Persistence approximation property for maximal Roe algebras and applications

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Operator propagation

- Let X be a proper metric space(i.e. closed balls are compact) and let $\pi : C_0(X) \to \mathcal{L}(\mathcal{H})$ be a representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$.
- Example: $\mathcal{H} = L^2(X, \mu)$ for μ Borelian measure on X and π the pointwise multiplication.

Definition

- If T is an operator of $\mathcal{L}(\mathcal{H})$, then $\mathrm{Supp}T$ is the complementary of the open subset of $X \times X$ $\{(x, y) \in X \times X$ such that there exists f and g such that $f(x) \neq 0, g(y) \neq 0$ and $\pi(f) \cdot T \cdot \pi(g) = 0\}$
- T has propagation less than r if $d(x, y) \leq r$ for all $(x, y) \in \mathrm{Supp}T$.

Propagation and indices

- Let D be an elliptic differential operator on a compact manifold M.
- Let Q be a parametrix for D.
- Then $S_0 := Id - QD$ and $S_1 := Id - DQ$ are smooth kernel operators on $M \times M$:
- $$P_D = \begin{pmatrix} S_0^2 & S_0(Id + S_0)Q \\ S_1 D & Id - S_1^2 \end{pmatrix} \tag{1.1}$$

  is an idempotent and we can choose Q such that P_D has arbitrary small propagation.

- D is a Fredholm operator and

$$\mathrm{Ind}\,(D) = [P_D] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right] \in K_0(K(L^2(M))) \cong \mathbb{Z} \tag{1.2}$$

- How can we keep track of the propagation and have homotopy invariance ?

Quasi-projection

Definition

If X is a proper metric space and $\pi : C_0(X) \to L(H)$ is a representation of $C_0(X)$ on a Hilbert space H, let $0 < \epsilon < \frac{1}{4}$ (control) and $r > 0$ (propagation). Then $q \in \mathcal{L}(\mathcal{H})$ is an ϵ-r-projection if $q = q^*$, $\|q - q^2\| < \epsilon$ and q has propagation less than r .

- If q is an ϵ-r-projection, then its spectrum has a gap around $\frac{1}{2}$.
- Hence there exists $k_0 : \sigma(q) \to \{0, 1\}$ continuous and such that $k_0(t) = 0$ if $t < \frac{1}{2}$ and $k_0(t) = 1$ if $t > \frac{1}{2}$.
- By continuous functional calculus,$k_0(q)$ is a projection such that $\|k_0(q) - q\| \leq 2\epsilon$.

Quasi-projections and indices

- Let D be a differential elliptic operator on a manifold, let Q be a parametrix. Set $S_0 := Id - QD$ and $S_1 := Id - DQ$ and

$$P_D = \begin{pmatrix} S_0^2 & S_0(Id + S_0)Q \\ S_1 D & Id - S_1^2 \end{pmatrix} \tag{1.3}$$

Then $((2P_D^* - 1)(2P_D - 1) + 1)^{\frac{1}{2}} P_D ((2P_D^* - 1)(2P_D - 1) + 1)^{-\frac{1}{2}}$ is a projection conjugated to the idempotent P_D;

- Choosing $Q = Q_{\epsilon,r}$ with propagation small enough and approximating $((2P_D^* - 1)(2P_D - 1) + 1)^{\frac{1}{2}} P_D((2P_D^* - 1)(2P_D - 1) + 1)^{-\frac{1}{2}}$ by a power series, we can for all $0 < \epsilon < \frac{1}{4}$ and $r > 0$, construct a ϵ-r-projection $q_D^{\epsilon,r}$ such that D such that

$$\text{Ind } (D) = [k_0(q_D^{\epsilon,r})] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right] \in K_0(K(L^2(M))) \cong \mathbb{Z} \tag{1.4}$$

Filtered C^*-algebras

Definition

A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces indexed by positive numbers such that:

- $A_r \subseteq A_{r'}$ if $r \leq r'$;
- A_r is stable by involution,i.e. for any $x \in A_r$, then $x^* \in A_r$;
- $A_r \cdot A_{r'} \subseteq A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A.

If A is unital, we impose that $1 \in A_r$, for any $r > 0$. If A is non unital filtered C^*-algebra, then its unitization $\tilde{A}$ is filtered by $(A_r + \mathbb{C})_{r>0}$. We can define the homomorphism

$$\rho_A : \tilde{A} \to \mathbb{C}; a + z \to z$$

for $a \in A$ and $z \in \mathbb{C}$

Definition

Let A and B be two C^*-algebras filtered by $(A_r)_{r>0}$ and $(B_r)_{r>0}$. A *-homomorphism $\phi : A \to B$ is said to be filtered if $\phi(A_r) \subseteq B_r$ for all $r > 0$.

