Hypergeometric function and Modular curvature

Yang Liu

MPIM, Bonn

Workshop on Noncommutative Geometry and Representation Theory
May 13, 2018

Notations for $\mathbb{T}^2_\theta$

- Smooth NC torus:

$$C^\infty(\mathbb{T}^2_\theta) = (C^\infty(\mathbb{T}^2), \times_\theta) = \left\{\sum a_{n,m} U^n V^m\right\}$$

  where
  - $UU^* = 1$, $VV^* = 1$ and $UV = e^{2\pi i\theta} VU$.
  - $a_{n,m}$ is of rapidly decay in $(n, m) \in \mathbb{Z}^2$.
  - $U \mapsto e^{ix}$, $V \mapsto e^{iy}$.
- Canonical trace $\varphi_0 : C^\infty(\mathbb{T}^2_\theta) \to \mathbb{C}$:

$$\varphi_0\left(\sum a_{n,m} U^n V^m\right) = a_{0,0}.$$

- $\mathcal{H}_0$ is the Hilbert space completion of $C^\infty(\mathbb{T}^2_\theta)$ with respect to the inner product:

$$\langle a, b\rangle = \varphi_0(b^* a).$$

- $C^\infty(\mathbb{T}^2_\theta) \subset B(\mathcal{H}_0)$ via left multiplication.

Notations for $\mathbb{T}^2_\theta$

- $\mathbb{T}^2$ action: $r = (r_1, r_2) \in \mathbb{R}^2/\mathbb{Z}^2$,

$$\alpha_r(U^n V^m) = e^{2\pi i(r_1 n + r_2 m)} U^n V^m$$

- Basic derivations: δ_1 and δ_2, play the role of $-i\partial_x$ and $-i\partial_y$ respectively:

$$\delta_1(U) = U,\ \delta_1(V) = 0,\ \ \delta_2(U) = 0,\ \delta_2(V) = V.$$

- Let $\tau \in \mathbb{C}$ with $\Re\tau > 0$ be the modular parameter of complex structures on $\mathbb{T}^2$. The analog complex structures on $\mathbb{T}^2_\theta$ is given by the $\bar\partial$-operator:

$$\bar\partial = \delta_1 + \bar\tau\delta_2,\ \ \bar\partial^* = \delta_1 + \tau\delta_2,$$

and the flat Dolbeault Laplacian for $\tau = \sqrt{-1}$:

$$\Delta = \delta_1^2 + \delta_2^2.$$

Conformal change of metric $g' = e^h g$

- Weyl factor: $k = e^h$, where $h = h^* \in C^\infty(\mathbb{T}^2_\theta)$.
- Rescaled volume functional:

$$\varphi(a) = \varphi_0(ae^{-h}), \;\; \forall a \in C^\infty(\mathbb{T}^2_\theta)$$

- φ is no longer a trace. It is a weight with the KMS property:

$$\varphi(ab) = \varphi(b\Delta(a)), \;\; \forall a, b \in C^\infty(\mathbb{T}^2_\theta),$$

  where Δ is called the modular operator, its logarithm is denoted by $\nabla = \log \Delta$:

$$\Delta(a) = k^{-1}ak, \;\; \nabla(a) = [a, h].$$

Conformal change of metric $g' = e^h g$

- New metric g' is implemented by the Dolbeault Laplacian with respect to φ:

$$\Delta_\varphi = \bar{\partial}_\varphi^* \bar{\partial} = k^{1/2} \Delta k^{1/2},$$
$$\Delta_k = k\Delta = k^{1/2} \Delta_\varphi k^{-1/2}.$$

- Local invariants for $(C^\infty(\mathbb{T}_\theta^m), \Delta_k)$ are recorded in the small time $(t \to 0)$ heat asymptotic:

$$\mathrm{Tr}(ae^{-t\Delta_k}) \backsim \sum_{j=0}^{\infty} V_j(a, \Delta_k) t^{(j-m)/2}, \ \forall a \in C^\infty(\mathbb{T}_\theta^m).$$

- Each V_j is a linear functional in a and is determined by its functional density $v_j \in C^\infty(\mathbb{T}_\theta^m)$:

$$V_j(a, \Delta_k) = \varphi_0(a v_j), \ \forall a \in C^\infty(\mathbb{T}_\theta^m).$$

Spectral Geometry for Riemannian manifolds

Let (M, g) be a closed Riemannian manifold and Δ be the scalar Laplacian operator.

