Fuzzy sphere and the Ginsparg-Wilson index

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Plan of Talk

- Dirac operators in Lattice Gauge Theory
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Dirac operators in Lattice Gauge Theory

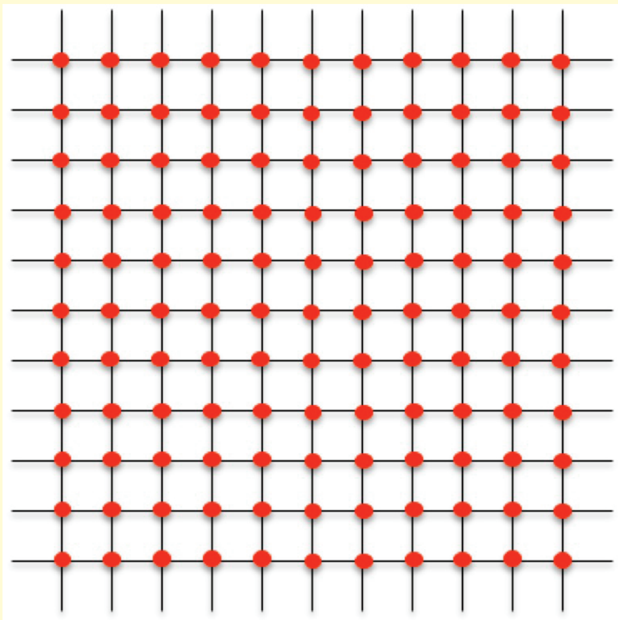
A naive Dirac operator in lattice gauge theory is given by

$$D = \gamma_\mu \left(\nabla^*_\mu + \nabla_\mu\right)/2,$$

where ∇_μ and ∇^*_μ denote backward and forward finite differences:

$$(\nabla_\mu \psi)(n) = \left[U_{n,\mu}\psi(n+\widehat{\mu}) - \psi(n)\right]/a$$
$$(\nabla^*_\mu \psi)(n) = \left[\psi(n) - U_{n,\mu}\psi(n-\widehat{\mu})\right]/a$$

with a the lattice spacing (Here D is skew-adjoint).



But it encountered a problem of **fermion doubling**, extra freedom of fermions that appears in lattice gauge theory. In order to overcome this point Wilson proposed the **Wilson fermion operator** :

$$D_W = D + W,$$

where $W = -\frac{a}{2}(\nabla_\mu^* \nabla_\mu) \geq 0$. But D_W has another difficulty, namely it breaks the chirality; $\Gamma D_W + D_W \Gamma \neq 0$. Then the third Dirac operator, the **overlap operator**, was found by Neuberger ,which is given by

$$D = \frac{1}{a}(1 + U)$$

with $U = \dfrac{aD_W - 1}{|aD_W - 1|}$. It still breaks the chirality but satisfies the Ginsparg-Wilson relation, which is a perturbation of the chirality.

The Ginsparg-Wilson relation

Γ: an involution, namely $\Gamma^* = \Gamma$, $\Gamma^2 = 1$

An operator D satisfies the **Ginsparg-Wilson relation** if

$$D\Gamma + \Gamma D = aD\Gamma D$$

with $a \in \mathbb{R}$. At $a = 0$, it means that D anti-commutes with Γ.

Proposition 1 *Let U be a unitary operator with $\Gamma U \Gamma = U^*$. Set $D = \frac{1}{a}(1 \pm U)$. Then D satisfies the Ginsparg-Wilson relation.*

In fact,

$$D\Gamma + \Gamma D = (1 \pm U)\Gamma/a + \Gamma(1 \pm U)/a = (2\Gamma \pm U\Gamma \pm \Gamma U)/a$$

is equal to

$$aD\Gamma D = (1 \pm U)\Gamma(1 \pm U)/a = (\Gamma \pm U\Gamma \pm \Gamma U + U\Gamma U)/a$$

since $U\Gamma U = \Gamma$.

Remark. spec $\left(\frac{1}{a}(1 \pm U)\right)$ tends to the imaginary axis $i\mathbb{R}$ as $a \to 0$.

