




On the Heat Equation

Yin Jingxue

A Talk at Sichuan University, March 28, 2018

$$\frac{\partial u}{\partial t} = \Delta u + f,$$

$$u(x, 0) = u_0(x).$$

-  Omega Limit Set
-  Chaotic Properties
-  Periodic Solutions
-  Small Perturbation
-  Life Span

Omega Limit Set

For a function $u(\cdot, t) : X \rightarrow \mathbb{R}$ with $u(\cdot, 0) = u_0$, the so called ω -limit set $\omega(u_0)$ is defined as follows

$$\omega(u_0) \equiv \{\varphi \in X; \exists t_n \rightarrow \infty \text{ such that } u(\cdot, t_n) \rightarrow \varphi \text{ in } X\}.$$

Examples:

1. If $u(\cdot, t) = e^{-t}$, then

$$\omega(u_0) = \{0\};$$

2. If $u(\cdot, t) = \sin t$, then

$$\omega(u_0) = [-1, 1];$$

3. If $u(\cdot, t) = t \sin t$, then

$$\omega(u_0) = (-\infty, +\infty) \equiv \mathbb{R}.$$



ω -Limit Set with Single Point



Consider the special case

$$\frac{\partial u}{\partial t} = \Delta u, \quad x \in \mathbb{R}^N, \quad t > 0,$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^N,$$

which has only trivial solution $u(x, t) \equiv 0$. So,

$$\omega(0) = \{0\}.$$

ω -LIMIT SET WITH SINGLE POINT

With a “small” perturbation $u_0(x)$, we further consider the problem

$$\begin{aligned}\frac{\partial u}{\partial t} - \Delta u &= 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^N.\end{aligned}$$

Then the solution decays to zero, namely

$$\lim_{t \rightarrow \infty} u(x, t) = 0 \text{ uniformly.}$$

So,

$$\omega(u_0) = \{0\}.$$

ω -LIMIT SET WITH SINGLE POINT

Indeed, if $0 \leq u_0 \in L^1(\mathbb{R}^N)$, then for some $\alpha > 0$

$$u(x, t) \leq Ct^{-\alpha}, \quad t > 0,$$

ω -LIMIT SET WITH SINGLE POINT

Indeed, if $0 \leq u_0 \in L^1(\mathbb{R}^N)$, then for some $\alpha > 0$

$$u(x, t) \leq Ct^{-\alpha}, \quad t > 0,$$

and the best decay exponent

$$\alpha_0 = \frac{N}{2},$$

namely

$$u(x, t) \leq Ct^{-N/2}, \quad t > 0.$$

Notice that

$$u(x, t) \leq Ct^{-N/2}, \quad t > 0$$

implies

$$0 \leq t^{N/2}u(x, t) \leq C, \quad t > 0.$$

To understand a better asymptotic behaviour, we should discuss the asymptotic behaviour of

$$t^{N/2}u(x, t)$$

as t goes to infinity.

In fact, we have

$$\lim_{t \rightarrow \infty} \|t^{\frac{N}{2}} u(\cdot, t) - G_M(\cdot, 1)\|_{\infty} = 0,$$

where M is the integral of the initial value u_0 and

$$G_M(x, t) = M(4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

It should be noticed that the condition $0 \leq u_0 \in L^1(\mathbb{R}^N)$ is not a necessary one for insuring the decay of solutions. In fact, if $0 \leq u_0 \in L^1(\mathbb{R}^N)$ and

$$u_0(x) \sim \frac{1}{|x|^\sigma}, \quad x \rightarrow \infty,$$

then $\sigma > N$. However, we need only to restrict σ to be a positive number

$$\sigma > 0,$$

due to an early work by Zhao.

✿ J. N. Zhao, The asymptotic behaviour of solutions of a quasilinear degenerate parabolic equation, *J. Diff. Eqns.* **102**(1993), 33–52.

