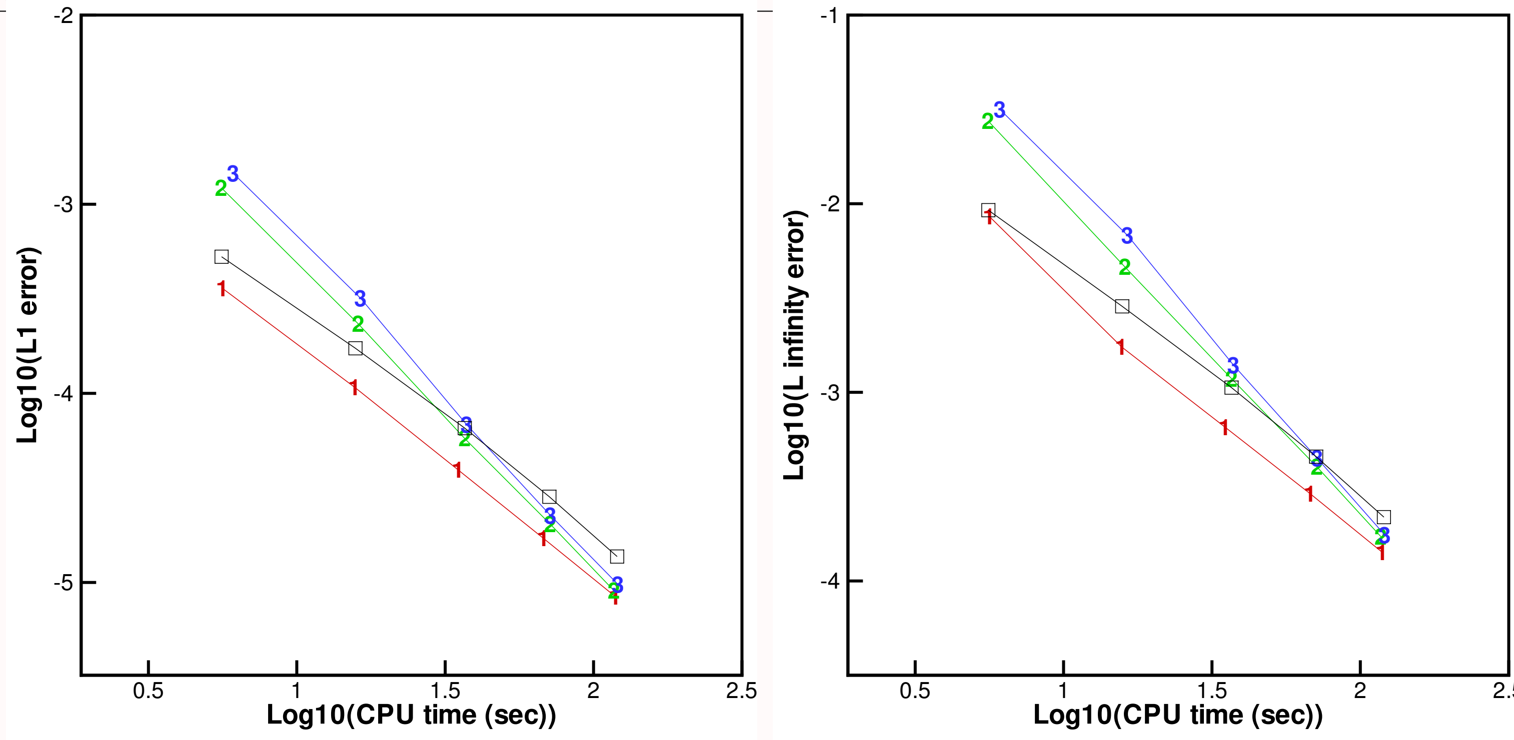
In which ρ is density; μ and ν are the velocities in the x and y-directions, respectively; E is total energy; and p is pressure. The initial conditions are: The initial conditions are: $\rho(x,y,0)=1+0.2\sin(\pi(x+y))$, $\mu(x,y,0)=0.5$, $\nu(x,y,0)=0.5$, $p(x,y,0)=1$ and $\gamma=1.4$. The computing domain is $(x,y)\in[0,2]\times[0,2]$. Periodic boundary conditions are applied in both directions. $T=2.0$

Numerical results

	WENO-ZQ (1) scheme				WENO-JS scheme			
grid cells	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
10×10	8.68E-4		1.84E-3		3.98E-3		6.50E-3	
20×20	2.71E-5	5.00	4.91E-5	5.23	1.98E-4	4.32	3.62E-4	4.16
30×30	3.63E-6	4.95	6.11E-6	5.14	2.57E-5	5.04	5.23E-5	4.77
40×40	8.69E-7	4.98	1.44E-6	5.01	6.13E-6	4.99	1.26E-5	4.93
50×50	2.85E-7	4.98	4.70E-7	5.03	2.00E-6	5.01	4.06E-6	5.09
	WENO-ZQ (2) scheme				WENO-ZQ (3) scheme			
grid cells	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
10×10	3.71E-3		7.54E-3		4.66E-3		9.15E-3	
20×20	4.71E-5	6.30	2.02E-4	5.22	6.72E-5	6.11	2.73E-4	5.06
30×30	3.66E-6	6.30	1.21E-5	6.94	3.79E-6	7.09	1.50E-5	7.15
40×40	8.68E-7	5.00	2.33E-6	5.72	8.68E-7	5.12	2.77E-6	5.88
50×50	2.85E-7	4.98	6.12E-7	5.99	2.85E-7	4.98	6.83E-7	6.28

The fifth order scheme on structured meshes

Numerical results



2D-Euler equations. Computing time and error. Number signs and a solid red line denote the results of WENO-ZQ scheme with different linear weights (1), (2) and (3); squares and a solid line denote the results of WENO-JS scheme.

