On coarse embedding and equivariant coarse Baum-Connes conjecture

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Outline

Coarse embedding into Hilbert space

Let H be a Hilbert space. A map

$$f : X \to H$$

from X to a Hilbert H is a coarse embedding if there exists non-decreasing function $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \rho_\pm(t) = \infty$ such that

$$\rho_-(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_+(d(x, y))$$

for all $x, y \in X$.

Gromov's suggestion

Let $\Gamma = \pi_1(M)$ be the fundamental group of a closed manifold M, quipped with the word metric.

M. Gromov suggested that coarse embeddability of Γ into Hilbert space would be helpful to attack the Novikov conjecture for M.

Affirmative answer

Let X be a discrete metric space with bounded geometry.

Theorem (Guoliang Yu 2000)

If X is coarsely embeddable into Hilbert space, then the coarse Baum-Connes conjecture holds for X.

Applications

- The Novikov conjecture.
- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.
- $\cdots$

Property A

Definition (Yu 2000)

A discrete metric space X has property A if for all $R, \varepsilon > 0$ there exists a family of finite non-empty subset A_x of $X \times \mathbb{N}$, indexed by x in X, and a number $S > 0$ such that

- $\frac{\#(A_x \triangle A_y)}{\#(A_x \cap A_y)} < \varepsilon$ if $d(x, y) \leq R$;
- $A_x \subseteq B(x, S) \times \mathbb{N}$ for every $x \in X$.

Examples:

- trees.
- amenable groups; hyperbolic groups; discrete linear groups; groups acting on finite dimensional CAT(0) cube complexes...
- metric spaces with finite asymptotic dimension, or finite decomposition complexity, etc.

Property A implies coarse embedding

Theorem (G. Yu 2000)

If X has property A then X is coarsely embdeddable in Hilbert space

Corollary

The coare Baum-Connes conjecture holds for amenable groups; hyperbolic groups; discrete linear groups; groups acting on finite dimensional CAT(0) cube complexes, etc.

More answers

Theorem (Kasparov-Yu 2006)

Let X be a discrete metric space with bounded geometry. If X admits a coarse embedding into a uniformly convex Banach space, then the Coarse Novikov Conjecture is true for X.

Definition

A Banach space E is uniformly convex if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$1 - \left\| \frac{x+y}{2} \right\| > \delta$$

for all $x, y \in S(E) := \{x \in E : \|x\| = 1\}$ with $\|x - y\| \geq \varepsilon$.

Example

H, ℓ^p, L^p, C_p (the Schatten p classes), for $1 < p < \infty$.

Notations

X: proper metric space with bounded geometry;
Γ: a countable discrete group;
H: separable Hilbert space;
A group action of Γ on X is a homomorphism

$$\alpha : \Gamma \to \text{Isometry}(X).$$

An action α is proper if for any $x \in X$ there is a compact neighborhood U such that the set

$$\{\gamma \in \Gamma | \gamma \cdot U \cap U \neq \emptyset\}$$

is finite. We call X a Γ-*space* if X admits a proper action of Γ.

Let ϕ be a $*$-representation from $C_0(X)$ to $\mathcal{B}(H)$. $T \in \mathcal{B}(H)$.

- The *support* $Supp(T)$ of T is the complement of the set of points $(x, y) \in X \times X$ for which there exists $f, g \in C_0(X)$ s.t.

$$\phi(f)T\phi(g) = 0, \quad f(x) \neq 0, g(y) \neq 0.$$

- The *propagation* of T is defined to be

$$\sup\{d(x, y), x, y \in Supp(T)\}.$$

- T is said to be *locally compact* if $\phi(f)T$ and $T\phi(f)$ are compact for all $f \in C_0(X)$.

