Sichuan University

Chengdu, March-April 2018

DERIVATIONS ON OPERATOR ALGEBRAS

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Introduction

- The theory of algebras of operators acting on Hilbert space began in 1930s with a series of papers by von **Murray and Neumann** (*On Rings of Operators, I-IV*). The principal motivations of these authors were the theory of unitary group representations and certain aspects of the quantum mechanical formalism. They analyzed in great details the structure of the family of algebras which are referred nowadays as *von Neumann algebras* or *W*-algebras*. These algebras have the distinctive property of being closed in the weak operator topology.
- In 1943 **Gelfand and Naimark** characterized and analyzed uniformly closed operator algebras, the so-called *C*-algebras*.

- Nowadays the theory of Operator Algebras is an intensively developed field, which is characterized by interlace of pure mathematical and application aspects. An important role which is played by this theory in the arsenal of methods of mathematical physics is motivated by the fact that in terms of operator algebras, their states, representations, groups of automorphisms, and derivations one can describe and investigate the properties of model systems with infinite number of degrees of freedom, studied in quantum field theory and statistical physics.

ALGEBRAS

A – a **complex algebra**, i.e. linear space over C equipped with an associative (but not commutative in general) multiplication $(x, y) \to xy$, with usual relations between algebraic operations $+, \lambda, \cdot$. Equivalently

A – is an associative ring which is also a complex linear space.

EXAMPLES
Commutative case

1. $P[t]=\{a_0+a_1t+\cdots a_nt^n\}, a_i\in C, i=1,2,\ldots$ – algebra of all complex polynomial on the variable *t.*

2. $C[0,1]$ (resp. $C^\infty[0,1]$) algebra of all continuous (resp. infinitely differentiable) complex functions on [0,1].

3. $L^0[0,1]$ (resp. $L^\infty[0,1]$) algebra of all measurable (resp. essentially bounded measurable) complex functions on [0,1].

EXAMPLES
Non commutative case

4. $M_n(C)$ – algebra of matrices over C;

5. $B(H)$ – algebra of all bounded linear operators on a complex Hilbert space H;

6. $B(X)$ – algebra of all bounded linear operators on a complex Banach space X.

DERIVATIONS

Definition. A linear operator $d : A \to A$ is called a *derivation* on the algebra A if

$$d(xy) = d(x)y + xd(y) \quad \text{(Leibniz rule)}$$

for all $x, y \in A$.

Examples. Commutative case:

1. $A = P[t]$

$$d(a_0 + a_1 t + \cdots a_n t^n) = a_1 + 2a_2 t + \cdots n a_n t^{n-1}.$$

2. $A = C^\infty[0,1], d(f) = \frac{df}{dt}, f \in C^\infty(0,1).$

3. A – non commutative, $a \in A$ fixed.

$$d_a(x) = [a, x] = ax - xa, x \in A.$$

Such derivation is called *inner derivation*.

4. $A \subset B$ subalgebra (or ideal) $a \in B.$

If $d = d_a : A \to A,$

then d is called *spatial derivation* on $A.$

Algebras which *have only inner derivations*:

– finite dimensional simple central algebras;

– simple unital C^*-algebras;

– von Neumann algebras;

– $B(X)$ – algebra of all bounded linear operators on a Banach space X.

MORE GENERAL PROBLEM:

Given an algebra A , does there exist an algebra B such that:

(*i*) A is an ideal in B, so that any element $a \in B$ defines a derivation on A by $d_a(x) = [a, x], x \in A$;

(*ii*) any derivation of B is inner;

(*iii*) any derivation of the algebra A is spatial and implemented by an element from B?

Examples of such algebras

– simple (non unital) C^*-algebras;

– $F(X)$ – finite rank operators on an infinite dimensional Banach space X;

– standard operator algebras on X, i.e. subalgebras of $B(X)$ which contain $F(X)$, etc.