Numerical results

	WENO3 (1)				WENO3 (2)			
h	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
2/5	1.09E-1		1.73E-1		1.09E-1		1.73E-1	
2/10	3.98E-2	1.45	7.30E-2	1.25	4.05E-2	1.43	7.51E-2	1.21
2/20	6.59E-3	2.60	1.41E-2	2.37	6.60E-3	2.62	1.43E-2	2.39
2/40	8.49E-4	2.96	1.81E-3	2.97	8.49E-4	2.96	1.81E-3	2.98
2/80	1.04E-4	3.03	2.19E-4	3.04	1.04E-4	3.03	2.19E-4	3.04
	WENO3 (3)				WENO3 (4)			
h	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
2/5	1.09E-1		1.73E-1		1.09E-1		1.73E-1	
2/10	4.06E-2	1.43	7.56E-2	1.20	3.99E-2	1.45	7.33E-2	1.24
2/20	6.60E-3	2.62	1.43E-2	2.40	6.59E-3	2.60	1.41E-2	2.37
2/40	8.49E-4	2.96	1.81E-3	2.98	8.49E-4	2.96	1.81E-3	2.97
2/80	1.04E-4	3.03	2.19E-4	3.04	1.04E-4	3.03	2.19E-4	3.04

The third order scheme on unstructured meshes

Numerical results

	WENO4 (1)				WENO4 (2)			
h	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
2/5	2.98E-2		5.86E-2		6.41E-2		1.09E-1	
2/10	1.10E-3	4.75	4.43E-3	3.73	3.16E-3	4.34	1.82E-2	2.59
2/20	4.43E-5	4.64	2.06E-4	4.42	1.06E-4	4.90	1.08E-3	4.07
2/40	1.93E-6	4.52	9.08E-6	4.51	2.14E-6	5.63	1.34E-5	6.34
2/80	9.80E-8	4.30	4.76E-7	4.25	9.80E-8	4.45	4.91E-7	4.77
	WENO4 (3)				WENO4 (4)			
h	L^1 error	order	L^∞ error	order	L^1 error	order	L^∞ error	order
2/5	6.74E-2		1.13E-1		3.45E-2		6.89E-2	
2/10	3.72E-3	4.18	2.09E-2	2.44	1.20E-3	4.84	6.65E-3	3.37
2/20	1.27E-4	4.86	1.28E-3	4.03	4.74E-5	4.67	3.44E-4	4.27
2/40	2.23E-6	5.84	1.54E-5	6.38	1.92E-6	4.62	9.07E-6	5.25
2/80	9.81E-8	4.51	4.95E-7	4.96	9.79E-8	4.30	4.78E-7	4.24

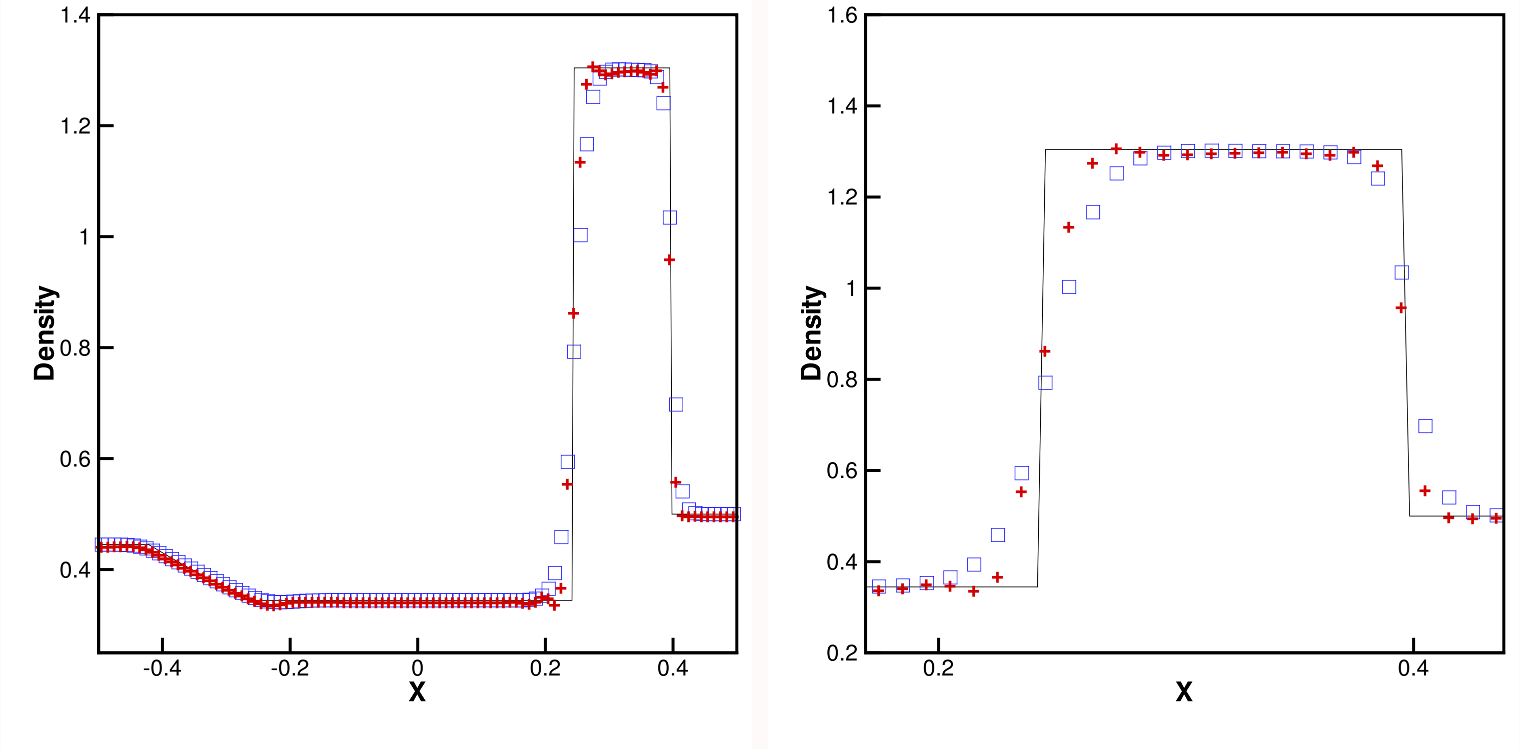
The fourth order scheme on unstructured meshes