Example $X = \mathbb{Z}$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} \cdots & \cdots & -R & \cdots & -1 & 0 & 1 & \cdots & R & \cdots \end{array} \\
\begin{array}{c} \vdots \\ \vdots \\ -R \\ \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \\ -R \\ \vdots \end{array} &
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
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0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
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0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots
\end{pmatrix}
\end{array}
$$

Covariant representation

Let X be a Γ-space. For $\gamma \in \Gamma$ and $f \in C_0(X)$, define $\gamma(f) \in C_0(X)$ by

$$\gamma(f)(x) = f(\gamma^{-1}x).$$

Let H be a Γ-Hilbert space and ϕ be the $*$-representation from $C_0(X)$ to $\mathcal{B}(H)$.

$\rho : \Gamma \to \mathcal{U}(H)$ is a group homomorphism from Γ to the the set of all unitary elements in $\mathcal{B}(H)$

Definition

ϕ is called a *covariant representation* if for all $v \in H$, $f \in C_0(X)$ and $\gamma \in \Gamma$,

$$\gamma(\phi(f))(v) = \rho(\gamma)\phi(f)\rho(\gamma)^*(v).$$

$(C_0(X), \Gamma, \phi)$ is called a *covariant system*.

The "definition" of equivariant Roe algebra

For $\gamma \in \Gamma$, $T \in \mathcal{B}(H)$, define $\gamma(T) \in \mathcal{B}(H)$ as

$$\gamma(T)(v) = \rho(\gamma) T \rho(\gamma)^*(v), \quad v \in H.$$

T is called Γ-*invariant* if $\gamma(T) = T$ for all $\gamma \in \Gamma$.

Definition

The equivariant Roe algebra $C^*(X, H)^\Gamma$ is the norm closure of the algebra of locally compact, Γ-invariant operator on H with finite propagation.

The "definition" of equivariant Roe algebra

Let $X = pt$, $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $H' = \ell^2(\Gamma) \otimes H$.
For $\gamma \in \Gamma$, $f \in \ell^2(\Gamma)$, $\gamma(f)(x) = f(\gamma^{-1}x)$.
Then

$$C^*(X, H)^\Gamma = \mathcal{K}(H)$$

but for $T \in C^*(X, H')^\Gamma$,

$$T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a, b \in \mathcal{K}(H).$$

Hence

$$C^*(X, H)^\Gamma \ncong C^*(X, H')^\Gamma.$$

The definition of equivariant Roe algebra $C^*(X, H)^\Gamma$ depends on the choice of Hilbert spaces!

Admisible representation

The covariant system $(C_0(X), \Gamma, \phi)$ is said to be *admissible* if there exist a Γ-Hilbert space H_X and a separable and infinite dimensional Γ-Hilbert space V such that

- H is isomorphism to $H_X \otimes V$ as Γ-Hilbert space;
- $\phi = \phi_0 \otimes I$ for some Γ-equivariant $*$-homomorphism $\phi_0 : C_0(X) \to \mathcal{B}(H_X)$.
- for all finite subgroup F of Γ, and F-invariant Borel subset E of X, $V \cong \ell^2(F) \otimes H_E$ as F-Hilbert space for some Hilbert space H_E with trivial action and

$$(\gamma\xi)(z) = \xi(\gamma^{-1}z), \gamma \in F, \xi \in \ell^2(F).$$

Equivariant Roe algebra

Let $(C_0(X), \Gamma, \phi)$ be an admissible covariant system.

Definition

The *equivariant Roe algebra* $C^*(X, H)^\Gamma$ is defined to be the operator norm clousure of the $*$-algebra of all locally compact and Γ-invariant operators with finite propagation in $\mathcal{B}(H)$.

Proposition

The equivariant Roe algebra $C^*(X, H)^\Gamma$ does not depends on the choice of Hilbert space H.

$C^*(X, H)^\Gamma$ can be abbreviated as $C^*(X)$.

Equivariant Roe algebra

Let X and Y be two metric spaces with proper Γ-action.

Definition

X and Y are *equivariant coarse equivalent* if there exists a coarse embedding $f: X \to Y$ such that $f(X)$ is an ε-net of Y for some $\varepsilon > 0$ and $f(\gamma x) = \gamma f(x)$ for all $x \in X$ and $\gamma \in \Gamma$.