Examples

- $K(L^2(X,\mu)$ for X a metric space and μ probability measure on X. More generally $A \otimes K(L^2(X,\mu)$ for A is a C^*-algebra.
- Roe algebras:
    - Σ proper discrete metric space, $\mathcal{H}$ separable Hilbert space.
    - $C[\Sigma]_r$: space of locally compact operators on $l^2(\Sigma) \otimes \mathcal{H}$ (i.e T satisfies fT and Tf are compact for all $f \in C_c(\Sigma)$) and with propagation less than r.
    - The Roe algebra of Σ is $C^*(\Sigma) = \overline{\bigcup_{r>0} C[\Sigma]_r} \subset \mathcal{L}(l^2(\Sigma) \otimes \mathcal{H})$ (filtered by $(C[\Sigma]_r)_{r>0}$).
- Also the maximal Roe algebras is filtered C^*- algebras.
- C^*-algebras of groups and cross-products.

ϵ-r-projections and ϵ-r-unitaries

Definition

Let A be a unital filtered C* algebra. for any $r > 0$ and $\epsilon \in (0, \frac{1}{4})$,we call:

- an element p in A an ϵ-r- projection if p belongs to A_r,$p = p^*$ and $\|p^2 - p\| < \epsilon$.The set of ϵ-r- projections will be denoted by $P^{\epsilon,r}(A)$.
- an element u in A is an ϵ-r- unitary if u belongs to A_r,$\|u^*u - 1\| < \epsilon$ and $\|uu^* - 1\| < \epsilon$. The set of ϵ-r-unitaries in A will be denoted by $U^{\epsilon,r}(A)$.

We can construct a projection by continuous functional calculus on $\sigma(p)$ denoted by $k_0(p)$ and a unitary $k_1(u) = u(u^*u)^{-\frac{1}{2}}$.

Definition

For a unital filtered C* algebra A,we can define the following equivalent relation on $P_\infty^{\epsilon,r}(A)\times\mathbb{N}$ and $U_\infty^{\epsilon,r}(A)$:

- if p and q are elements of $P_\infty^{\epsilon,r}(A)$, l and l' are positive integers,$(p,l)\sim(q,l')$ if there exists a positive integer k and an element h of $P_\infty^{\epsilon,r}(A[0,1])$ such that $h(0)=\mathrm{diag}(p,I_{k+l'})$ and $h(1)=\mathrm{diag}(q,I_{k+l})$
- if u and v are elements of $U_\infty^{\epsilon,r}(A)$,$u\sim v$ if there exists an element h of $U_\infty^{3\epsilon,2r}(A[0,1])$ such that $h(0)=u$ and $h(1)=v$.

If p is an element of $P_\infty^{\epsilon,r}(A)$ and l is an integer,we denote by $[p,l]_{\epsilon,r}$ the equivalent class of (p,l) modulo $\sim$. And if u is an element of $U_\infty^{\epsilon,r}(A)$ we denote by $[u]_{\epsilon,r}$ its equivalent class modulo $\sim$.

Quantitative K-theory

Definition

Let $r > 0$ and $\epsilon \in (0, \frac{1}{4})$. We define:

(i) $K_0^{\epsilon,r}(A) = P_\infty^{\epsilon,r}(A) \times \mathbb{N}/\sim$ unital and
$K_0^{\epsilon,r}(A) = P_\infty^{\epsilon,r}(\tilde{A}) \times \mathbb{N}/\sim$ such that $\dim k_0(\rho_A(p)) = l$ for A non unital.

(ii) $K_1^{\epsilon,r}(A) = U_\infty^{\epsilon,r}(\tilde{A})/\sim$ (with $A = \tilde{A}$ if A is already unital).

Then $K_0^{\epsilon,r}(A)$ turns to be an abelian group where

$$[p, l]_{\epsilon,r} + [p', l']_{\epsilon,r} = [\mathrm{diag}(p, p'), l + l']_{\epsilon,r}$$

$K_1^{\epsilon,r}(A)$ is also an abelian group with

$$[u]_{\epsilon,r} + [u']_{\epsilon,r} = [\mathrm{diag}(u, u')]_{\epsilon,r}$$

.

Lemma

If A is a filtered C^*- algebra, then $K_*^{\epsilon,r}(A) = K_0^{\epsilon,r}(A) \oplus K_1^{\epsilon,r}(A)$ is a $\mathbb{Z}_2$- graded abelian group.
For any filtered C* algebra A and any positive numbers ϵ, ϵ' and r, r' with $\epsilon \leq \epsilon' < \frac{1}{4}$ and $r \leq r'$,there exists natural group homomorphisms:

- $\iota_0^{\epsilon,r} : K_0^{\epsilon,r}(A) \to K_0(A); [p,l]_{\epsilon,r} \mapsto [k_0(p)] - [I_l];$
- $\iota_1^{\epsilon,r} : K_1^{\epsilon,r}(A) \to K_1(A); [u]_{\epsilon,r} \to [k_1(u)];$
- $\iota_*^{\epsilon,r} = \iota_0^{\epsilon,r} \oplus \iota_1^{\epsilon,r};$
- $\iota_0^{\epsilon,\epsilon',r,r'} : K_0^{\epsilon,r}(A) \to K_0^{\epsilon',r'}(A); [p,l]_{\epsilon,r} \mapsto [p,l]_{\epsilon',r'};$
- $\iota_1^{\epsilon,\epsilon',r,r'} : K_1^{\epsilon,r}(A) \to K_1^{\epsilon',r'}(A); [u]_{\epsilon,r} \to [u]_{\epsilon',r'};$
- $\iota_*^{\epsilon,\epsilon',r,r'} = \iota_0^{\epsilon,\epsilon',r,r'} \oplus \iota_1^{\epsilon,\epsilon',r,r'}.$

Control pair

Definition

A control pair is a pair (λ, h),where

- $\lambda > 1$;
- $h : (0, \frac{1}{4\lambda}) \to (1, +\infty); \epsilon \mapsto h_\epsilon$ is a map such that exists a non-increasing map $g : (0, \frac{1}{4\lambda}) \to (1, +\infty)$, with $h \leq g$.

The set of control pairs is equipped with a partial order:$(\lambda, h) \leq (\lambda', h')$ if $\lambda \leq \lambda'$ and $h_\epsilon \leq h'_\epsilon$ for all $\epsilon \in (0, \frac{1}{4\lambda'})$.

Definition

For any filtered C^*- algebra A,define the families
$\mathcal{K}_0(A) = (K_0^{\epsilon,r}(A))_{0<\epsilon<\frac{1}{4},r>0}$, $\mathcal{K}_1(A) = (K_1^{\epsilon,r}(A))_{0<\epsilon<\frac{1}{4},r>0}$,
$\mathcal{K}_*(A) = (K_*^{\epsilon,r}(A))_{0<\epsilon<\frac{1}{4},r>0}$.

Controlled morphism

Definition

Let (λ, h) be a control pair, A and B be two filtered C^*-algebras, and i, j be elements of $\{0, 1, *\}$. A (λ, h)-controlled morphism

$$\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$$

is a family $\mathcal{F} = (F^{\epsilon,r})_{0<\epsilon<\frac{1}{4},r>0}$ of group homomorphisms

$$F^{\epsilon,r} : K_i^{\epsilon,r}(A) \to K_j^{\lambda\epsilon,h_\epsilon r}(B)$$

such that for any positive numbers ϵ, ϵ' and r, r' with $0 < \epsilon \leq \epsilon' < \frac{1}{4\lambda}$, $r \leq r'$ and $h_\epsilon r \leq h_{\epsilon'} r'$, we have

$$F^{\epsilon',r'} \circ \iota_i^{\epsilon,\epsilon',r,r'} = \iota_j^{\lambda\epsilon,\lambda\epsilon',h_\epsilon r,h_{\epsilon'} r'} \circ F^{\epsilon,r}.$$

The composition of controlled morphism

Definition

If (λ, h) and (λ', h') are two control pairs,define

$$h * h' : (0, \frac{1}{4\lambda\lambda'}) \to (0, +\infty); \epsilon \mapsto h_{\lambda'\epsilon} h'_{\epsilon}.$$

Then $(\lambda\lambda', h * h')$ is a control pair. Let A, B_1 and B_2 be filtered C^*-algebras, i, j and l in $\{0, 1, *\}$. Let $\mathcal{F} = (F^{\epsilon,r})_{0<\epsilon<\frac{1}{4\alpha_{\mathcal{F}}}, r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B_1)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism,let $\mathcal{G} = (G^{\epsilon,r})_{0<\epsilon<\frac{1}{4\alpha_{\mathcal{G}}}, r>0} : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2)$ be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism. Then $\mathcal{G} \circ \mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_l(B_2)$ is the $(\alpha_{\mathcal{G}}\alpha_{\mathcal{F}}, k_{\mathcal{G}} * k_{\mathcal{F}})$-controlled morphism defined by the family $(G^{\alpha_{\mathcal{F}}\epsilon, k_{\mathcal{F},\epsilon} E} \circ F^{\epsilon,r})_{0<\epsilon<\frac{1}{4\alpha_{\mathcal{F}}\alpha_{\mathcal{G}}}, r>0}$.