- Lax problem

We solve the 1D Euler equations with Riemann initial condition for the Lax problem:

$$(\rho, u, p, \gamma)^T = \begin{cases} (0.445, 0.698, 3.528, 1.4)^T, & x \in [-0.5, 0), \\ (0.5, 0, 0.571, 1.4)^T, & x \in [0, 0.5]. \end{cases} \tag{25}$$

Numerical results



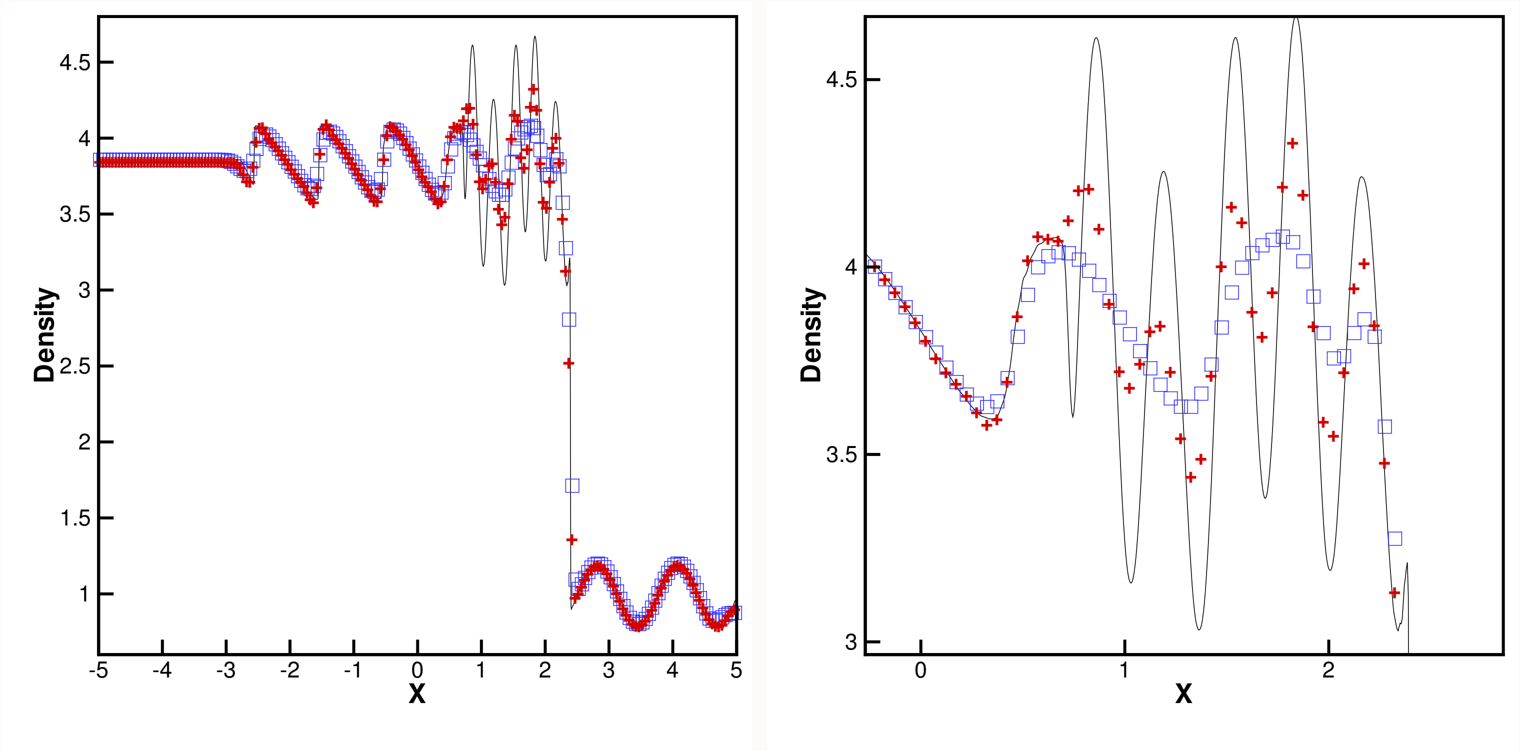
The Lax problem. T=0.16. From left to right: density; density zoomed in. Solid line: the exact solution; plus signs: the results of WENO-ZQ scheme; squares: the results of WENO-JS scheme. Cells: 100.

Numerical results

- The shock density wave interaction problem.

We solve the Euler equations with a moving Mach = 3 shock interacting with sine waves in density: $(\rho, \mu, p, \gamma)^T = (3.857143, 2.629369, 10.333333, 1.4)^T$ for $x \in [-5, -4)$; $(\rho, \mu, p, \gamma)^T = (1 + 0.2\sin(5x), 0, 1, 1.4)^T$ for $x \in [-4, 5]$.