- All the odd coefficients vanish since the manifold has no boundary.
- In general, the even heat coefficients involve complicated combinations of the components of the curvature tensor and all its derivatives.
- Upto a universal constant, the first one equals the volume functional, that is, $v_0 = 1$:

$$V_0(a, \Delta) = \int_M a d\mu_g, \ \ \forall a \in C^\infty(M).$$

- The second one recovers the scalar curvature function: $v_2 = S_g/6$:

$$V_2(a, \Delta) = \int_M a(S_g/6) d\mu_g, \ \ \forall a \in C^\infty(M).$$

Modular Scalar Curvature

Definition

We define the scalar curvature $R_{\Delta_k} \in C^\infty(\mathbb{T}_\theta^m)$ to be the functional density of the second heat coefficient:

$$V_2(a, \Delta_k) = \varphi_0\left(aR_{\Delta_k}\right), \forall a \in C^\infty(\mathbb{T}_\theta^m).$$

The full expression of R_{Δ_k} has been computed in various settings: Connes-Moscovici, Khalkhali-Fathizadeh, Moscovici-Lesch, Liu.

Modular Scalar Curvature

Theorem (Full local expression of R_{Δ_k})

Recall $k = e^h$ is the Weyl factor, $\Delta(a) = k^{-1}ak$ and $\nabla(a) = [a,h]$.

$$\begin{aligned} R_{\Delta_k} &= k^{-m/2}K(\Delta)(\nabla^2 k)\cdot g^{-1} + k^{-m/2-1}H(\Delta^{(1)},\Delta^{(2)})(\nabla k\nabla k)\cdot g^{-1} \\ &= e^{(-m/2+1)h}\left(\tilde{K}(\nabla)(\nabla^2 h)\cdot g^{-1} + \tilde{H}(\nabla^{(1)},\nabla^{(2)})(\nabla h\nabla h)\cdot g^{-1}\right) \end{aligned}$$

- g^{-1} is the metric tensor on the cotangent bundle so that

$$-(\nabla^2 k)\cdot g^{-1} = \Delta k = \sum_1^m \delta_j^2(k)$$

$$(\nabla h\nabla h)\cdot g^{-1} = \langle dh, dh\rangle_{g^{-1}} = \sum_1^m [\delta_j(k)]^2.$$

- In Connes-Moscovici's paper, the one variable function is a generating function of Bernoulli numbers:

$$\tilde{K}(u) = 8 \sum_{1}^{\infty} \frac{B_{2n}}{(2n)!} u^{2n-2}.$$

- Connes-Moscovici relation:

$$\tilde{H}(u,v) = \frac{\tilde{K}(v) - \tilde{K}(u)}{v+u} + \frac{\tilde{K}(v+u) - \tilde{K}(v)}{u} - \frac{\tilde{K}(v+u) - \tilde{K}(u)}{v}.$$

- Prominent role of divided difference (Lesch): since $\tilde{K}$ is a even function, the first term in the RHS above is indeed a divided difference:

$$\tilde{H}(u,v) = -[u+v,u]\tilde{K} + [u+v,v]\tilde{K} + [-u,v]\tilde{K},$$

where

$$[u,v]K \triangleq \frac{K(u) - K(v)}{u - v}.$$

Pseudo differential calculus

- The heat operator can be defined using holomorphic functional calculus:

$$e^{-t\Delta_k} = \frac{1}{2\pi i} \int_C e^{-\lambda} (\Delta - \lambda)^{-1} d\lambda,$$