The equivalence of controlled morphism

Definition

Let A and B be filtered C^*-algebras,and (λ, h) is a control pair. Let $\mathcal{F} = (F^{\epsilon,E})_{0<\epsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$(resp. $\mathcal{G} = (G^{\epsilon,r})_{0<\epsilon<\frac{1}{4\alpha_{\mathcal{G}}},r>0}$) be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$- controlled morphism (resp. a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$- controlled morphism). Then we write $\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$ if

- $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)$ and $(\alpha_{\mathcal{G}}, k_{\mathcal{G}}) \leq (\lambda, h)$
- for any $\epsilon \in (0, \frac{1}{4\lambda})$ and $r > 0$, then

$$\iota_j^{\alpha_{\mathcal{F}}\epsilon,\lambda\epsilon,k_{\mathcal{F},\epsilon}r,h_\epsilon r} \circ F^{\epsilon,r} = \iota_j^{\alpha_{\mathcal{G}}\epsilon,\lambda\epsilon,k_{\mathcal{G},\epsilon}r,h_\epsilon r} \circ G^{\epsilon,r}.$$

Controlled isomorphism

Definition

Let (λ, h) be a control pair, and $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism with $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)$.

- $\mathcal{F}$ is called left (λ, h)-invertible if there exists a controlled morphism
$$\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$$
such that $\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_i(A)}$. and $\mathcal{F} \circ \mathcal{G} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{K}_j(B)}$.
- $\mathcal{F}$ is (λ, h)-isomorphism if there exists a controlled morphism
$$\mathcal{G} : \mathcal{K}_j(B) \to \mathcal{K}_i(A)$$
which is a (λ, h)-inverse for $\mathcal{F}$.

Let (λ, h) be a control pair and let $\mathcal{F} : \mathcal{K}_i(A) \to \mathcal{K}_j(B)$ $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism.

Definition

- $\mathcal{F}$ is called (λ, h)-injective if $(\alpha_{\mathcal{F}}, k_{\mathcal{F}}) \leq (\lambda, h)$ and for any $0 < \epsilon < \frac{1}{4\lambda}$, any $r > 0$ and any $x \in K_i^{\epsilon,r}(A)$, then $F^{\epsilon,r}(x) = 0$ in $K_j^{\alpha_{\mathcal{F}}\epsilon, k_{\mathcal{F},\epsilon} r}(B)$ implies that $\iota_i^{\epsilon,\lambda\epsilon,r,h_\epsilon r}(x) = 0$ in $K_i^{\lambda\epsilon,h_\epsilon r}(A)$;
- $\mathcal{F}$ is called (λ, h)-surjective ,if for any $0 < \epsilon < \frac{1}{4\lambda\alpha_{\mathcal{F}}}$, any $r > 0$ and $y \in K_j^{\epsilon,r}(B)$, there exists an element $x \in K_i^{\lambda\epsilon,h_\epsilon r}(A)$ such that $F^{\lambda\epsilon,h_{\lambda\epsilon} r}(x) = \iota_j^{\epsilon,\alpha_{\mathcal{F}}\lambda\epsilon,r,k_{\mathcal{F},\lambda\epsilon}h_\epsilon r}(y)$ in $K_j^{\alpha_{\mathcal{F}}\lambda\epsilon,r,k_{\mathcal{F},\lambda\epsilon}h_\epsilon r}(B)$.

Controlled exact sequence

Definition

Let (λ, h) be a control pair. Let $\mathcal{F} = (F^{\epsilon,r})_{0<\epsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{K}_i(A) \to \mathcal{K}_j(B_1)$ be a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism,and let $\mathcal{G} = (G^{\epsilon,r})_{0<\epsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{K}_j(B_1) \to \mathcal{K}_l(B_2)$ be a $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism,where i, j and l are in $\{0, 1, *\}$ and A, B_1, B_2 are filtered C^*-algebras. Then the composition

$$\mathcal{K}_i(A) \xrightarrow{\mathcal{F}} \mathcal{K}_j(B_1) \xrightarrow{\mathcal{G}} \mathcal{K}_l(B_2)$$

is said to be (λ, h)-exact at $\mathcal{K}_j(B_1)$ if $\mathcal{G} \circ \mathcal{F} = 0$

and if for any $0 < \epsilon < \frac{1}{4\max\{\lambda\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}\}}$, any $r > 0$ and $y \in K_j^{\epsilon,r}(B_1)$ such that $G^{\epsilon,r}(y) = 0$ in $K_j^{\alpha_{\mathcal{G}}\epsilon, k_{\mathcal{G},\epsilon} r}(B_2)$,
there exists an element x in $K_i^{\lambda\epsilon, h_\epsilon r}(A)$ such that

$$F^{\lambda\epsilon, h_{\lambda\epsilon} r}(x) = \iota_j^{\epsilon, \alpha_{\mathcal{F}}\lambda\epsilon, r, k_{\mathcal{F},\lambda\epsilon} h_\epsilon r}(y)$$

in $K_j^{\alpha_{\mathcal{F}}\lambda\epsilon, k_{\mathcal{F}\lambda\epsilon} h_\epsilon r}(B_1)$.