A universal Ginsparg-Wilson algebra

Given an involution Γ and a unitary U with $\Gamma U \Gamma = U^*$.
Set $\Gamma' = \Gamma U$, which gives another involution:

$$(\Gamma')^* = U^*\Gamma = \Gamma U \Gamma U = 1, \qquad (\Gamma')^2 = \Gamma U \Gamma U = U^* U = 1$$

Conversely, given two involutions Γ and Γ', we set $U = \Gamma\Gamma'$. Then U is a unitary with $\Gamma U \Gamma = U^*$. This argument implies:

Proposition 2 *Let S^1 be the unit circle in $\mathbb{C}$ and $\epsilon \in \mathbb{Z}_2$ an involution that acts on S^1 by the complex conjugation: $\epsilon(z) = \bar{z}$ ($z \in S^1$). The resulting crossed product $C(S^1) \rtimes \mathbb{Z}_2$ is isomorphic to the universal C^*-algebra generated by two involutions.*

Definition 1 *The crossed product $C(S^1) \rtimes \mathbb{Z}_2$ called a* **universal Ginsparg-Wilson algebra** *and denoted by C^*_{GW}.*

The K-theory of C^*_{GW}

Let $x \in \mathbb{R}$ be the Cayley coordinate for the unit circle $S^1 \subset \mathbb{C}$:

$$e^{i\theta} = \frac{x-i}{x+i} \in S^1.$$

Thus $\mathbb{R}$ is identified with $S^1 \setminus \{1\}$ and there is a short exact sequence:

$$0 \longrightarrow C_o(\mathbb{R}) \rtimes \mathbb{Z}_2 \longrightarrow C^*_{GW} \xrightarrow{\pi} C^*\mathbb{Z}_2 \longrightarrow 0$$

where π is the evaluation at $z = 1 \in S^1$. Then one has the 6-term exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

Thus the K-group of C^*_{GW} are:

$$K_0(C^*_{GW}) \cong \mathbb{Z}^3, \qquad K_1(C^*_{GW}) \cong 0.$$

Proposition 3 *Define projections in* C^*_{GW} *such as*

$$p = (1+\epsilon)/2, \qquad p' = (1+\epsilon e^{i\theta})/2.$$

$K_0(C^*_{GW})$ *admits a basis given by* $[p]-[p']$, $[p]-[1-p']$, $[1]$, *where*

$$[p]-[p'] \in K_0(C_o(\mathbb{R}) \rtimes \mathbb{Z}_2) \cong \mathbb{Z},$$
$$\pi_*([p]-[1-p']), \pi_*[1] \in K_0(C^*\mathbb{Z}_2) \cong \mathbb{Z}^2$$

generate those groups. Let $e_x = (x+\epsilon)^{-1}p(x+\epsilon)$ *be the* **graph projection**. *Then the (index) class*

$$[e_x]-[1-p] \in K_0(C_o(\mathbb{R}) \rtimes \mathbb{Z}_2)$$

coincides with $[p]-[p']$.

Definition 2 *Set*

$$\mathrm{Ind}(U,-1) := [p]-[p'], \quad \mathrm{Ind}(U,+1) := [p]-[1-p'].$$

We call $\mathrm{Ind}(U,-1)$ *a (universal)* **index class** *and* $\mathrm{Ind}(U,+1)$ *a (universal)* **doubler class**. *A life is not simple but the* K*-theory is.*

The Ginsparg-Wilson Index theorem

Given an involution Γ and a unitary U with $\Gamma U \Gamma = U^*$ on a Hilbert space $\mathcal{H}$, there exists a $*$-homomorphism

$$\rho : C^*_{GW} \to \mathcal{L}(\mathcal{H})$$

to the bounded operators on $\mathcal{H}$ in such a way that $\rho(e^{i\theta}) = U,\ \rho(\epsilon) = \Gamma$. Associated to Γ and U, the **Ginsparg-Wilson index** can be defined: $\rho_*(\mathrm{Ind}(U, -1))$ is called the **index class** and $\rho_*(\mathrm{Ind}(U, +1))$ the **doubler class**, denoted often suppressing $\rho *$.

Recall $\Gamma' = \Gamma U$ is an involution. Define selfadjoint operators such as

$$H_+ = (\Gamma + \Gamma')/2, \qquad H_- = (\Gamma - \Gamma')/2.$$

Let $V(U, \pm 1)$ be the eigenspace of U of eigenvalue ± 1, respectively. One has

$$V(U, \pm 1) = \{\xi | \, \Gamma \xi = \pm \Gamma' \xi\} = \ker H_\mp$$

since $1 \pm U = \Gamma(\Gamma \pm \Gamma')$.