Proposition

If X and Y are Γ-equivariant coarse equivalent, then

$$C^*(X)^\Gamma \cong C^*(Y)^\Gamma.$$

Equivariant Roe algebra

Two special cases

- If X/Γ is compact, then

$$C^*(X)^\Gamma \overset{\text{Morita}}{\cong} C_r^*(\Gamma).$$

- If Γ is trivial, then

$$C^*(X)^\Gamma = C^*(X).$$

The localization algebra

The *localization algebra* $C_L^*(X)^\Gamma$ associated to a metric space is the norm closure of the set of functions

$$f : [0, \infty) \to C^*(X)^\Gamma$$

such that

- f is uniformly continuous;
- f is bounded under $\|f\| = \sup_{t \in [0,\infty)} \|f(t)\|$, and
- propagation of f converges to 0 as $t \to \infty$.

Equivariant Baum-Connes conjecture

Let X be a proper metric space with a proper Γ action. For any $d > 0$, let e be the evaluation map from $C_L^*(P_d(X))^\Gamma$ to $C^*(P_d(X))^\Gamma$ defined by $e(f) = f(0)$, where $P_d(X)$ is the Rip's complex. Then

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X))^\Gamma) \to \lim_{d\to\infty} K_*(C^*(P_d(X))^\Gamma) = K_*(C^*(X)^\Gamma).$$

Equivariant Baum-Connes conjecture

$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X))^\Gamma) \to K_*(C^*(X)^\Gamma)$ is an isomorphism.

Two special cases of equivariant Baum-Connes conjecture

If X/Γ is compact, then

- The Baum-Connes Conjecture asserts that e_* is an isomorphism.
- The Strong Novikov Conjecture states that e_* is an injection.

If Γ is the trivial group, then

- The Coarse Baum-Connes Conjecture asserts that e_* is an isomorphism.
- The Coarse Novikov Conjecture states that e_* is an injection.

Some recent results

(2000) **G. Yu:** If X is coarsely embeddable into Hilbert space, then the coarse Baum-Connes conjecture holds for X.

(2006) **G. Kasparov, G. Yu:** ..uniformly convex Banach space, ... the coarse Novikov conjecture...

(2007) **L. Shan, Q. Wang:** ...non-positively curved manifold, ... the coarse Novikov conjecture...

(2008) **G. Gong, Q. Wang, G. Yu:** The maximal coarse Novikov conjecture holds for the expander graphs derived from some residually finite Property (T) groups.

(2010) **R. Willet, G. Yu:** The maximal coarse Baum-Connes conjecture holds for expander graphs with large girth.

(2013) **X. Chen, Q. Wang, G. Yu:** The maximal coarse Baum-Connes conjecture for spaces which admit a fibred coarse embedding into Hilbert space.

(2014) **X. Chen, Q. Wang, Z. Wang:** Fibred coarse embedding into non-positively curved manifolds and higher index problem

(2015) **X. Chen, Q. Wang, Z. Wang:** The coarse Novikov conjecture and Banach spaces with Property (H)

Results on equivariant coarse Baum-Connes conjecture

Theorem (L. Shan 2008)

If Γ acts properly, freely, on a non-positively curved simply connected manifold, then equivariant assembly map is injective.

Remark

THIS is the first result on non-cocompact actions.

Equivariant coarse embedding

Definition

Let X be a metric space, B a Banach space. A map $f : X \to B$ is a coarse embedding, if there exist two non-decreasing functions $\rho_-, \rho_+ : \mathbb{R}_+ \to \mathbb{R}_+$ such that

- $\rho_-(d(x,y)) \leq \|f(x) - f(y)\| \leq \rho_+(d(x,y))$, for all $x, y \in X$.
- $\lim_{t\to\infty} \rho_-(t) = \infty$.

Definition

If Γ is group acting properly on X, A map $f : X \to B$ is equivariant coarse embedding, if there exists a proper action α of Γ on B such that

- $f(\gamma \cdot x) = \alpha(\gamma) \cdot f(x)$ for all $x \in X$ and $\gamma \in \Gamma$.