Completely filtered extension

Definition

Let A be a filtered C^*-algebra. let J be an ideal of A and set $J_r = J \cap A_r$. The extension of C^*- algebras

$$0 \to J \to A \to A/J \to 0$$

is called a completely filtered extension of C^*- algebras if the bijection continuous linear map

$$A_r/J_r \to (A_r + J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is a complete isometry i.e for any integer n, any $r > 0$ and $x \in M_n(A_r)$,then

$$\inf_{y \in M_n(J_r)} \|x + y\| = \inf_{y \in M_n(J)} \|x + y\|.$$

Controlled six term exact sequence

Theorem Oyono-Oyono and G.Yu

There exists a control pair (λ, h) such that for any completely filtered extensions of C^*-algebras

$$0 \to J \xrightarrow{j} A \xrightarrow{q} A/J \to 0,$$

the following six-term sequence is (λ, h)-exact

$$\begin{tikzcd}
\mathcal{K}_0(J) \arrow[r, "j_*"] & \mathcal{K}_0(A) \arrow[r, "q_*"] & \mathcal{K}_0(A/J) \arrow[d, "\mathcal{D}_{J,A}"'] \\
\mathcal{K}_1(A/J) \arrow[u, "\mathcal{D}_{J,A}"] & \mathcal{K}_1(A) \arrow[l, "q_*"'] & \mathcal{K}_1(J) \arrow[l, "j_*"']
\end{tikzcd}$$

Controlled Roe transformation

For any $z \in KK_1(A,B)$,then z can be represent by a triple (H_A, π, T) where:

- $\pi : A \to \mathcal{L}_B(H_B)$ is a $*$-representation of A on H_B;
- $T \in \mathcal{L}_B(H_B)$ is a self-adjoint operator;
- $[T, \pi(a)], \pi(a)[T^2 - Id_{H_B}]$ are compact operators in $K(H_B) \cong K(H) \otimes B$

Let $P = (\frac{1+T}{2}) \in \mathcal{L}_B(H_B)$ and

$$E^{(\pi,T)} = \{(a, P\pi(a)P + y) : a \in A, y \in B \otimes K(H)\}$$

Then we have a semi-split exact extension:

$$0 \to B \otimes K(H) \to E^{(\pi,T)} \to A \to 0$$

where the completely positive section is $s : A \to E^{\pi,T}; a \mapsto (a, P\pi(a)P)$.

By the functor property of $C^*_{max}(X, \cdot)$,then we have a semi-split exact extension:

$$0 \to C^*_{max}(X, B) \to E^{\pi,T}_{X,max} \to C^*_{max}(X, A) \to 0$$

where $E^{\pi,T}_{X,max} = C^*_{max}(X, E^{\pi,T})$

Proposition

The controlled boundary map $\mathcal{D}^{\pi,T} = \mathcal{D}_{C^*_{max}(X,B),E^{\pi,T}_{X,max}}$ of the extension

$$0 \to C^*_{max}(X, B) \to E^{\pi,T}_{X,max} \to C^*_{max}(X, A) \to 0$$

only depends on the class z.

Odd case

Let A and B be two C^*-algebra.then there exists a control pair(α_X, k_X)such that for any $z \in KK_1(A, B)$,there exists a (α_X, k_X)- controlled morphism

$$\hat{\sigma}_{X,max}(z) : \mathcal{K}_*(C^*_{max}(X, A)) \to \mathcal{K}_{*+1}(C^*_{max}(X, B))$$

Even case

Using Bott periodicity theorem, Let$A$ and B be two C^*-algebras,for any $z \in KK_0(A, B)$,there exists a control pair $(\alpha_X, k_X)$and even degree (α_X, k_X)-controlled morphism

$$\hat{\sigma}_{X,max}(z) : \mathcal{K}_*(C^*_{max}(X, A)) \to \mathcal{K}_*(C^*_{max}(X, B))$$

For any positive number d and probability η of the Rips complex $P_d(X)$ can be written as $\eta = \sum_{x\in X} \lambda_x(\eta)\delta_x$, where δ_x is the Dirac probability at x, and $\lambda_x : P_d(X) \to [0,1]$ is a continuous function. Let

$$h_d : \begin{cases} X \times X \to C_0(P_d(X)) \\ \qquad\qquad (x,y) \mapsto \lambda_x^{\frac{1}{2}} \lambda_y^{\frac{1}{2}} \end{cases}$$

Let $(e_x)_{x\in X}$ be the canonical basis of $l^2(X)$, e be a rank one projection in $\mathcal{H}$, and P_d be defined as the extension by linearity and continuity of

$$P_d(e_x \otimes \xi \otimes f) = \sum_{y\in X} e_y \otimes (e\xi) \otimes (h(x,y)f)$$

for every $x \in X, \xi \in \mathcal{H}$ and $f \in C_0(P_d(X))$. As $\sum_{x\in X} \lambda_x = 1$, P_d is projection of $K(l^2(X)) \otimes C_0(P_d(X))$ of propagation less than d. Hence, P_d define a class $[P_d, 0]_{\epsilon,r'} \in K_0^{\epsilon,r'}(C^*_{max}(X, C_0(P_d(X))))$ for any $\epsilon \in (0, \frac{1}{4})$ and $r' \geq d$.