Theorem 4 (The Ginsparg-Wilson index theorem) *Suppose that the spectrum* $\pm 1 \in \mathrm{spec}(U)$ *are isolated and the corresponding projections are finite. Then one has* $\rho_* \mathrm{Ind}(U, \pm 1) \in K_0(\mathcal{K})$ *with* $\mathcal{K}$ *the ideal of compact operators and*

$$\mathrm{Tr}(\rho_* \mathrm{Ind}(U, \pm 1)) = \mathrm{Tr}(\Gamma | V(U, \pm 1)) = \mathrm{Tr}(\Gamma | \ker H_{\mp})$$

Moreover, if $H_{\pm}$ *is traceable, then*

$$\mathrm{Tr}(\rho_* \mathrm{Ind}(U, \pm 1)) = \mathrm{Tr}(H_{\pm}/2).$$

The second identity is easily follows from

$$\rho_*(p - p') = (1 + \Gamma)/2 - (1 + \Gamma')/2 = H_-$$
$$\rho_*(p - (1 - p')) = (1 + \Gamma)/2 - (1 - \Gamma')/2 = H_+$$

Theorem justifies the notation $\mathrm{Ind}(U, \pm 1)$ since they are associated to the spectral projection of U corresponding to ± 1. Also note that $H_{\pm}/2$ are not projections in general, but the trace turned out to be integers.

Examples of the Ginsparg-Wilson index

Example 1 *Let D be a Dirac operator on a closed manifold and Γ the grading operator such as $\Gamma D + D\Gamma = 0$. The Cayley transform $U = (D-i)(D+i)^{-1}$ satisfies*

$$\Gamma U \Gamma = \Gamma \frac{D-i}{D+i} \Gamma = \frac{D+i}{D-i} = U^*.$$

Then one has

$$U\xi = -\xi \iff (D-i)\xi = -(D+i)\xi \iff \xi \in \ker D$$

and thus

$$\mathrm{Tr}(\mathrm{Ind}(U,-1)) = \mathrm{Tr}(\Gamma | V(U,-1)) = \mathrm{Tr}(\Gamma | \ker D) = \mathrm{Ind}(D^+).$$

On the other hand, one has $V(U,+1) = 0$ since $(D-i)\xi = (D+i)\xi$ if and only if $\xi = 0$. Thus $\mathrm{Tr}(\Gamma | V(U,+1)) = 0$ although the class $\mathrm{Ind}(U,+1)$ does not belong to $K_0(\mathcal{K})$. Replacing D by $f(D)$ with f a tempered function of $x/|x|$ on $\mathbb{R}$, one has $0 = \mathrm{Ind}(U,+1) \in K_0(\mathcal{K})$.

Example 2 (Finite dimensional case) *Set*

$$\Gamma = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}, \qquad \Gamma' = \begin{pmatrix} 1_k & 0 \\ 0 & 1_l \end{pmatrix}$$

with $n+m = k+l$. *Then one has*

$$P = (1+\Gamma)/2 = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}, \quad P' = (1+\Gamma')/2 = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}, \quad 1-P' = \begin{pmatrix} 0 & 0 \\ 0 & 1_l \end{pmatrix}$$

and thus

$$\begin{aligned} \mathrm{Tr}(\mathrm{Ind}(U,-1)) &= \mathrm{Tr}(P-P') = n-k \\ \mathrm{Tr}(\mathrm{Ind}(U,+1)) &= \mathrm{Tr}(P-(1-P')) = n-l. \end{aligned}$$

Note that the Fredholm index of $F : V \to W$ *does not depend on the choice of* F *if* V *is finite dimensionl. However, we can get a nontrivial index class* $\mathrm{Ind}(U,-1)$ *with the doubler class* $\mathrm{Ind}(U,+1)$ *trivial (take n=ℓ) even if* $\dim V < \infty$.