Numerical results



The shock density wave interaction problem. T=1.8. From left to right: density; density zoomed in. Solid line: the exact solution; plus signs: the results of WENO-ZQ scheme; squares: the results of WENO-JS scheme. Cells: 200.

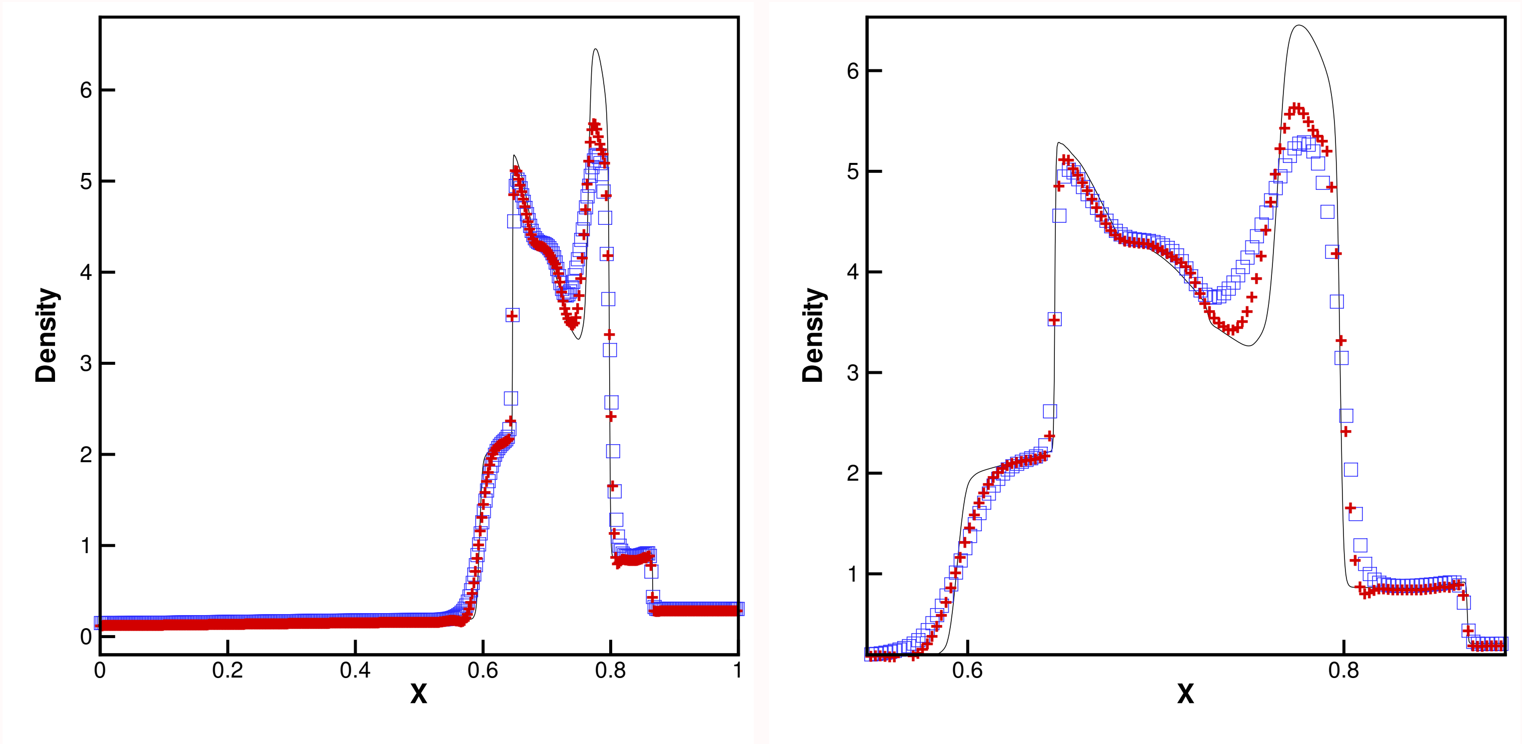
Numerical results

- The blast wave problem.

We now consider the interaction of two blast waves. The initial conditions are:

$$(\rho, u, p, \gamma)^T = \begin{cases} (1, 0, 10^3, 1.4)^T, & 0 < x < 0.1, \\ (1, 0, 10^{-2}, 1.4)^T, & 0.1 < x < 0.9, \\ (1, 0, 10^2, 1.4)^T, & 0.9 < x < 1. \end{cases}$$