Zhao consider another working space for u_0 :

$$W_\sigma(\mathbb{R}^N) \equiv \{\varphi \in L^1_{\text{loc}}(\mathbb{R}^N); |\cdot|^\sigma \varphi \in L^\infty(\mathbb{R}^N)\}.$$

ω -LIMIT SET WITH SINGLE POINT

He showed that if $0 < \sigma < N$ and the nonnegative initial value $u_0 \in W_\sigma(\mathbb{R}^N)$ with

$$\lim_{|x| \rightarrow \infty} |x|^\sigma u_0(x) = A,$$

then the solution satisfies

$$\lim_{t \rightarrow \infty} u(x, t) = 0,$$

and so

$$\omega(u_0) = \{0\}.$$

We can also get the best decay estimate

$$0 \leq u(x, t) \leq Ct^{-\sigma/2}, \quad t > 0.$$

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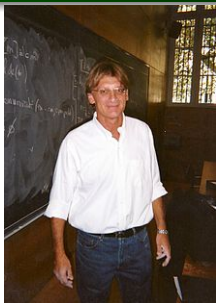
Furthermore, we have

$$\lim_{t \rightarrow \infty} \sup_{\{|x| \leq Ct^{1/2}\}} t^{\sigma/2} |u(x, t) - W_A(x, t)| = 0,$$

where $W_A(x, t)$ is the solution of the heat equation with the initial value

$$u_0(x) = A|x|^{-\sigma}.$$

A NONLINEAR CASE



Lions, P. L., Asymptotic behavior of some nonlinear heat equations, Nonlinear phenomena, **Physica D**, 5(1982), 293–306.

Consider bounded solutions of the heat equation with sources

$$\frac{\partial u}{\partial t} - \Delta u = f(u). \quad (1)$$

If $u_0 \in X = W_0^{1,\infty}(\Omega)$, $f(s)$ is strictly convex with $f(0) = 0$ and $f'(0) < \lambda_1$, then the ω -limit set $\omega(u_0)$ contains only stationary solution.



ω -Limit Set with More Points



ω -LIMIT SETS WITH MORE POINTS

国际数学家大会主要报告人



国际自动控制联合会主席



We turn the case for omega limit set with more points.

✿ Vázquez J. L. and Zuazua E., Complexity of large time behaviour of evolution equations with bounded data, *Chin. Ann. Math. Ser. B*, **23**(2002), 293–310.

They investigate the complexity of

rescaled solutions $v(x, t) \equiv u(t^{1/2}x, t)$

ω -LIMIT SETS WITH MORE POINTS

They showed that if the nonnegative initial value $u_0 \in L^\infty(\mathbb{R}^N)$, then the ω -limit set may contain infinite number of points.

For further investigation, they introduce the concept of omega limit set for initial data, namely

$$\Omega(u_0) \equiv \{f \in L^\infty(\mathbb{R}^N); \exists \lambda_n \rightarrow \infty \text{ s. t. } u_0(\lambda_n \cdot) \xrightarrow{\star} f \text{ in } L^\infty(\mathbb{R}^N)\}.$$

ω -LIMIT SETS WITH MORE POINTS

To express their new results, they use the semigroup associated with the heat operator. For this purpose, we rewrite the heat equation into

$$\frac{1}{u} \frac{\partial u}{\partial t} = \Delta,$$

that is

$$\frac{\partial \ln u}{\partial t} = \Delta.$$

So,

$$\ln u = t\Delta + C.$$

Taking $t = 0$, we get $C = \ln u_0$. Therefore

$$u(x, t) = e^{t\Delta} u_0 \equiv S(t)u_0.$$

They prove that

$$\omega(v(x, 1)) = S(1)\Omega(u_0),$$

where

$$v(x, t) \equiv u(t^{1/2}x, t).$$

Some relevant works about the study of the ω -limit sets can be found in several papers, for example

✿ T. Cazenave, F. Dickstein and F.B. Weissler, Universal solutions of the heat equation on \mathbb{R}^N . *Discrete Contin. Dyn. Sys.* **9**(2003), 1105-1132.

✿ T. Cazenave, F. Dickstein and F.B. Weissler, Universal solutions of a nonlinear heat equation on \mathbb{R}^N , *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **5**(2003) 77–117.

✿ T. Cazenave, F. Dickstein and F.B. Weissler, Chaotic behavior of solutions of the Navier-Stokes system in \mathbb{R}^N , *Adv. Differ. Equations* **10**(2005) 361–398.

✿ T. Cazenave, F. Dickstein and F.B. Weissler, Nonparabolic asymptotic limits of solutions of the heat equation on \mathbb{R}^N , *J. Dyn. Differ. Equations* **19**(2007) 789–818.

✿ J.X. Yin, L.W. Wang and R. Huang, Complexity of asymptotic behavior of Solutions for the porous medium equation with Absorption, *Acta Mathematica Scientia* 30B(6)(2010) 1865–1880.