Affine actions

Let Γ be a group and let V be a vector space. An affine action of Γ on V is a homomorphism $\alpha : \Gamma \to \mathsf{Aff}(V)$, where $\mathsf{Aff}(V)$ is the group of affine bijections $V \to V$.
There exist a representation $\pi : \Gamma \to GL(V)$ and a map $b : \Gamma \to V$ such that

- $\alpha(g)v = \pi(g)v + b(g)$ for $g \in \Gamma$ and $v \in V$ and
- b satisfies the 1-cocycle relation

$$b(gh) = \pi(g)b(h) + b(g) \forall g, h \in \Gamma.$$

Affine actions

Theorem (Mazur-Ulam)

If V is a real Banach space, any isometry of V is affine.

Definition

A locally compact group Γ is a-T-menable if Γ admits a proper affine isometric action on a Hilbert space.

Fact

If metric space X is Γ-equivariant coarse embeddable in Hilbert space, then Γ is a-T-menable.

Examples of equivariant coarse embeddability

Example

Let X be a discrete metric space with a proper Γ action. Then X has a Γ-equivariant coarse embedding into $\ell^\infty(X)$.

Fix $x_0 \in X$ and set $\eta(x) = d(x, x_0)$. Let

$$f : X \to \ell^\infty(X), \qquad f(x)(y) = d(y, x) - \eta(y).$$

Let π be the regular action of Γ on $\ell^\infty(X)$ defined by

$$((\pi(g))\xi)(\gamma) = \xi(g^{-1}\gamma)$$

and let the affine isometric action to be

$$\alpha(g)\xi = \pi(g)\xi + \pi(g)\eta - \eta$$

for every $\xi \in \ell^\infty(X)$, all g and γ in Γ.

Examples of equivariant coarse embeddability

Theorem (N. Brown, E. Guentner+U. Haagerup, A.Przybyszewska)

Let X be a discrete metric space with bounded geometry and Γ be a discrete group acts on X properly. Then there exists a sequence of positive real numbers $\{p_n\}$ such that X admits a Γ-equivariant coarse embedding into ℓ^2-direct sum $\oplus_n \ell^{p_n}(X)$.

Remark

Banach space $\oplus_n \ell^{p_n}(X)$ is strictly convex not uniformly convex.

Positive vs. negative definite kernel

Definition

A map $k : X \times X \to \mathbb{C}$ (or $\mathbb{R}$) is of positive type if for all finite sequences $x_1, \cdots, x_n$ of elements of X and $\lambda_1, \cdots, \lambda_n$ of complex (or real) numbers, $\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} k(x_i, x_j) \geq 0$.

Definition

A map $k : X \times X \to \mathbb{R}$ is of negative type if for all finite sequences $x_1, \cdots, x_n$ of elements of X and $\lambda_1, \cdots, \lambda_n$ of real numbers such that $\sum_{i=1}^{n} \lambda_i = 0$, $\sum_{i,j=1}^{n} \lambda_i \lambda_j k(x_i, x_j) \leq 0$.

A positive (negative) type kernel is normalised if for all $x \in X$, $k(x, x) = 1$ $(k(x, x) = 0)$.

Equivariant coarse embedding into Hilbert space

Theorem (E. Guentner and J. Kaminker+ J. Roe)

Let X be a discrete metric space with a proper Γ-action. TFAE

- *X admits a Γ-equivariant coarse embedding into H.*
- *There exists a conditional negative definite kernel k on X and maps $\rho_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} \rho_-(t) = \infty$, satisfying*
    - i) $\rho_-(d(x,y)) \leq k(x,y) \leq \rho_+(d(x,y))$
    - ii) $k(x,y) = k(\gamma x, \gamma y)$ *for all* $\gamma \in \Gamma$.
- *For all $R > 0, \varepsilon > 0$ there exists a normalised symmetric kernel $k : X \times X \to \mathbb{R}$ of positive type such that*
    - i) $k(x,y) = k(\gamma x, \gamma y)$ *for all* $\gamma \in \Gamma$,
    - ii) $|k(x,y) - 1| < \varepsilon$ *if* $d(x,y) \leq R$,
    - ii) $\lim_{S\to\infty} \sup\{|k(x,y)| : d(x,y) \geq S\} = 0$.