  where the contour C is chosen to be the imaginary axis, from $-\infty i$ to $i\infty$.
- $\sigma(\Delta_k) = p_2 + p_1 + p_0$, $p_2 = k\,|\xi|^2$ and $p_1 = p_0 = 0$.
- Pseudo differential calculus provides a recursive algorithm to construct an approximation of the resolvent symbol:

$$\sigma\left((\Delta_k - \lambda)^{-1}\right) \backsim b_0 + b_1 + b_2 + \dots,$$

  starting with the resolvent of the leading symbol:

$$b_0 = (p_2 - \lambda)^{-1} = (k\,|\xi|^2 - \lambda)^{-1}.$$

Rearrangement Lemma

- The trace can be recovered by integrating the symbol over the cotangent bundle, in particular,

$$R_{\Delta_k} = \int_{\mathbb{R}^m} \frac{1}{2\pi i} \int_C e^{-\lambda} b_2(\xi, \lambda) d\lambda d\xi$$
$$= (*) \int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} b_2(r, \lambda) d\lambda (r^{m-1} dr).$$

- b_j is a finite sum of terms of the form:

$$b_0^{a_0} \rho_1 b_0^{a_0} \cdots \rho_n b_0^{a_n} \triangleq (b_0^{a_0} \otimes \cdots \otimes b_0^{a_n}) \cdot (\rho_1 \cdots \rho_n)$$
$$= (k^{(0)} r^2 - \lambda)^{-a_0} \cdots (k^{(n)} r^2 - \lambda)^{-a_n} \cdot (\rho_1 \cdots \rho_n),$$

where ρ_j's are the derivatives of the symbols of Δ_k. For the b_2-term, ρ_j is either ∇k or $\nabla^2 k$.

Rearrangement Lemma

- For the b_2 term:

$$b_2 = \sum \left(b_0^a (\nabla^2 k) b_0^b + b_0^{\tilde{a}} (\nabla k) b_0^{\tilde{b}} (\nabla k) b_0^{\tilde{c}} \right) \cdot g^{-1}.$$

-

$$\begin{aligned}
&\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} b_0^a (\nabla^2 k) b_0^b d\lambda (r^{m-1} dr) \\
=&\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} (k^{(0)} r^2 - \lambda)^{-a} (k^{(1)} r^2 - \lambda)^b d\lambda (r^{m-1} dr) \cdot (\nabla^2 k) \\
&= k^{-m/2} K_{a,b}(\not\triangle; m)(\nabla^2 k)
\end{aligned}$$

- $H_{a,b,c}(\not\triangle^{(1)}, \not\triangle^{(2)}; m)(\nabla k \nabla k)$ is defined in a similar fashion.
- We have used the substitution:

$$k^{(1)} = k^{(0)} \not\triangle^{(1)}, \;\; k^{(2)} = k^{(0)} \not\triangle^{(1)} \not\triangle^{(2)}.$$

It follows that the modular curvature is of the form:

$$\sum k^j K_{a,b}(\not\triangle; m)(\nabla^2 k) \cdot g^{-1}$$
$$+ \sum k^{j-1} H_{\tilde{a},\tilde{b},\tilde{c}}(\not\triangle^{(1)}, \not\triangle^{(2)}; m)(\nabla k \nabla k) \cdot g^{-1},$$

where $j = -m/2$, m is the dimension.

Hypergeometric Functions

Proposition (Liu, 17)

Let $d_m = a + b + m/2 - 2$,

$$K_{a,b}(y;m) = \frac{\Gamma(d_m)}{\Gamma(a+b)}\, {}_2F_1(d_m, b; a+b; 1-y)$$

where

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{c-b-1} t^{b-1} (1-zt)^{-a} dt.$$

Similarly, set $d_m = a + b + c + m/2 - 2$,

$$H_{a,b,c}(u,v;m) = \frac{\Gamma(d_m)}{\Gamma(a+b+c)} F_1(d_m; c, b; a+b+c; 1-uv, 1-u).$$

Differential and Contiguous Relations

- Examples for the one-variable family:

$$K_{a,b}(u;m+2) = (d_m + ud/du)K_{a,b}(u;m),$$
$$K_{a,b+1}(u;m) = (b^{-1}d/du)K_{a,b}(u;m),$$
$$K_{a,b+1}(u;m) = (1 + b^{-1}ud/du)K_{a+1,b}(u;m)$$

  and

$$K_{a,b}(u;m+2) = aK_{a+1,b}(u;m) + bK_{a,b+1}(u;m).$$

  Similar relations holds for $H_{a,b,c}(u,v;m)$.