Numerical results



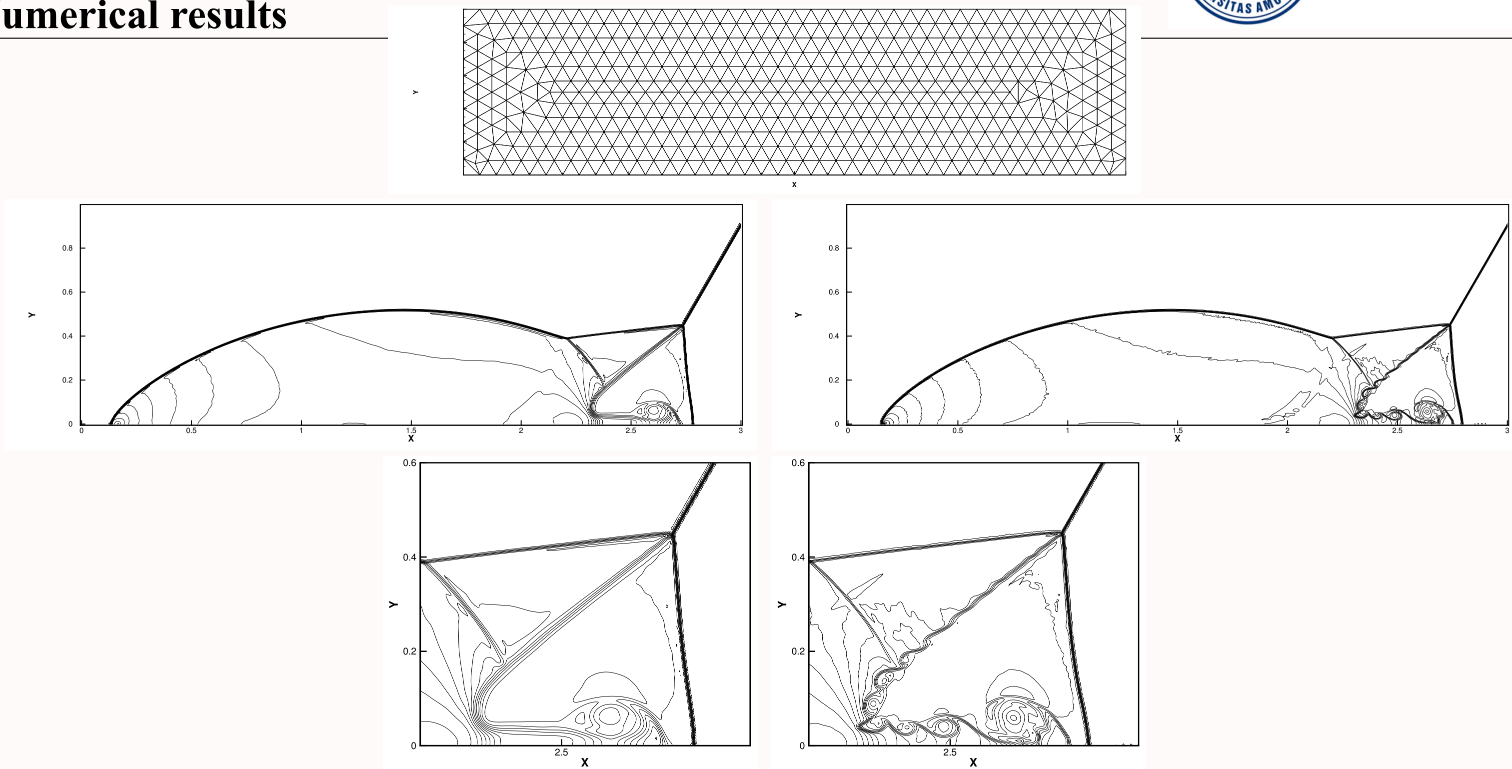
The blast wave problem. T=0.038. From left to right: density; density zoomed in. Solid line: the exact solution， cells: 400.

Numerical results

- Double mach reflection problem

The computational domain for this problem is $[0,4] \times [0,1]$. The reflecting wall lies at the bottom, starting from $x = \frac{1}{6}$. Initially a right-moving Mach 10 shock is positioned at $x = \frac{1}{6}$, $y = 0$ and makes a 60° angle with the x-axis. For the bottom boundary, the exact post-shock condition is imposed for the part from $x = 0$ to $x = \frac{1}{6}$ and a reflective boundary condition is used for the rest. At the top boundary, the flow values are set to describe the exact motion of a Mach 10 shock. We compute the solution up to $t = 0.2$.

Numerical results



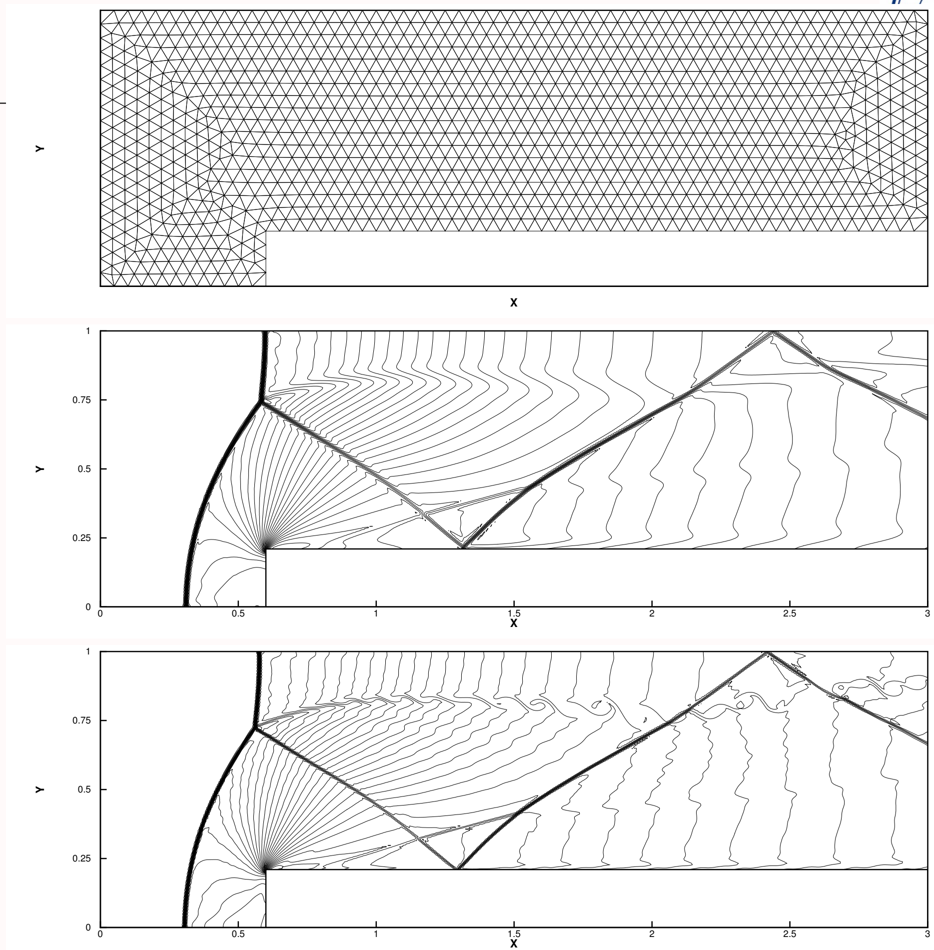
Double Mach reflection problem. T=0.2. The mesh points on the boundary are uniformly distributed with cell length $h = 1/320$. 30 equally spaced density contours from 1.5 to 22.7. Top： Sample mesh; Zoom-in pictures around the Mach stem (Bottom). The third order WENO (left); The fourth order WENO (right).