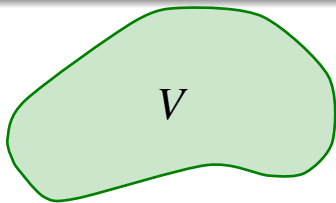
✿ J.X. Yin, L.W. Wang and R. Huang, Complexity of asymptotic behavior of the porous medium equation in \mathbb{R}^N , *J. Evol. Equ.* **11**(2011) 429–455.

Chaotic Properties

A continuous map $F : X \rightarrow X$ is called **transitive**, if for all non-empty open subsets U and V of X , there exists a natural number k such that

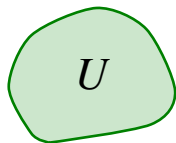
$$F^k(U) \cap V \neq \emptyset.$$

DYNAMICAL PROPERTIES — TRANSITIVE

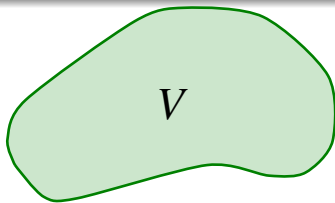


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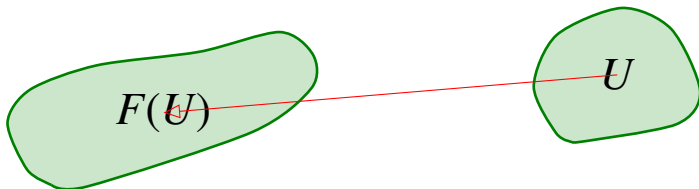


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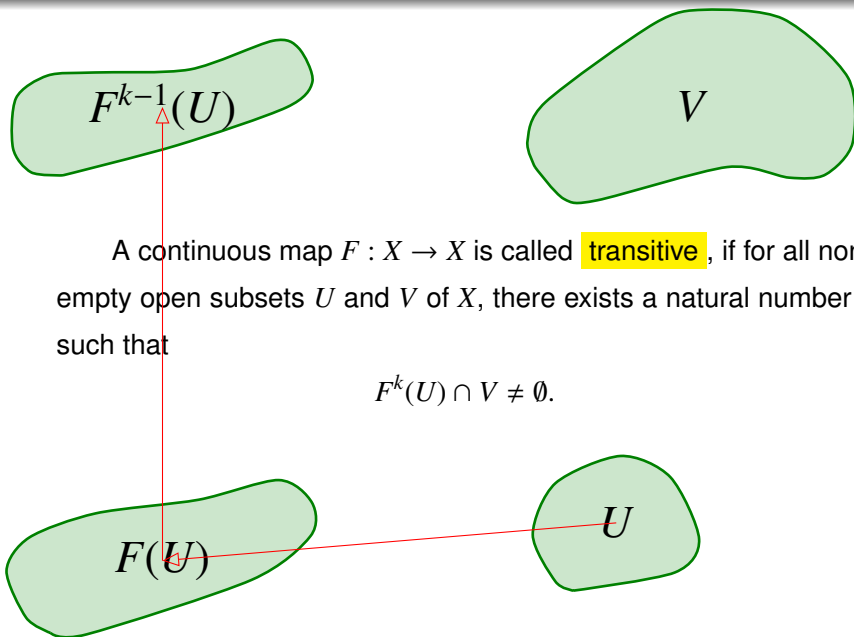


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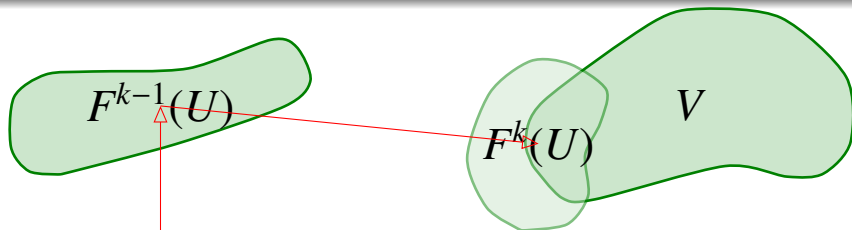
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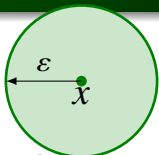
$$F^k(U) \cap V \neq \emptyset.$$