References

I.N.Herstein, (2005) Non commutative rings, The Carus Mathematical Monographs.

P. Chernoff (1973), Representation, automorphisms and derivations on some operators algebras, *J. Funct. Anal.* 12, 275-289.

S. Sakai (1971), C*-algebras and W*-algebras. Springer-Verlag.

S. Sakai (1991), Operator algebras in dynamical systems. Cambridge University Press.

H. G. Dales (2000), Banach algebras and automatic continuity. Clarendon Press, Oxford.

C^*- ALGEBRAS AND VON NEUMANN ALGEBRAS

H – a complex Hilbert space

$B(H)$– algebra of all bounded linear operators on H

$M \subset B(H)$ a *-subalgebra.

Definition. M is called a C^**-algebra* if it is closed in the norm topology.

Examples: 1) $B(H)$, $K(H)$– compact operators on H.

2) Commutative case. $C[0,1], C(Q), Q$ – compact Hausdorf space. *Conversely* any commutative C^*-algebra is isomorphic to some $C(Q)$.

Definition. A C^*-algebra $M \subset B(H)$ is said to be a *von Neumann algebra*, if it is closed in the weak operator topology and $1 \in M$ (identity operator).

Equivalently if $M = M'' = (M')'$, where

$$M' = \{x \in B(H) : xy = yx, \forall y \in M\}$$

– commutant of M.

Examples: 1) $B(H)$

2) Commutative case. $L^\infty[0,1]$ or more general $L^\infty(\Omega, \Sigma, \mu)$ for a measure space (Ω, Σ, μ).

Conversely, any commutative von Neumann algebra is isomorphic to $L^\infty(\Omega, \Sigma, \mu)$ for an appropriate measure space.

STATES AND TRACES

M – C*-algebra. A linear functional $f: M \to C$ is called a *state* if

i) f – positively defined, i.e. $f(x^* x) \geq 0, \forall x \in M;$

ii) $f(1) = 1.$

A state f is said to be *normal*, if $f(x_\alpha) \to 0$ for any monotone net $\{x_\alpha\} \subset M$ decreasing to $\theta \in M$.

A C*-algebra M is a von Neumann algebra if and only if M is monotone complete and admits a separating family of normal states.

Definition. A map $\tau : M^+ \to [0, +\infty]$ on a von Neumann algebra M is a *trace* if

i) $\tau(x+y) = \tau(x) + \tau(y), \forall x, y \in M^+ = \{a^* a : a \in M\};$

ii) $\tau(\lambda x) = \lambda \tau(x), \forall x, \lambda \in R^+, x \in M^+;$

iii) $\tau(x^* x) = \tau(x x^*), \forall x \in M.$

A trace τ is *normal* if $\tau(x_\alpha) \to \tau(x)$ for any increasing net $\{x_\alpha\} \subset M$ which monotone increases and $\sup x_\alpha = x$.

A trace τ is *faithful*, if $\tau(x)=0, x \in M^+$ implies $x=0$

A trace τ is called *semifinite*, if the set $M_\tau = \{x \in M^+ : \tau(x) < \infty\}$ is weakly dense in M.

A trace τ is said to be *finite*, if $M_\tau = M^+$, equivalently, $\tau(1) < +\infty$.

Any finite trace can be extended to a positive linear functional τ on M with $\tau(x^* x) = \tau(x x^*), \forall x \in M^+$.

PROJECTIONS AND CLASSIFICATION OF VON NEUMANN ALGEBRAS

M – a von Neumann algebra.

An element $e \in M$ is called a *projection*, if $e^* = e = e^2$ (i.e. self-adjoint idempotent).

The set $P(M)$ of all projections in M form a complete orthomodular lattice.

Two projection $e, f \in M$ are *equivalent* ($e \sim f$) if there exists $u \in M$ such that $e = u^*u, f = uu^*$.

A projection $e \in M$ is said to be *finite* if it is not equivalent to a proper subprojection, otherwise it is said to be *infinite*.