- We also have some reduction formulas for double integrals via divided difference:

$$F_1(a;1,1;b;x,y) = [x,y](z\,{}_2F_1(a,1;b;z))$$

Geometric functionals

- Conformal change of metric $g \mapsto g' = e^h g$ has the modular analogue: $\Delta \mapsto \Delta_k = k\Delta$ with $k = e^h$ and $h = h^* \in C^\infty(\mathbb{T}_\theta^m)$.
- Einstein-Hilbert Functional:

$$F_{\mathrm{EH}}(h) = F_{\mathrm{EH}}(k) = V_2(1, \Delta_k) = \varphi_0(R_{\Delta_k}).$$

- When $m = 2$: OPS (Osgood-Phillips-Sarnak) functional, a scaling invariant version of the Ray-Singer determinant functional:

$$F_{\mathrm{LogDet}'}(k) = -\operatorname{LogDet}'(\Delta_k) + \log \varphi_0(k) = \zeta'_{\Delta_k}(0) + \log \varphi_0(k).$$

- Facts, in demission two:

$$F_{\mathrm{EH}}(k) = \zeta_{\Delta_k}(0) + 1,$$
$$F_{\mathrm{LogDet}'}(k) = \zeta'_{\Delta}(0) - \int_0^1 V_2(h, \Delta_{k_s}) ds,$$

where $k_s \triangleq k^s$ for $s \in \mathbb{R}$.

Spectral zeta functions

For $a \in C^\infty(\mathbb{T}^m_\theta)$,

$$\zeta_{\Delta_k}(z;a) = \mathrm{Tr}(a\Delta_k^{-z}(1-P_k)),$$

when $\Re z$ is sufficiently large and P_k is the orthonormal projection to the kernel of Δ_k, with a meromorphic continuation via the Mellin's transform and heat asymptotic:

$$\zeta_{\Delta_k}(z;a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\, \mathrm{Tr}(ae^{-t\Delta_k}(1-P_k))dt.$$

Proposition (Gauss-Bonnet Thm for $\mathbb{T}^2_\theta$)

The EH-action $F_{\mathrm{EH}}(k)$ is a constant functional in k, in other words,

$$\zeta_{\Delta_k}(0) = \zeta_\Delta(0).$$

Connes-Tretkoff, Khalkhali-Fathizadeh.

Modular Curvature as Functional Gradients

Let $a = a^* \in C^\infty(\mathbb{T}^m_\theta)$, we consider variation along a in the following way:

$$h_\varepsilon = h + a\varepsilon, \;\; k_\varepsilon = e^{h+a\varepsilon}, \;\; \delta_a \triangleq \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$$

Definition

Let F be a functional in h or $k = e^h$, the functional gradient $\operatorname{grad}_k F \in C^\infty(\mathbb{T}^m_\theta)$ is the uniquely determined by the property:

$$\delta_a F(k) = \varphi_0(\delta_a(k) \operatorname{grad}_k F), \;\; \forall a \in C^\infty(\mathbb{T}^m_\theta).$$

Similarly, we define $\operatorname{grad}_h F$ via:

$$\delta_a F(h) = \varphi_0(h \operatorname{grad}_h F), \;\; \forall a \in C^\infty(\mathbb{T}^m_\theta).$$

Modular Curvature as Functional Gradients

By studying the variation of the heat operator, we obtain that for $m \geq 2$

$$\delta_a F_{\mathrm{EH}}(k) = \frac{2-m}{2} V_2(\delta_a(k)k^{-1}, \Delta_k),$$

therefore: (compare to the variation of the scalar curvature)

$$\boxed{\operatorname{grad}_k F_{\mathrm{EH}} = \tfrac{2-m}{2} k^{-1} R_{\Delta_k}.}$$