Numerical results

- Forward step problem

A Mach 3 wind tunnel with a step. The wind tunnel is 1 length unit wide and 3 length units long. The step is 0.2 length units high and is located 0.6 length units from the left-hand end of the tunnel. The problem is initialized by a right-going Mach 3 flow. Reflective boundary conditions are applied along the wall of the tunnel and in/out flow boundary conditions are applied at the entrance/exit. We compute the solution up to $t = 4$.

Numerical results



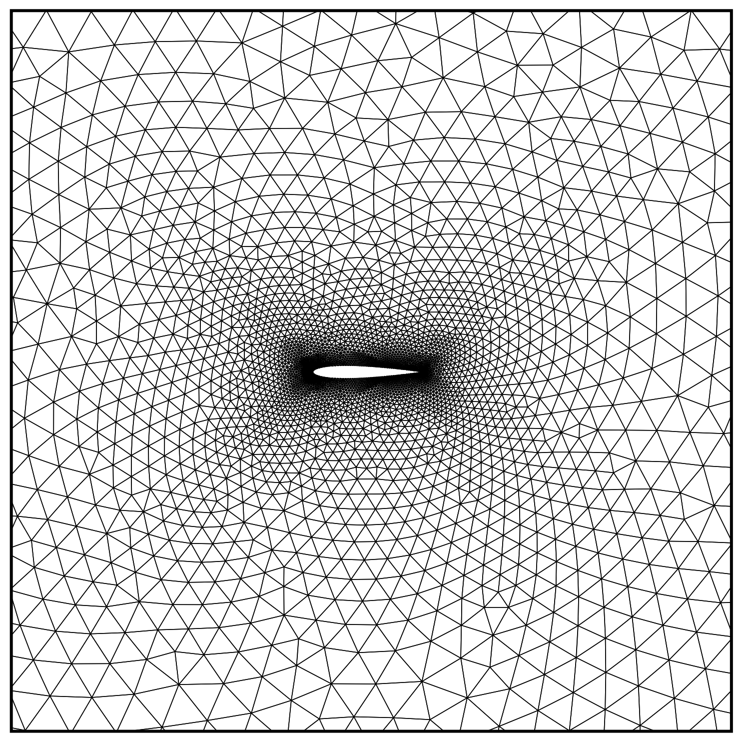
Forward step problem. 30 equally spaced density contours from 0.32 to 6.15. Top: Sample mesh; Middle the third order WENO scheme; bottom: the fourth order WENO scheme. The mesh points on the boundary are uniformly distributed with cell length $h = 1/160$.

Numerical results

- Inviscid Euler transonic flow past a single NACA0012 airfoilproblem

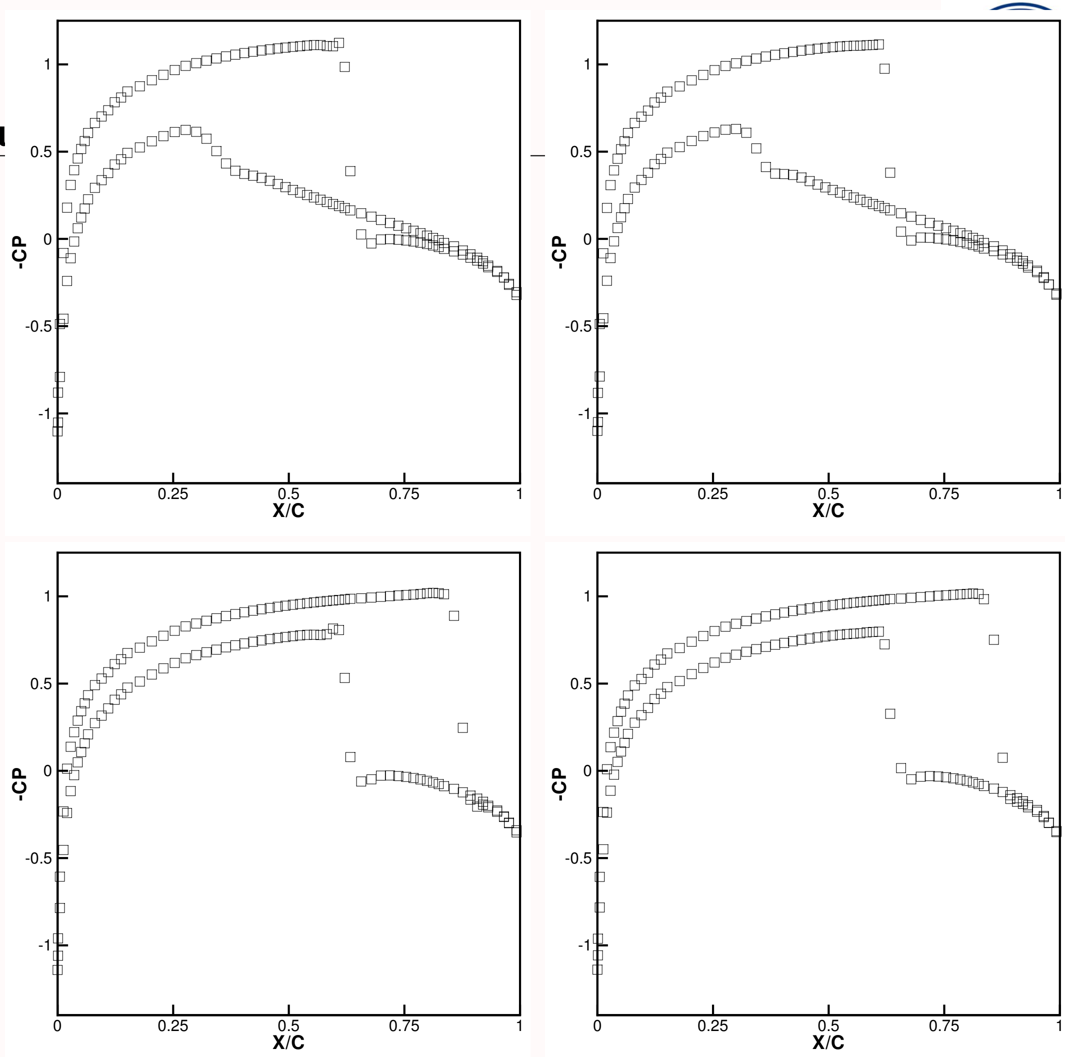
We consider inviscid Euler transonic flow past a single NACA0012 airfoil configuration with Mach number $M_\infty = 0.8$, angle of attack $\alpha = 1.25^\circ$ and with $M_\infty = 0.85$, angle of attack $\alpha = 1^\circ$. The computational domain is $[-15, 15] \times [-15, 15]$.
The mesh used in the computation consists 9340 elements with the maximum diameter of the circumcircle being 1.4188 and the minimum diameter being 0.0031 near the airfoil.