Quantitative maximal coarse Baum-Connes assembly map

Definition

Let A be a C^*-algebra,$\epsilon \in (0, \frac{1}{4})$ and positive numbers d, r satisfying that $k_X(\epsilon)d \leq r$.The quantitative assembly map $\hat{\mu}_{X,A,max,*} = (\mu^{\epsilon,d,r}_{X,A,max,*})_{\epsilon,r}$ is defined as the family of maps

$$\mu^{\epsilon,d,r}_{X,A,max,*} : \begin{cases} KK_*(C_0(P_d(X)), A) \to K^{\epsilon,r}_*(C^*_{max}(X, A)) \\ z \mapsto \iota_*^{\alpha_X \epsilon', \epsilon, k_X(\epsilon')r', r} \circ \hat{\sigma}_{X,max}(z)[P_d, 0]_{\epsilon', r'} \end{cases}$$

where ϵ' and r' satisfy:

- $\epsilon' \in (0, \frac{1}{4})$ such that $\alpha_X \epsilon' \leq \epsilon$.
- $d \leq r'$ such that $k_X(\epsilon')r' \leq r$.

Let $KK_*(P_d(X), A)$ denote $KK_*(C_0(P_d(X)), A)$.

Definition

Let A be a G-algebra, We say that:

- (Quantitative injectivity) $\mu_{X,A,max,*}$ is quantitative injective if for any $d > 0$, there exists $\epsilon \in (0, \frac{1}{4})$ such that for any $r > 0$ satisfying $k_X(\epsilon)d \leq r$, there exists $d' > d$ such that for any $z \in KK_*(P_d(X), A)$, $\mu^{\epsilon,d,r}_{X,A,max,*}(z) = 0$ implies that $(q_d^{d'})^*(z) = 0$.
- (Quantitative surjectivity) $\mu_{X,A,max,*}$ is quantitative surjective if there exists $\epsilon \in (0, \frac{1}{4})$ such that for any $r > 0$ such that, there exists $\epsilon' \in (\epsilon, \frac{1}{4})$ and positive numbers d, r' such that $r \leq r'$ and $k_X(\epsilon')d \leq r'$, for any $y \in K^{\epsilon,r}_*(C^*_{max}(X, A))$ there exists $z \in KK_*(P_d(X), A)$ such that $\mu^{\epsilon',d,r'}_{X,A,max,*}(z) = \iota^{\epsilon,\epsilon',r,r'}_*(y)$

Proposition

Let X be a discrete metric space with bounded geometry and A be a C^*-algebra.

- If $\mu_{X,A,max,*}$ is quantitative injective then $\mu_{X,A,max,*}$ is one-to-one.
- If $\mu_{X,A,max,*}$ is quantitative surjective then $\mu_{X,A,max,*}$ is onto.

Definition

- $QI_{X,A,max,*}(d,d',\epsilon,r)$:for any $x \in KK_*((P_d(X),A)$, then $\mu^{\epsilon,d,r}_{X,A,max,*}(x) = 0$ implies $(q_d^{d'})^*(x) = 0$ in $KK_*(P_{d'}(X),A)$
- $QS_{X,A,max,*}(d,\epsilon,\epsilon',r,r')$: for any $y \in K_*^{\epsilon,r}(C^*_{max}(X,A)$,then there exist a $x \in KK_*(P_d(X),A))$ such that $\mu^{\epsilon',d,r'}_{X,A,max,*}(x) = \iota_*^{\epsilon,\epsilon',r,r'}(y)$.

Theorem

Let X be a discrete matric space with bounded geometry and A is a C^*-algebra. The following are equivalent:

(1) $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),max,*}$ is one to one,

(2) For any $d>0, \epsilon\in(0,\frac{1}{4})$ and $r>0$ with $k_X(\epsilon)d\leq r$,there exists d' such that $d\leq d'$ and $QI_{X,A,max,*}(d,d',\epsilon,r)$ holds.

Theorem

Let X be a discrete metric space with bounded geometry and A is a C^*-algebra. Then there exist $\lambda>1$ such that the following are equivalent:

(1) $\mu_{X,l^\infty(\mathbb{N},K(H)\otimes A),max,*}$ is onto;

(2) For any positive numbers ϵ with $\epsilon<\frac{1}{4\lambda}$ and $r>0$,there exist $d>0$ and $r'>0$ with $k_X(\epsilon)d\leq r$ and $r\leq r'$ for which $QS_{X,A,max,*}(d,r,r',\epsilon,\lambda\epsilon)$ is satisfied.