When $m = 2$, (for closed surfaces, the result is knows as Polyakov's conformal anomaly):

$$\delta_a F_{\mathrm{LogDet}'}(k) = -V_2(\delta_a(k)k^{-1}, \Delta_k),$$

thus

$$\boxed{\operatorname{grad}_k F_{\mathrm{LogDet}'} = -k^{-1} R_{\Delta_k}.}$$

It suggests that $\mathrm{grad}_k F$ is of the form:

$$k^j K(\not\triangle)(\nabla^2 k) \cdot g^{-1} + k^{j-1} H(\not\triangle^{(1)}, \not\triangle^{(2)})(\nabla k \cdot \nabla k) \cdot g^{-1},$$

and

$$K = K_{\Delta_k}, \;\; H = H_{\Delta_k}$$

by comparing the spectral functions.

Computing $\mathrm{grad}_k F$ via its local formula

- Upto a constant, both $F_{\mathrm{EH}}(k)$ and $F_{\mathrm{LogDet}'}(k)$ are of the form:

$$F(k) = \varphi_0\left(k^j T(\triangle)(\nabla k)\nabla k\right) \cdot g^{-1}, \;\; j \in \mathbb{R}.$$

Proposition

Let m be the dimension parameter,

$$F_{\mathrm{EH}}(k) = \varphi_0\left(k^{-m/2-1} T_{\Delta_k}(\triangle)(\nabla k)\nabla k\right) \cdot g^{-1},$$

with

$$T_{\Delta_k}(u) = -K_{\Delta_k}(1)\frac{u^{-m/2}-1}{u-1} + H_{\Delta_k}(u, u^{-1})$$

Some integration by parts formulas

$$\varphi_0(k^j K(\Delta)(\rho)) = K(1)\varphi_0(k^j \rho)$$
$$\varphi_0(k^j \rho_1 \Delta(\rho_2)) = \varphi_0(k^j \Delta^{-1}(\rho_1)\rho_2)$$

$$\varphi_0(k^j \nabla^2 k) = -\varphi_0(\nabla k^j \nabla k),$$

where

$$\nabla k^j = k^{j-1} \frac{\Delta^j - 1}{\Delta - 1}(\nabla k)$$

In $\dim M = 2$,

$$F_{\mathrm{LogDet}'}(k) = \varphi_0 \left(k^{-2} T_{\zeta'_{\Delta - k}}(\not\Delta)(\nabla k) \nabla k \right) \cdot g^{-1},$$

where

$$\begin{aligned} -T_{\zeta'_{\Delta - k}}(u) &= K_{\Delta_k}(1) \frac{\ln y}{(1-y)^2} \int_0^1 \left(y^{-s} - 1 \right) ds \\ &\quad - \frac{1}{2} \int_0^1 \left(\frac{y^s - 1}{y - 1} \right)^2 T_{\Delta_k}(y^s) \ln y ds \end{aligned}$$

Variational Formula

Theorem (Liu, 17)

Let $j \in \mathbb{R}$, consider $F(k) = \varphi_0\left(k^j T(\Delta)(\nabla k)\nabla k\right) \cdot g^{-1}$, then the functional gradient at point k is given by:

$$\operatorname{grad}_k F = k^j K_T(\Delta)(\nabla^2 k) \cdot g^{-1} + k^{j-1} H_T(\Delta^{(1)}, \Delta^{(2)})(\nabla k \cdot \nabla k) \cdot g^{-1}.$$

where

$$-K_T(u) = T(u) + u^j T(u^{-1}),$$
$$-H_T(u,v) = u^{j-1}[u^{-1}, v]K_T - (uv)^{j-1}[(uv)^{-1}, v^{-1}]K_T - [uv, u]K_T.$$