Numerical results



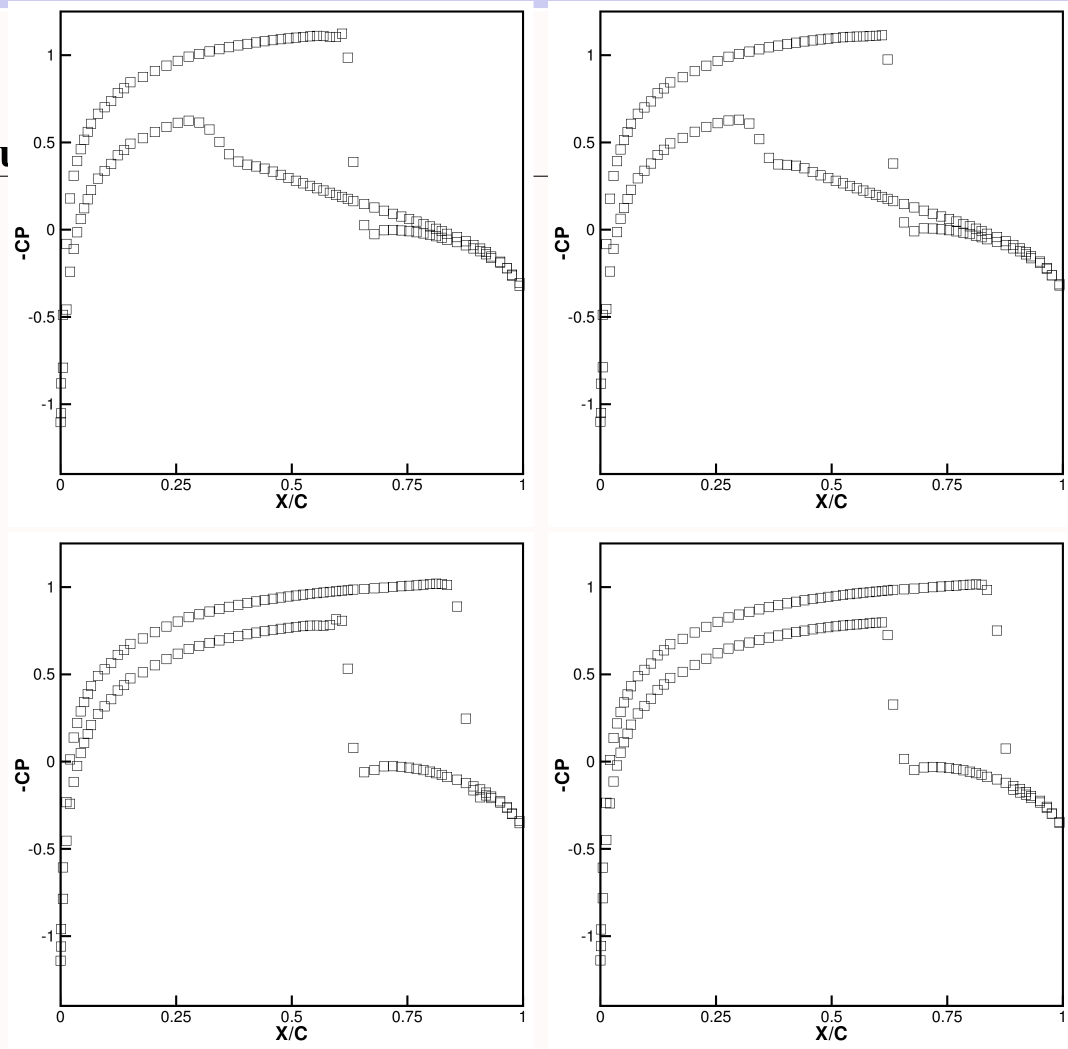
NACA0012 airfoil. Sample mesh

Numerical resu



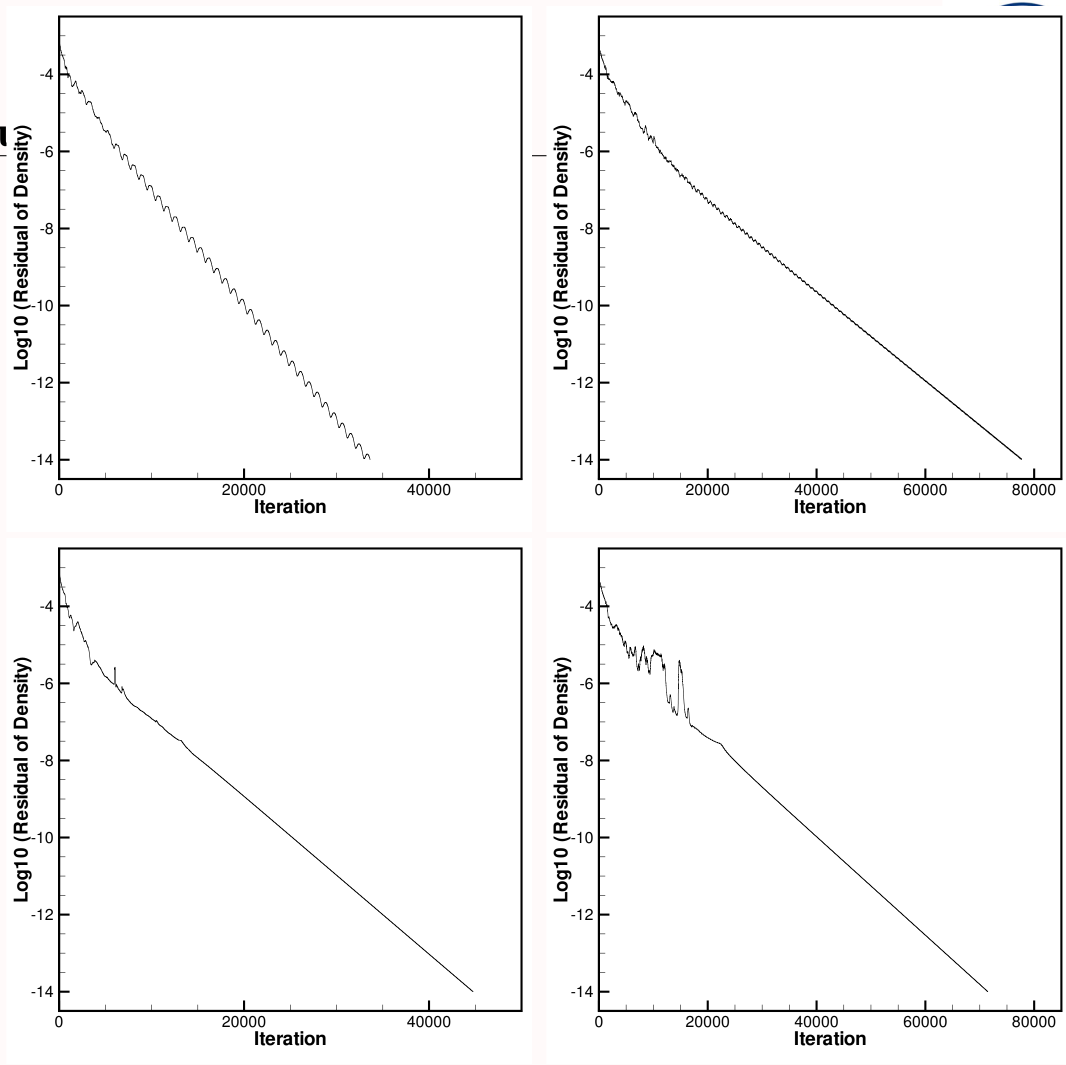
NACA0012 airfoil. Pressure distribution, from top to bottom: $M_\infty = 0.8$, angle of attack $\alpha = 1.25^\circ$ and $M_\infty = 0.85$, angle of attack $\alpha = 1^\circ$. Left: the third order WENO scheme; right: the fourth order WENO scheme.

Numerical resu



NACA0012 airfoil. Top: $M_\infty = 0.8$, angle of attack $\alpha = 1.25°$, 30 equally spaced mach number contours from 0.172 to 1.325; Bottom: $M_\infty = 0.85$, angle of attack $\alpha = 1°$, 30 equally spaced mach number contours from 0.158 to 1.357. Left: the third order WENO scheme; right: the fourth order WENO scheme.

Numerical resu



NACA0012 airfoil. Reduction of density residual as a function of the number of iterations, from top to bottom: $M_\infty = 0.8$, angle of attack $\alpha = 1.25^\circ$ and $M_\infty = 0.85$, angle of attack $\alpha = 1^\circ$. Left: the third order WENO scheme; right: the fourth order WENO scheme.

4 Conclusions

- A new simple finite volume WENO scheme is constructed for solving the hyperbolic conservation laws on structure and unstructured meshes.
- The crucial advantages of WENO-ZQ scheme are its simplicity and simultaneously obtaining optimal order accuracy with unequal spatial stencils.
- The procedure of this new WENO methodology is adopted by artificially setting linear weights.
- Comparing it with the classical WENO schemes, the new WENO scheme is very simple in the computation of problems with strong shocks, can obtain the same order accuracy in the same big stencil simultaneously having less absolute numerical errors in L^1 and L^∞ norms.

The End

THANK YOU!

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