The algebra M is called *semifinite*, if any projection in M contains a nonzero finite projection.

M is called *finite* if **1** is finite projection, otherwise M is called *infinite*.

M is *properly infinite* if all nonzero projections in the center are infinite.

M is *purely infinite* or *type III* if all nonzero projection in M are infinite.

A von Neumann algebra M is finite if and only if it admits a separating family of finite normal traces

A von Neumann algebra is semifinite if and only if it admits a faithful normal semifinite trace.

A von Neumann algebra is of type III if and only if it has no nonzero normal semifinite trace.

A projection $e \in M$ is called *abelian*, if eMe is commutative. A projection e is said to be *faithful*, if the smallest central projection majorazing e is $\mathbf{1}$.

A von Neumann algebra M is said to be of *type I* if it has a faithful abelian projection. M is said to be *continuous* if it contains no nonzero abelian projection.

Given any von Neumann algebra M there exist five orthogonal central projection with sum 1

(i.e. $e_i e_j = 0, i \neq j, e_1 + ... + e_5 = 1$) such that

$$M = e_1 M \oplus e_2 M \oplus e_3 M \oplus e_4 M \oplus e_5 M$$

and

$e_1 M$ – is of type I finite (type I_{fin})

$e_2 M$ – is of type I infinite and semifinite (type I_∞)

$e_3 M$ – is finite and continuous (type II_1)

$e_4 M$ – is properly infinite, semifinite and continuous (type II_∞)

$e_5 M$ – is purely infinite (type *III*).

S. Sakai (1971), *C**-algebras and *W**-algebras. Springer-Verlag.

M. Takesaki (1991), Theory of Operator Algebras. I, Springer-Verlag, New-York; Heildelberg; Berlin.

DERIVATIONS AND AUTOMORPHISMS OF C^*-ALGEBRAS AND VON NEUMANN ALGEBRAS

M – C^*-algebra. One-to-one mapping $\alpha : M \to M$ is called a ** -automorphism* of M if

$$\alpha(x+y) = \alpha(x) + \alpha(y),$$
$$\alpha(\lambda x) = \lambda \alpha(x),$$
$$\alpha(xy) = \alpha(x)\alpha(y),$$
$$\alpha(x^*) = \alpha(x)^* \text{ for all } x, y \in M, \lambda \in C.$$

A (one-parameter) *group of automorphisms* of M is a mapping $t \to \alpha_t : R \to Aut(M)$ which is a group homomorpism of the additive group R into the group $Aut(M)$ of all * -automorphisms of M i.e.

$\alpha_0 = I$ – identical automorphism of M;

$\alpha_{t+s} = \alpha_t \circ \alpha_s$ for all $t, s \in R$.

The *infinitesimal generator* of the group α_t is the linear operator d on M with domain $D(d)$ = $\{x \in M$: there exists the (norm) limit

$$\lim_{t \to 0} \frac{\alpha_t(x) - x}{t} = d(x)\}.$$

From algebraic properties of the group α_t it easily follows that d is a *symmetric derivations* on $D(d)$, i.e.

(i) $d(x^*) = d(x)^*, x \in D(d);$

(ii) $d(xy) = d(x)y + xd(y), x, y \in D(d).$

The most studied is the case when $D(d) = M$.

Theorem 1. *Let M be a commutative C*-algebra. Then any derivation on M is identically zero.*

Theorem 2. *Let d be a derivation on an arbitrary C*-algebra. Then d is automatically bounded i.e. norm continuous.*

Theorem 3. *Let M be a von Neumann algebra. Then any derivation d on M is inner, i.e. there exists an element $a \in M$ such that* $d(x) = d_a(x) = [a, x] = ax - xa, \forall x \in M$.

S. Sakai (1971), *C**-algebras and *W**-algebras. Springer-Verlag.
S. Sakai (1991), Operator algebras in dynamical systems. Cambridge University Press.