In particular, we observe the following symmetries:

$$K_T(u) = u^j K_T(u^{-1}), \;\; H_T(u,v) = (uv)^{j-1} H_T(v^{-1}, u^{-1})$$

Symbolic verification

In terms of Hypergeometric functions:

$$K_{\Delta_k}(y;m) = \frac{4}{m}K_{3,1}(y;m) - K_{2,1}(y;m),$$

$$H_{\Delta_k}(y_1,y_2;m) = (\frac{4}{m}+2)H_{2,1,1}(y_1,y_2;m) - \frac{4y_1 H_{2,2,1}(y_1,y_2;m)}{m} - \frac{8H_{3,1,1}(y_1,y_2;m)}{m}.$$

Theorem (Liu, 17)

Let $j = -m/2 - 1$, Connes-Moscovici relations:

$$-H_{\Delta_k}(u,v) = u^{j-1}[u^{-1},v]K_{\Delta_k} - (uv)^{j-1}[(uv)^{-1},v^{-1}]K_{\Delta_k} - [uv,u]K_{\Delta_k}$$

holds for all $m \geq 2$.

Symmetry Breaking

- In the commutative world, metrics are symmetric:

$$g' = e^h g, \;\; g = e^{-h} g'.$$

- For a noncommutative coordinate h, the appearance of the spectral functions shows vividly the some fundemental difference bewteen g and g'.

Symmetry bending along a projection p in $C^\infty(\mathbb{T}^2_\theta)$

- Let $s \in \mathbb{R}$, $h \triangleq h_s = sp$, then ∇h has the following eigen-decomposition with respect to the modular derivation $\nabla\!\!\!/ = [\cdot, p]$:

$$\nabla h = \mathrm{I}_1(h) + \mathrm{I}_2(h) = sp(\nabla p) + s(\nabla p)p,$$

  where $\mathrm{I}_1(h)$ and $\mathrm{I}_2(h)$ corresponds to eigenvalues $-s$ and s respectively, that is $\nabla\!\!\!/(p(\nabla h)) = -sp(\nabla h)$.
- For $\nabla^2 h$, we have the following decomposition with eigenvalues $0, -s, s, 0$: $\nabla^2 h = \mathrm{II}_1(h) + \cdots + \mathrm{II}_4(h)$, where

$$\mathrm{II}_1(h) = sp(\nabla^2 p)p, \ \mathrm{II}_2(h) = sp(\nabla^2 p)(1-p),$$

  and

$$\mathrm{II}_3(h) = s(1-p)(\nabla^2 p)p, \ \mathrm{II}_4(h) = s(1-p)(\nabla^2 p)(1-p),$$

- Let g be the flat metric on $\mathbb{T}^2$ and $g' = e^h g$, then the Gaussian curvature $\mathcal{R}_{g'} = \Delta h = s\Delta p$.
- Let $\mathcal{R}(s) \triangleq \mathcal{R}(h) = e^{-h}\,\mathrm{grad}_h\, F_{\mathrm{LogDet}'}$ be the normalized Gaussian curvature for $(C^\infty(\mathbb{T}^2_\theta), \Delta_k)$.

Proposition (Connes-Moscovici,14)

Upto a factor $\pi/4$,

$$\mathcal{R}(s) = s\tilde{K}(s)\Delta(p) + \frac{s^2}{2}\tilde{K}'(s)\left[p\Delta(p)p + (1-p)\Delta(p)(1-p)\right]$$

with

$$\frac{1}{4}s\tilde{K}(s) = 2\left(\frac{1}{e^s-1} - \frac{1}{s}\right) + 1,$$
$$\frac{1}{8}s^2\tilde{K}'(s) = \frac{-se^s}{(e^s-1)^2} - \frac{1}{e^s-1} + \frac{2}{s} - \frac{1}{2}.$$

In particular, $\mathcal{R}(s)$ is bounded in s.

In the calculation, CM functional relation

$$\tilde{H}(u,v) = -[u+v,u]\tilde{K} + [u+v,v]\tilde{K} + [-u,v]\tilde{K}$$

is used to derive:

$$\frac{s}{2}\tilde{H}(s,-s) = s\tilde{K}'(s) + 2(\tilde{K}(s) - \tilde{K}(0)).$$