The case Γ is an amenable group

Let X be a discrete metric space with bounded geometry. Let Γ be a discrete group acting on X properly.

Theorem

If Γ is amenable and X is coarse embeddable in Hilbert space, then X is Γ-equivariant coarse embeddable in Hilbert space.

Idea of proof

For any $R > 0$, $\delta > 0$, there exists $k : X \times X \to \mathbb{R}$ such that

- $|1 - k(x, y)| \leq \delta$ if $d(x, y) \leq R$,
- $\lim\limits_{S\to\infty} \sup\{|k(x, y)| : d(x, y) \geq S\} = 0$.

For any $x, y \in X$, define $f^{(x,y)}(g) = k(gx, gy) \in \ell^\infty(\Gamma)$. Since Γ is amenable, there is a right invariant mean $m : \ell^\infty(\Gamma) \to \mathbb{C}$. Define

$$\tilde{k}(x, y) = m(f^{(x,y)}).$$

Then $\tilde{k}(x, y)$ has the required property.

The case Γ is not an amenable group

Let $\Gamma = SL(2,\mathbb{Z})$. Then $\Gamma \cong \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

Let X be the Bass-Serre tree of Γ. Precisely,

- the vertex sets $V = \Gamma/\mathbb{Z}_6 \cup \Gamma/\mathbb{Z}_4$,
- the edge sets $E = \Gamma/\mathbb{Z}_2$.
- the endpoints of an edge are the vertices that contain it.

Then Γ acts properly on the tree X and X admits a Γ-equivariant coarse embedding into Hilbert space.

Bass-Serre tree

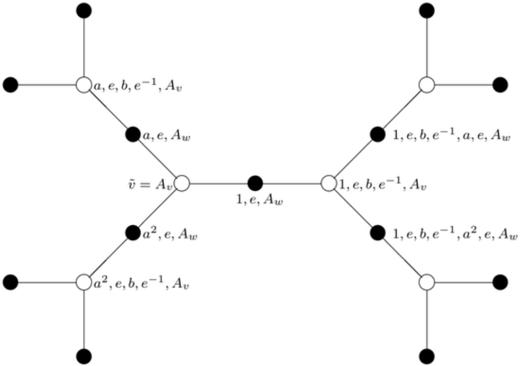

Figure: The Bass-Serre tree of $\mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$.

$SL_2(\mathbb{Z})$ is not amenable

Let

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Then

$$\mathbb{F}_2 = \langle a, b \rangle.$$

is a free subgroup of $SL_2(\mathbb{Z})$.

Outline of proof

Let $X = (V, E)$ be a tree. Let $\mathbb{E}$ be the set of oriented edges in X. Let $H = \ell^2(\mathbb{E})$ and Γ be a discrete group acting on X properly. Set

$$c(x,y)(e) = \begin{cases} 0 & e \notin [x,y]; \\ +1 & e \in [x,y] \text{ and } e \text{ ponts from } x \text{ to } y; \\ -1 & e \in [x,y] \text{ and } e \text{ ponts from } y \text{ to } x;. \end{cases}$$

Then $c : X \times X \to H$ is a continuous map satisfying

- $c(\gamma x, \gamma y) = \gamma c(x,y)$ for all $x, y \in X$ and $\gamma \in \Gamma$;
- $c(x,z) = c(x,y) + c(y,z)$ for all $x, y, z \in X$.

Let $b(\gamma) = c(\gamma x_0, x_0)$. Set

$$\alpha(\gamma)\xi = \gamma \cdot \xi + b(\gamma), \xi \in H, \gamma \in \Gamma$$

we get an isometric action α of Γ on H.
Fix a base point $x_0 \in X$. Define

$$f : V \to \ell^2(\mathbb{E}), \quad f(x) = c(x_0, x).$$

Then $\|f(x) - f(y)\| = \sqrt{2d(x,y)}$ is a equivariant coarse embedding with $\rho_-(t) = \sqrt{2t}$.