PHYSICAL BACKGROUND

Physical theories consist of two elements:

1) **Kinematical structure** describing the instantaneous states and observables of the system.

2) **Dynamical rule** describing the change of these states and observables with time.

KINEMATICAL STRUCTURE

Classical mechanics of point particle:

states = points of a differentiable manifold X,

observables = functions over the manifold X.

Quantum mechanics (systems with a finite number of degrees of freedom):

states = unit vectors (rays) in a Hilbert space H;

observables = operators acting on H.

Quantum field theory (systems with the infinite number of degrees of freedom):

states = states on an Operator Algebra M;

observables = elements of the algebra M.

DYNAMICAL EVOLUTION

Classical mechanics:

group of diffeomorphisms of the manifold X;

Quantum mechanics:

group of unitary operators on the Hilbert space H;

Quantum field theory:

group of *-automorphism of the operator algebra M.

INFINITESIMAL MOTION:

The infinitesimal motion is described by some form of *Hamiltonian formalism*, incorporating the interparticle interaction.

Classical mechanics:

vector field on the manifold X = a derivation on the algebra $C^\infty(X)$ of all infinitely differentiable functions on X;

Quantum mechanics:

self-adjoint Hamiltonian operator h on the Hilbert space H which gives a spatial derivation $d(x) = [ih, x], x \in B(H)$;

Quantum field theory:

a derivation d on the operator algebra M of observables.

The *basic problem* which occurs in this approach – is the integration of these infinitesimal motion in order to obtain the dynamical flow.

In terms of *operator algebras* this means:

to prove that the given *derivation* on the algebra of observables is the infinitesimal generator of a one-parameter automorphisms group, moreover it is *spatial* (i.e. defined by some Hamiltonian operator) or even *inner* (i.e. the Hamiltonian operator itself is an observables in the considered physical system).

O. Bratteli, D. Robinson (1979), Operator algebras and quantum statistical mechanics, Vol. 1, Springer-Verlag.

Physical Theories

Kinematical Structure	
Observables	States

Dynamical Evolution	
Dynamical Flow	Infinitesimal Motion

Classical Mechanics

Differentiable Manifold	
Function over Manifold X	Points of the Manifold X

Group of Automorphisms	
Group of Diffeomor-phisms of X	Vector Field on X

Quantum Mechanics

Hilbert Space H	
Operators on H	Unit vectors in H

Group of Automorphisms	
Group of Unitary operators on H	Self-adjoint Ha-miltonian on H

Quantum Field Theory

Operator Algebra M	
Elements of the Algebra M	States on the Algebra M

Group of Automorphisms	
Group of * automor-phisms of M	Derivation on the Algebra M

C^*-MODELS

Let M be a C^*-algebra in $B(H)$.

Suppose that the dynamical evolution is given by a Hamiltonian operator $h = h^*$ on H,

$$\alpha_t(x) = e^{-iht} x e^{iht}$$

where $e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ – the operator exponent.

Generator

$$d(x) = \lim_{t \to 0} \frac{\alpha_t(x) - x}{t} = i[h, x] = [ih, x]$$

is a *spatial* derivation implemented by the (unbounded) operator ih.

In particular, if $h \in M$ (i.e. the Hamiltonian operator h is an observable), then the derivation d is *inner*.

Now Theorem 3 implies the following solution of the problem of integration of “infinitesimal dynamics” in the case $D(d) = M$.

Theorem 4. *Let* d *be a a derivation on a von Neumann algebra* M. *Then* d *is the infinitesimal generator of a norm continuous one-parameter automorphism group* $\{\alpha_t\}$ *of the algebra* M, *where*

$$\alpha_t(x) = e^{-iht} x e^{iht}$$

and h *is a self-adjoint element of* M.