Corollary

Let X be a discrete metric space with bounded geometry and A is a C^*-algebra. Then we have the following results:

- $\mu_{X,A,max,*}$ is one to one. Then for any $\epsilon \in (0, \frac{1}{4})$ and every $d > 0, r > 0$ such that $k_X(\epsilon)d \leq r$,there exists d' with $d \leq d'$ such that $QI_{X,A,max,*}(d, d', \epsilon, r)$ holds.
- $\mu_{X,A,max,*}$ is onto. Then for some $\lambda \geq 1$ and any $\epsilon \in (0, \frac{1}{4\lambda})$ and every $r > 0$, there exists $d > 0, r' > 0$ such that $k_X(\epsilon)d \leq r$ and $r \leq r'$ such that $QS_{X,A,max,*}(d, r, r', \epsilon, \lambda\epsilon)$ holds.

Persistence approximation property

Persistence approximation property was introduced by Oyono-Oyono and Guoliang Yu. It provides the geometric obstruction to Baum-Connes conjecture.

Definition

Let B be a filtered C^*-algebra. we sat that $K_*(B)$ has persistence approximation property if: for any $\epsilon \in (0, \frac{1}{4})$ and $r > 0$, there exists $\epsilon' \in (\epsilon, \frac{1}{4})$ and $r' \geq r$ such that for any $x \in K_*(B)$, then $\iota_*^{\epsilon,\epsilon',r,r'}(x) \neq 0$ in $K_*^{\epsilon',r'}(B)$ implies that $\iota_*^{\epsilon,r}(x) \neq 0$ in $K_*(B)$.
$\mathcal{PA}_*(B, \epsilon, \epsilon', r, r')$: for any $x \in K_*^{\epsilon,r}(B)$, then $\iota_*^{\epsilon,r}(x) = 0$ in $K_*(B)$ implies that $\iota_*^{\epsilon,\epsilon',r,r'}(x) = 0$ in $K_*^{\epsilon',r'}(B)$.

Persistence approximation property for crossed with groups

Theorem Oyono-Oyono and G.Yu

Let Γ be a finite generated group and A be a C^*-algebra.Assume that:

- $\mu_{\Gamma, l^\infty(\mathbb{N}, A\otimes K(H))}$ is onto and $\mu_{\Gamma,A}$ is one to one.
- Γ admits a cocompact universal example for proper actions.

Then for some universal constant $\lambda_{PA} \geq 1$, any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$,any $r > 0$, and any Γ-C^*-algebra A there exists $r' \geq r$ such that $\mathcal{PA}_*(A \rtimes_{red} \Gamma, \epsilon, \lambda_{PA}\epsilon, r, r')$ holds.

Persistence approximation property for crossed product with groupoids

Theorem Clément Dell'Aiera

Let G be an étale groupoid such that:

- $G^{(0)}$ is compact.
- G admits a cocompact example for universal space for proper actions.

Then there exists a universal constant $\lambda_{PA} \geq 1$ such that for any G-algebra A,if $\mu_{G,l^\infty(\mathbb{N},A\otimes K(H))}$ is onto and $\mu_{G,A}$ is one to one, then for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and every $F \in \mathcal{E}$,there exists a F such that $F \subseteq F'$ and $\mathcal{PA}_*(A \rtimes_{red} G, \epsilon, \lambda_{PA}\epsilon, F, F')$ holds.

For the metric space, we need a condition to replace that the group(groupoid) admits a cocompact universal example for proper actions.

Definition

A discrete metric space is coarsely uniformly contractible: if for every $d > 0$,there exists $d' \geq d$ such that any compact subset of $P_d(X)$ lies in a contractible invariant compact subset of $P_{d'}(X)$.

Example 2.5 D.Meintrup and T.Schick

Any discrete hyperbolic metric space is coarsely uniformly contractible.

Persistence approximation property for maximal Roe algebras

Theorem Q.Wang and Z.Wang

Let X be a discrete metric space with bounded geometry and A is a C^*-algebra. Assume that:

- X is coarsely uniformly contractible.
- $\mu_{X,l^\infty(\mathbb{N},A\otimes K(H)),max,*}$ is onto and $\mu_{X,A,max,*}$ is one to one

Then there exists a universal constant $\lambda_{PA} \geq 1$ such that for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and every $r > 0$,there exists a $r' > 0$ such that $r \leq r'$ and $\mathcal{PA}_*(C^*_{max}(X, A), \epsilon, \lambda_{PA}\epsilon, r, r')$ holds.

Theorem Q.Wang and Z.Wang

Let X be a discrete metric space with bounded geometry. Assume that X admits a fibred coarse embedding into Hilbert space and X is coarsely uniformly contractible. then there exists a universal constant $\lambda_{PA} \geq 1$ such that: for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{PA}_*(C^*_{max}(X), \epsilon, \lambda_{PA}\epsilon, r, r')$ holds.

Theorem Q.Wang and Z.Wang

Let Γ be a finite generated residually finite group with haagerup property and admits a cocompact universal example for proper actions. Then there exists a universal constant $\lambda_{PA} \geq 1$ for any $\epsilon \in (0, \frac{1}{4\lambda_{PA}})$ and any $r > 0$ there exists $r' > r$ such that $\mathcal{PA}_*(C^*_{max}(X(\Gamma)), \epsilon, \lambda_{PA}\epsilon, r, r')$ holds.