Example 3 *As in Example 1, let D be a selfadjoint elliptic operator on a closed manifold and take Γ a grading operator such as $\Gamma D + D\Gamma = 0$. Also take a selfadjoint operator A such that $DA = AD$ and $\Gamma A = A\Gamma$. Set*

$$U = \begin{cases} \dfrac{D - iA}{D + iA} & \text{on } (\ker D \cap \ker A)^{\perp} \\ 1 & \text{on } \ker D \cap \ker A \end{cases}$$

One has

$$\Gamma \frac{D - iA}{D + iA} \Gamma = \frac{D + iA}{D - iA}, \qquad \Gamma 1 \Gamma = 1$$

and thus $\Gamma U \Gamma = U^*$. It then follows that

$$\begin{aligned} \mathrm{Tr}(\mathrm{Ind}(U, -1)) &= \mathrm{Tr}(\Gamma | \ker D) = \mathrm{Ind}(D^{+}) \\ \mathrm{Tr}(\mathrm{Ind}(U, +1)) &= \mathrm{Tr}(\Gamma | \ker D \cap \ker A) = \mathrm{Tr}(\Gamma | \ker(D + iA)) \end{aligned}$$

Since $D + iA$ dose not anti-commute with Γ, the second index is not an ordinary index for an odd operator. In fact, we can obtain a nontrivial doubler class for suitable D and A. The evil twin is not bad at all. This is generalized to the case of Dolbeault index theorem on $SU(2)$.

A Fuzzy sphere

$S^2 \subset \mathbb{R}^3$: the standard sphere defined by $x_1^2 + x_2^2 + x_3^2 = 1$

$\mathcal{A}(S^2)$= the algebra of polynomials in x_i restricted to S^2

$G = SU(2)$ naturally acts on S^2 ($SO(3)$-action)

$\mathcal{A}(S^2)$ splits into the direct sum of irreducible representations of G:

$$\mathcal{A}(S^2) = \bigoplus_{l=0}^{\infty} V_l$$

where V_l is the representation space with highest weight $l \in \mathbb{N} \cup \{0\}$, a unique irreducible representation of $SU(2)$ with $\dim V = 2l + 1$ or the eigenspace of the Laplacian Δ on S^2 with eigenvalue $l(l+1)$.

Definition 3 *A* **fuzzy sphere** *is the C^*-algebra $\mathcal{A}_N = \mathrm{End}(V_l)$ with $N = 2l$, where it decomposes into irreducible G-representations:*

$$\mathcal{A}_N = \mathrm{End}(V_l) = V_l \otimes V_l = \bigoplus_{k=0}^{2l} V_k.$$

One has $\mathcal{A}_N \to \mathcal{A}(S^2)$ as $N = 2l \to \infty$ "naively".

The fuzzy Dirac operator

$L_i \in \mathrm{End}(V_l)$ $(i = 1, 2, 3)$: standard self-adjoint infinitesimal generators of $SU(2)$-action, namely L_i are the angular momentum operators such as $[L_i, L_j] = \sqrt{-1}\epsilon_{ijk}L_k$. They satisfy

$$L_1^2 + L_2^2 + L_3^2 = l(l+1)1_V$$

Set $X_i = L_i/\sqrt{l(l+1)} \doteqdot L_i/l$. One has

$$X_1^2 + X_2^2 + X_3^2 = 1, \qquad [X_i, X_j] = \frac{\sqrt{-1}}{\sqrt{l(l+1)}}\epsilon_{ijk}X_k \longrightarrow 0$$

as $l \to \infty$. Moreover, with $\hbar = 1/\sqrt{l(l+1)}$, one has

$$\frac{\sqrt{-1}}{\hbar}[X_i, X_j] \longrightarrow \{x_i, x_j\} : \text{the Poisson bracket on } S^2$$

as $\hbar \to 0$. Thus noncommutative coordinate X_i converges to ordinary coortinate x_i as $l \to \infty$ in a suitable sense (Rieffel, D'Andorea-Lizzi-Várilly, ...).

$\sigma_i \in \mathrm{End}(V_{1/2}) \cong M_2\mathbb{C}$ $(i = 1, 2, 3)$:the Pauli matrix, namely

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define operators on the fuzzy spinors $\mathcal{A}_N \otimes V_{1/2} = \mathrm{End}(V_l) \otimes V_{1/2}$ by

$$D_1 = 1/2 + \sum_i L_i^{(l)} \otimes \sigma_i, \qquad D_2 = 1/2 - \sum_i L_i^{(r)} \otimes \sigma_i$$

with $L_i^{(l)} X = L_i X$ and $L_i^{(r)} X = X L_i$ for $X \in \mathrm{End}(V_l) \otimes V_{1/2}$. Set

$$\Gamma_1 = D_1/|D_1|, \qquad \Gamma_2 = D_2/|D_2|,$$

which are involutions. One has

$$\frac{\Gamma_1 + \Gamma_2}{2} = \frac{1}{2l+1}\Big(1 + \sum [L_i, X] \otimes \sigma_i\Big) = \frac{\not{D}}{2l+1}.$$

Here $\not{D}$ is the (unbounded) Dirac operator on S^2, which preserves the fuzzy spinors $\mathcal{A}_N \otimes V_{1/2} \subset \mathcal{A}(S^2) \otimes V_{1/2}$.