DERIVATION ON UNBOUNDED OPERATOR ALGEBRAS

There are various classes of unbounded operator algebras which are important in analysis and mathematical physics:

Algebras of measurable operators

I. Segal (1953), A non commutative extension of abstract integration, *Ann. Math.* 57, 401-457.

E. Nelson (1975), Notes on non-commutative integration, *J. Funct. Anal*, 15, 91-102.

O^* **-algebras,**

EW^***-algebras,**

GB^***-algebras, etc.**

J. P. Antoine, A. Inoue, C. Trapani (2002), Partial *-Algebras and Their Operator Realizations, Kluwer AP, Dordrecht /Boston/ London.

K. Schmüdgen (1990), Unbounded Operator Algebras and Representation Theory. Akademie, Verlag. Berlin.

ALGEBRAS OF MEASURABLE OPERATORS AFFILIATED WITH VON NEUMANN ALGEBRAS

M – semi-finite von Neumann algebra on the Hilbert space H, $P(M)$-the lattice of projections in M.

Definition. A densely-defined closed operator x in H is said to be *affiliated with* M if $xu = ux$ for all unitary operators $u \in M'$, where M' – commutant M in $B(H)$.

Let τ be a faithful normal semifinite trace on M (i.e. M is semifinite von Neumann algebra).

Definition. An operator x affiliated with M is said to be τ *-measurable* if for each $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leq \varepsilon$ such that $e(H) \subset D(x)$ – domain of x, where $e^{\perp} = 1 - e$.

Denote by $L(M, \tau)$ – the set of all τ -measurable operators affiliated with M.

Definition. An operator x affiliated with M is said to be *measurable* if there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in M such that $p_n \uparrow 1,\ p_n(H) \subset D(x)$ and $p_n^{\perp} = 1 - p_n$ is finite for all $n \in N$.

Definition. A closed linear operator x in H is said to be *locally measurable with respect to* M if $x \eta M$ and there exists a sequence $\{z_n\}$ of central projections in M such that $z_n \uparrow 1$ and $z_n x$ is measurable for all $n \in N$.

Denote by $S(M)$ (resp. $LS(M)$) the set of all measurable (resp. locally measurable) operators with respect to M. It is known that $LS(M)$ is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator, and contains $S(M)$ as a solid *-subalgebra, while $S(M)$ contains $L(M,\tau)$ as a solid *-subalgebra.

The *measure topology,* t_τ, in $L(M,\tau)$ is the one given by the following system of neighborhoods of zero:

$$V(\varepsilon,\delta)=\{x\in L(M,\tau): \exists e\in P(M), \tau(e^\perp)\le\delta, xe\in M, \|xe\|\le\varepsilon\},$$

where $\varepsilon>0, \delta>0$ and $\|\cdot\|$ denotes the uniform norm on M.

$L(M,\tau)$ equipped with the measure topology t_τ is *a complete metrizable topological * -algebra.*

The center *Z* of *M* is a commutative von Neumann algebra

$\Rightarrow Z=L^\infty(\Omega,\Sigma,\mu)$, where Ω is a measurable space which a

σ -finite measure μ.

Moreover the *center* of $L(M,\tau)$ is *-isomorphic with $L^0(\Omega,\Sigma,\mu)$.

1. DERIVATIONS ON ALGEBRAS OF MEASURABLE OPERATORS

Sh. A. Ayupov (2000), Derivations in measurable operator algebras, ***DAN RUz**,* № 3, 14-17.
Sh. A. Ayupov (2005), Derivations on unbounded operators algebras, Abstracts of the international conference ***Operators Algebras and Quantum Probability***. Tashkent 2005.

Problems: Are theorem 1-3 remain valid for the algebra $L(M,\tau)$.

Commutative case. If $M = L^{\infty}(\Omega, \Sigma, \mu)$, then (Theorem 1) any derivation on M is zero.

Consider $L(M,\tau) = L^{0}(\Omega, \Sigma, \mu)$ – all measurable functions on Ω.

In particular $L^{0}(0,1)$.