Let X be a discrete metric space with bounded geometry. Let Γ be an a-T-menable discrete group acting on X properly.

Question

Under what conditions is X Γ-equivariant coarse embeddable into Hilbert space?

Alain Valette's construction

Let (X, d) be a metric space with an proper action of Γ. Suppose we have

- a Hilbert space H with a unitary representation π of Γ,
- a continuous map $c : X \times X \to H$ such that
  - $\forall x, y \in X, g \in \Gamma : c(gx, gy) = \pi(g)c(x, y)$ (equivariance),
  - $\forall x, y, z \in X : c(x, y) + c(y, z) = c(x, z)$ (Chasles' relation),
  - there exists a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|c(x, y)\|^2 = \varphi(d(x, y)), \quad x, y \in X$$

(i.e. the norm of $c(x, y)$ depends only on $d(x, y)$).

Alain Valette's construction

Theorem

If $\varphi(t)$ is non-decreasing and $\lim_{t\to\infty} \varphi(t) = \infty$, Then X is Γ-equivariant coarse embedding into Hilbert space.

- finite dimensional $CAT(0)$ cube complex,
- spaces with measured walls,
- $\cdots$.

Main results

Let Γ be discrete group and X a discrete metric space with bounded geometry.

Theorem (Fu and Wang 2016)

If Γ acts properly on X and X is Γ-equivariant coarse embeddable into Hilbert space, then

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X))^\Gamma) \to \lim_{d\to\infty} K_*(C^*(P_d(X))^\Gamma)$$

is an isomorphism.

Main results

Let Γ be a discrete group and X a discrete metric space with bounded geometry.

Theorem (Fu, Wang and Yu 2017)

If Γ acts properly on X and X is coarse embeddable into Hilbert space, then

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X))^\Gamma) \to \lim_{d\to\infty} K_*(C^*(P_d(X))^\Gamma)$$

is injective.

Main results

Definition

Let X be a metric space with bounded geometry and Γ be a countable discrete group. If Γ acts on X properly and for any $\gamma \in \Gamma$, $\sup_{x \in X} d(\gamma x, x) < +\infty$, we say that X has *bounded distortion*.

Theorem (Fu, Wang and Yu 2018)

Let X be a Γ-space with bounded distortion. If X/Γ and Γ admit coarse embedding into Hilbert spaces, then

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X))^\Gamma) \to \lim_{d\to\infty} K_*(C^*(P_d(X))^\Gamma)$$

is injective.

Outline of proof

Let X be a Γ-space, $\xi : X/\Gamma \to H$ be the coarse embedding and $\pi : X \to X/\Gamma$ be the quotient map. Define the π-*localization algebra* $C^*_{\pi,L}(X)^\Gamma$ to be the norm closure of the set of functions

$$f : [0,\infty) \to \mathbb{C}^*(X)^\Gamma$$

, such that

- f is uniformly continuous;
- f is bounded under $\|f\| = \sup_{t\in[0,\infty)} \|f(t)\|$, and
- $\sup\{d(\pi(x),\pi(y)) : (x,y) \in \mathrm{Supp}(f(t))\} \to 0$ as $t \to \infty$.

Outline of proof

Let $\mathcal{S} = C_0(\mathbb{R})$ and V_a be a finite dimensional Euclidean subspace of H. Set

$$\mathcal{C}(V_a) = C_0(V_a, Cliff(V_a^0)), \quad \mathcal{A}(V_a) = \mathcal{S}\hat{\otimes}\mathcal{C}(V_a).$$

If $V_a \subset V_b$, there exists a $*$-homomorphism $\beta_{ba} : \mathcal{A}(V_a) \to \mathcal{A}(V_b)$.

Proposition

$$\begin{tikzcd}
\mathcal{A}(V_a) \arrow[r, "\beta_{ba}"] \arrow[d, "\gamma"] & \mathcal{A}(V_b) \arrow[d, "\gamma"] \\
\mathcal{A}(\gamma V_a) \arrow[r, "\beta_{ba}^{\gamma}"] & \mathcal{A}(\gamma V_a)
\end{tikzcd}$$

Let $\mathcal{A} = \lim\limits_{\rightarrow} \mathcal{A}(V_a)$.