In the commutative limit ($N = 2l \to \infty$), one has

$$H_+ = \frac{\Gamma_1 + \Gamma_2}{2} = \frac{1}{2l+1}\left(1 + \sum [L_i, X] \otimes \sigma_i\right) \to \frac{D\!\!\!/}{|D\!\!\!/|}$$

$$H_- = \frac{\Gamma_1 - \Gamma_2}{2} = \frac{1}{2l+1} \sum \left(L_i^{(l)} + L_i^{(r)}\right) \otimes \sigma_i \to \sum x_i \otimes \sigma_i = \epsilon$$

since $X_i \doteqdot L_i/l$ "converges" to x_i, where ϵ is the grading operator on the spinor bundle on S^2. Consider the GW index:

$$\mathrm{Ind}(U, \pm 1) = \mathrm{Tr}(H_\pm |\ker H_\mp) = \mathrm{Tr}(H_\pm)$$

Theorem 5 (The index theorem on fuzzy sphere)

$$\mathrm{Ind}(U, -1) = \mathrm{Tr}(H_-) = 0, \;\; \mathrm{Ind}(U, +1) = \mathrm{Tr}(H_+) = 4l + 2$$

Remark. 1) In the commutative limit ($l \to \infty$), one has $\mathrm{Tr}(H_- |\ker H_+) \to \mathrm{Tr}(\epsilon|\ker D\!\!\!/) = \mathrm{Ind}(D\!\!\!/^+) = 0$, thus $\mathrm{Ind}(U, -1)$ contains no doubler fermions.

2) In a similar way we can define the Dirac operator on $V_l \otimes V_m \otimes V_{1/2}$, where one has $\mathrm{Ind}(U, -1) = 2m - 2l$, $\mathrm{Ind}(U, +1) = 2(l + m) + 2$.

A GW index theorem on $SU(2)$

$SU(2) \cong S^3$: the 3-dimensional sphere

$\mathcal{E} = C^\infty(S^3) \otimes \mathbb{C}^2$: the space of spinors with $\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{End}(\mathbb{C}^2)$

X, Y, Z:left-invariant differential operators on S^3 corresponding to a basis of Lie algebra; Do not confuse $\mathrm{End}(\mathbb{C}^2)$ with $\mathrm{Lie}(SU(2))$.

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{Lie}(SU(2)).$$

A Dolbeault operator lifted from S^2 is defined by

$$D = \begin{bmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -X + iY \\ X + iY & 0 \end{bmatrix} \text{ on } \mathcal{E}.$$

Since $[Z, X] = 2Y$, $[Z, Y] = -2X$, the Dolbeault operator D commutes with the action by $T = \begin{bmatrix} Z & 0 \\ 0 & Z + 2i \end{bmatrix}$.

Set $U = \dfrac{D - iA_k}{D + iA_k}$ with $A_k = -iT - k$ $(k \in \mathbb{Z})$. One has $\Gamma U \Gamma = U^*$ as is proved in Example 3.

Theorem 6 (A GW index theorem on $SU(2)$) *The doubler class* $\mathrm{Ind}(U, +1)$ *belongs to* $K_0(\mathcal{K})$ *although the index class* $\mathrm{Ind}(U, -1)$ *not. One has*

$$\mathrm{Ind}(U, +1) = \mathrm{Tr}(\Gamma|\ker(D + iA_k)) = \mathrm{Ind}(\bar{\partial}^+ \otimes H(k)).$$

Here $H(k)$ denotes the ample line bundle on S^2 of degree k, namely the first Chern number $c_1(H(k))[S^2] = k$ and

$$\bar{\partial}^+ \otimes 1_{H(k)} : \Omega^{0,0}(S^2, H(k)) \to \Omega^{0,1}(S^2, H(k))$$

is the Dolbeault operator coupled with $H(k)$.

Thank you for your attention