Does $L^{0}(\Omega, \Sigma, \mu)$ *admit a non zero derivation?*

A. F. Ber, V. I. Chilin, F. A. Sukochev (2006), Non-trivial derivation on commutative regular algebras, ***Extracta Math***., 21, 107-147.
A. G. Kusraev (2006), Automorphisms and Derivations on a Universally Complete Complex f-Algebra, ***Sib. Math. Jour.*** 47, 77-85.

$L^{0}(0,1)$ admits a non-zero (and hence discontinuous and non inner) derivation.

Let $d : L(M,\tau) \to L(M,\tau)$ be a derivation.

If $e \in Z$ central projection, then $d(e) = d(e^2) = 2d(e) \Rightarrow d(e) = 0$, i.e. $d|_{P(Z)} \equiv 0$.

Since the linear span of $P(Z)$ is dense in Z in measure topology $\Rightarrow$

any derivation which is continuous in the measure topology (in particular, any inner derivation) is identically zero on Z, i.e. $d|_Z \equiv 0$.

Therefore $d(zx) = d(z)x + zd(x) = zd(x), \forall z \in Z, x \in L(M,\tau)$

i.e. d is necessary Z-*linear*.

Conjecture. A derivation on $L(M,\tau)$ is inner (or at least continuous) if and only if it is Z -linear.

Theorem 5. *If M is a von Neumann algebra of type I, then any Z-linear derivation on the algebra* $L(M,\tau)$ *is inner.*

Corollary. *Let M be a type I von Neumann algebra. Then*

(i) *any Z-linear derivation on* $L(M,\tau)$ *is automatically continuous in the measure topology.*

(ii) *a derivation on* $L(M,\tau)$ *is inner if and only if it is continuous in the measure topology.*

See for details

S. Albeverio, Sh. A. Ayupov, K. K. Kudaybergenov, Derivations on the Algebra of Measurable Operators Affiliated with a Type I von Neumann Algebra // ***Siberian Advances in Mathematics***, 18(2008),86-94.

EXAMPLE OF NON *Z*-LINEAR DERIVATION

Let δ be any of the non zero derivations on $L^0[0,1]$ constructed in

A. F. Ber, V. I. Chilin,F. A. Sukochev, Non-trivial derivation on commutative regular algebras, ***Extracta Math*.,** 21 (2006) 418-419.

Consider the von Neumann algebra $M = L^\infty[0,1]\overline{\otimes} M_n(C)$, = the algebra of all $n \times n$ matrices $(f_{i,j})_{i,j=1}^n$ with entries from $L^\infty[0,1]$.

Then the algebra $L(M,\tau) \approx \mathcal{M}_n$ of all $n \times n$ matrices with entries $f_{i,j}$ from the algebra $L^0[0,1]$.

Define the mapping D_δ: $\mathcal{M}_n \to \mathcal{M}_n$ by

$$D_\delta((f_{i,j})_{i,j=1}^n) = (\delta(f_{i,j}))_{i,j=1}^n.$$

Then D_δ is a derivation on $\mathcal{M}_n$ which is not Z-linear (where $Z = L^\infty[0,1]$), and D_δ is discontinuous and hence can not be inner.

Theorem 6. *If* M *is a von Neumann algebra of type* I_∞ *, then any derivation on the algebra* $L(M,\tau)$ *is inner, and in particular is continuous in the measure topology.*

- ***Sh.Ayupov, S.Albeverio, K.K.Kudaybergenov*** ,Structure of derivations on various algebras of measurable operators for type I von Neumann algebras. ***Journal of Functional Analysis,*** 256 (2009), 2917-2943.
- ***Sh.Ayupov, S.Albeverio, K.K.Kudaybergenov*** , Description of Derivations on Locally Measurable Operator Algebras of Type I. ***Extracta Mathematicae***, 24 (2009), No 1, 1-15.

The following result gives a complete description of derivations on the algebra of measurable operators in the type I case.