Example

Both $\mathbb{F}_2$ and $SL_2(\mathbb{Z})$ are finite generated group with Haagerup property. Since their classifying space is a tree and this tree is cocompact. So they admit a cocompact universal example for proper actions. Hence the maximal Roe algebra of their box space will have persistence approximation property.

Let $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ be a family of discrete metric space with bounded geometry and $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of C*-algebras. Denote $C^*_{max}(\mathcal{X},\mathcal{A})$ be the closure of $\bigcup_{r>0}(\prod_{i\in\mathbb{N}}\mathbb{C}[X_i,A_i])_r$ of $\prod_{i\in\mathbb{N}}C^*_{max}(X_i,A_i)$. Then $C^*_{max}(\mathcal{X},\mathcal{A})$ is filtered C*-algebra.

Lemma

There exist a control pair (λ, h) and a (λ, h)-controlled isomorphism

$$\mathcal{K}_*(C^*_{max}(\mathcal{X},\mathcal{A})) \to \prod_i \mathcal{K}_*(C^*_{max}(X_i,A_i))$$

Quantitative assembly map for a family of metric space

Definition

For any $\epsilon \in (0, \frac{1}{4})$ and $d, r > 0$ with $k_{\mathcal{X}}(\epsilon) \cdot d \leq r$. Define:

$$\mu^{\infty,\epsilon,d,r}_{\mathcal{X},max,*} : \begin{cases} \prod_{i\in\mathbb{N}} KK_*(C_0(P_d(X_i)), \mathbb{C}) \to K^{\epsilon,r}_*(C^*_{max}(\mathcal{X})) \\ z \mapsto \iota_*^{\alpha_{\mathcal{X}}\epsilon',\epsilon,k_{\mathcal{X}}(\epsilon')r',r} \circ \hat{\sigma}^{\infty}_{\mathcal{X},max}(z)[P^{\infty}_{d,\mathcal{X}}, 0]_{\epsilon',r'} \end{cases}$$

where ϵ' and r' satisfy:

- $\epsilon' \in (0, \frac{1}{4})$ such that $\alpha_{\mathcal{X}} \cdot \epsilon' \leq \epsilon$;
- $d \leq r'$ and $k_{\mathcal{X}}(\epsilon') \cdot r' \leq r$.

Definition

- $QI_{\mathcal{X},max,*}(d,d',r,\epsilon)$: for any $x \in \prod_{i\in\mathbb{N}} KK_*(P_d(X_i),\mathbb{C})$, then $\mu^{\infty,\epsilon,d,r}_{\mathcal{X},max,*}(x) = 0$ implies $(q_d^{d'})^*(x) = 0$ in $\prod_{i\in\mathbb{N}} KK_*(P_d(X_i),\mathbb{C})$
- $QS_{\mathcal{X},max,*}(d,r,r',\epsilon,\epsilon')$: for any $y \in K_*^{\epsilon,r}(C^*_{max}(\mathcal{X}))$, there exists $x \in \prod_{i\in\mathbb{N}} KK_*(P_d(X_i),\mathbb{C})$ such that $\mu^{\infty,\epsilon',d,r'}_{\mathcal{X},max,*}(x) = \iota_*^{\epsilon,\epsilon',r,r'}(y)$

Let $\Sigma = \bigsqcup_{i\in\mathbb{N}} X_i$, where $(X_i)_{i\in\mathbb{N}}$ is a family of metric space satisfying: for any $r>0$, there exists an integer N_r such that for any integer i, any ball of radius r in X_i is no more than N_r elements.

The metric d on Σ is defined to be:

- on each X_i, the metric is just the usual metric on X_i;
- $d(X_i,X_j) \geq i+j$ if $i \neq j$.

Theorem Q.Wang and Z.Wang

Let $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ be a family of discrete metric space with bounded geometry.Let $\Sigma = \bigsqcup_{i\in\mathbb{N}} X_i$ defined as before. Assume that:

- for any $\epsilon \in (0, \frac{1}{4})$ and positive numbers such that $\alpha_{\mathcal{X}}(\epsilon) \cdot d \leq r$, there exists d' with $d \leq d'$, such that $QI_{\mathcal{X},max,*}(d, d', \epsilon, r)$ is holds.
- For some $\lambda > 1$ and any $\epsilon \in (0, \frac{1}{4\lambda}), r > 0$, there exists $d > 0, r' > r$ with $\alpha_{\mathcal{X}}(\epsilon) \cdot d \leq r'$ such that $QS_{\mathcal{X},max,*}(d, r, r', \epsilon, \lambda\epsilon)$.

Then Σ satisfies the maximal coarse Baum-Connes conjecture.

Thank you !