Outline of proof

Proposition

Let Γ be a countable discrete group. TFAE

- Γ is embeddable into Hilbert space;
- $\exists$ compact, Hausdorff, second countable space Y with a right Γ-action which admits a continuous, proper negative type function F on $Y \rtimes \Gamma$.

Let Z be the space of probability measure on Y with w*-topology. The right Γ-action on Y induce a right Γ-action on Z.

$\tilde{F}(\mu, \gamma) = \int_Y F(y, \gamma) d\mu$ is a continuous proper negative type function on $Z \rtimes \Gamma$.

Hence, $Z \rtimes \Gamma$ admits a continuous, proper and affine isometric action on some Hilbert bundle $(H_\mu)_{\mu \in Z}$ with $H_\mu = H$.

Outline of proof

Note that $C(Z)\hat{\otimes}\mathcal{A}$ is a Γ-proper C^*-algebra with the Γ-action induced by the proper affine isometric action of $Z \rtimes \Gamma$ on $(H_\mu)_{\mu \in Z}$. We can show that

$$(\pi_L)_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X), C(Z)\hat{\otimes}\mathcal{A}) \to \lim_{d\to\infty} K_*(C_{\pi,L}^*(P_d(X), C(Z)\hat{\otimes}\mathcal{A})$$

and

$$(e'_\pi)_* : \lim_{d\to\infty} K_*(C_{\pi,L}^*(P_d(X), \mathcal{A})^\Gamma) \to \lim_{d\to\infty} K_*(C^*(P_d(X), \mathcal{A})^\Gamma)$$

are isomorphisms

Outline of proof

From the following commutative diagram

$$\begin{tikzcd}
\lim\limits_{d\to\infty} K_*(C^*_{\pi,L}(P_d(X))^\Gamma) \arrow[r, "i'_*"] & \lim\limits_{d\to\infty} K_*(C^*_{\pi,L}(P_d(X),C(Z))^\Gamma) \arrow[d, "(\beta_{\pi,L})_*"] \\
\lim\limits_{d\to\infty} K_*(C^*_L(P_d(X))^\Gamma) \arrow[u, "(\pi_L)_*"] \arrow[r, "\cong"] \arrow[d, "i_*"', "\cong"] & \lim\limits_{d\to\infty} K_*(C^*_{\pi,L}(P_d(X),C(Z)\hat{\otimes}\mathcal{A})^\Gamma) \\
\lim\limits_{d\to\infty} K_*(C^*_L(P_d(X),C(Z))^\Gamma) \arrow[r, "(\beta_L)_*", "\cong"'] & \lim\limits_{d\to\infty} K_*(C^*_L(P_d(X),C(Z)\hat{\otimes}\mathcal{A})^\Gamma) \arrow[u, "\cong", "(\pi_L)_*"']
\end{tikzcd}$$

then $(\pi_L)_*$ is injective.

Outline of proof

$$\begin{tikzcd}
\lim\limits_{d\to\infty} K_*(C^*(P_d(X),\mathcal{A})^\Gamma) & & \lim\limits_{d\to\infty} K_*(C^*_{\pi,L}(P_d(X),\mathcal{A})^\Gamma) \arrow[ll, "(e'_\pi)_*"', "\cong"] \\
\lim\limits_{d\to\infty} K_*(C^*(P_d(X))^\Gamma) \arrow[u, "\beta_*"] & & \lim\limits_{d\to\infty} K_*(C^*_{\pi,L}(P_d(X))^\Gamma) \arrow[ll, "(e_\pi)_*"'] \arrow[u, "(\beta_{\pi,L})_*", "\cong"'] \\
& & \lim\limits_{d\to\infty} K_*(C^*_L(P_d(X))^\Gamma) \arrow[ull, "e_*"] \arrow[u, "(\pi_L)_*", "injective"']
\end{tikzcd}$$

So e_* is injective.

Thank you!