Theorem 7. *If M is a von Neumann algebra of type I then every derivation D on $L(M,\tau)$, can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

Where D_a is inner and implemented by an element a from $L(M,\tau)$, and D_δ is a derivation generated by a derivation δ on the center of $L(M,\tau)$

Problem. *Obtain similar results for arbitrary von Neumann algebras.*

S. Albeverio, Sh. A. Ayupov, K. K. Kudaybergenov, Structure of Derivations on Various Algebras of Measurable Operators for Type I von Neumann Algebra. ***J. of Functional Analysis***, *256 (2009), 2917-2943.*

Sh. A. Ayupov, K. K. Kudaybergenov, Derivations on Algebras of Measurable Operators. *Infinite dimensional analysis, Quantum Probability and related* ***topics,13(2010)****, No. 2, 305-337.*

- Later the above result has been generalized for type III von Neumann algebras in the following sense.

- **Theorem 8.** *Let M be a direct sum of von Neumann algebras of type I_{∞} and type III. Then every derivation on the algebra $LS(M)$ of locally measurable operators affiliated with M is inner.*

- ***Sh.A.Ayupov, K.K.Kudaybergenov.*** Additive derivation on algebras of measurable operators. ***Journal of Operator Theory*** 62:2 (2012), 101-116.

Since $LS(M)$ contains $S(M)$ as a solid *-subalgebra, and $S(M)$ contains $L(M,\tau)$ as a solid *-subalgebra, from the above theorem 8 it follows that similar results are valid for derivations on the algebras $S(M)$ and $L(M,\tau)$ for type *I* and type *III* von Neumann algebras $\boldsymbol{M}$.

Thus the remaining case in the problem of description of derivations on the above algebras is the case of type *II* von Neumann algebras.

Recently ***A.Ber, V.Chilin and F.Sukochev*** in the paper "Continuity of derivations of algebras of locally measurable operators". ***Integral Equations and Operator Theory*** *75 (2013), 527 – 557*

have proved that any derivation on the algebra $LS(M)$ of all locally measurable operators affiliated with a properly infinite von Neumann algebra M is continuous with respect to the so called local measure topology. For type *I* and type *III* cases this follows from our theorem 8.

This is a new result for the type II_∞ case.

Later in the paper

"Continuous derivations on algebras of locally measurable operators are inner." ***Proc. London Math. Soc**. (2014)*
they proved the following extension of our Theorem 8 for the type II_∞ case.

Theorem 9. *Every derivation on the algebra $LS(M)$ is inner, provided that M is a properly infinite von Neumann algebra.*

Therefore, the only remaining case is type II_1 von Neumann algebra.

A partial answer to this case is given in the following theorem.

Theorem 10. If M is finite von Neumann algebra with a faithful normal semi-finite trace τ, equipped with the local measure topology *t*, then every *t*-continuous derivation $D : S(M) \to S(M)$ is inner.

Sh.Ayupov, K.Kudaybergenov. Innerness of continuous derivations on algebras of measurable operators affiliated with finite von Neumann algebras. ***J.Math. Anal. Appl.*** 408 (2013) 256-267.

The above theorem follows also from the above mentioned paper of ***A.Ber, V.Chilin and F.Sukochev*** in ***Proc. London Math. Soc***. *2014*.

Thus the only remaining problem concerning innerness of derivations on algebras of measurable operators is the case of type II_1 von Neumann algebras. In this case all the algebras $L(M,\tau)$, $S(M)$ and $LS(M)$ coincide with the algebra of all closed operators affiliated with the M (this is so called *Murray - von Neumann algebra*) – see also

R.Kadison, Zhe Liu A note on derivations of Murray - von Neumann algebras. PNAS. www.pnas.org/sgi/doi/10.1073/pnas.132158111.

Problem. Given a type II_1 von Neumann algebra M, does there exist a derivation on $S(M)$, which is not continuous in the local measure topology or is every derivation on $S(M)$ inner?