DYNAMICAL PROPERTIES — SENSITIVE DEPENDENCE

A map $F : X \rightarrow X$ is called to have the property of **sensitive dependence** on initial conditions if there is a $\delta > 0$ such that for every point $x \in X$ and every neighborhood Ω of x , there exists a point $y \in \Omega$ and a nonnegative integer k such that

$$\text{dist}(F^k(x), F^k(y)) > \delta.$$

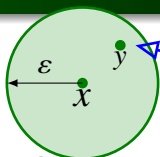
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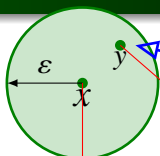


sensitive point

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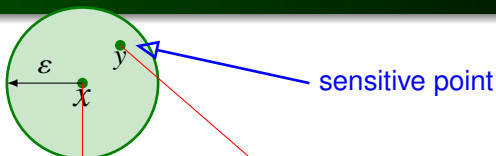
$F^k(x)$

A red line with an open downward-pointing triangle at the top and a green dot at the bottom connects the point x in the neighborhood diagram to the point $F^k(x)$.

$F^k(y)$

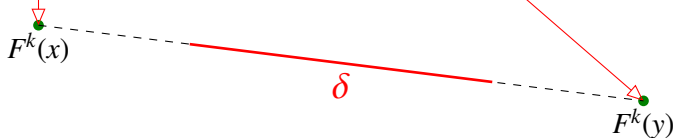
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DYNAMICAL PROPERTIES — SENSITIVE DEPENDENCE

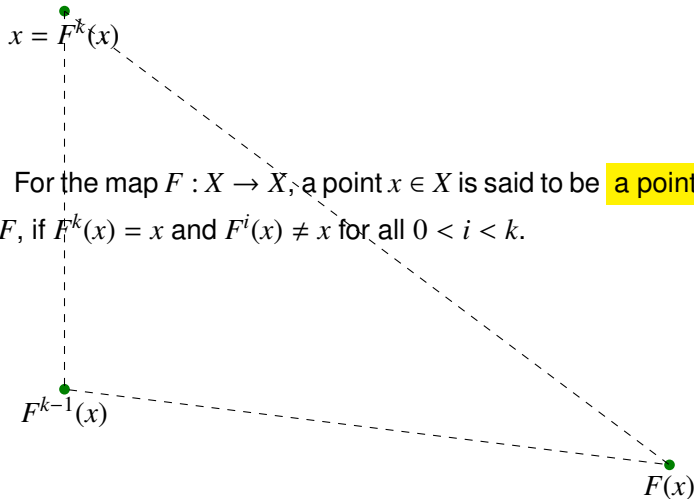


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DYNAMICAL PROPERTIES — PERIODIC POINT



The diagram illustrates a periodic orbit of period 3 for a point x under a map F . It shows three points: $x = F^k(x)$ at the top left, $F^{k-1}(x)$ at the bottom left, and $F(x)$ at the bottom right. Dashed lines connect these points in a cycle: from x to $F(x)$, from $F(x)$ to $F^{k-1}(x)$, and from $F^{k-1}(x)$ back to x . Each point is marked with a green dot.

$$x = F^k(x)$$

For the map $F : X \rightarrow X$, a point $x \in X$ is said to be a point of period k for F , if $F^k(x) = x$ and $F^i(x) \neq x$ for all $0 < i < k$.

$$F^{k-1}(x)$$

$$F(x)$$

Definition (Devaney's Definition of Chaos)

Let (X, d) be a metric space. A continuous map $F : X \rightarrow X$ is said to be chaotic on X if

1. F is transitive;
2. The periodic points of F are dense in X ;
3. F has sensitive dependence on initial conditions.

FINDING A MAP WITH DYNAMICAL PROPERTIES

Remember that

$$W_\sigma(\mathbb{R}^N) \equiv \{\varphi \in L^1_{\text{loc}}(\mathbb{R}^N); |\cdot|^\sigma \varphi \in L^\infty(\mathbb{R}^N)\}.$$

Let us construct a chaotic map. Let $M > 0, \lambda > 1$. Define

$$B_M^{\sigma,+} \equiv \{\varphi \in W_\sigma(\mathbb{R}^N); \|\varphi\|_{W_\sigma(\mathbb{R}^N)} \leq M \text{ and } \varphi \geq 0\}.$$

Consider the map $F_\lambda^\sigma: S(1)B_M^{\sigma,+} \mapsto S(1)B_M^{\sigma,+}$ defined by

$$F_\lambda^\sigma = D_\lambda^\sigma S(\lambda - 1) = S(1 - \frac{1}{\lambda})D_\lambda^\sigma,$$

where $S(t)$ is the semigroup operator, and D_λ^σ is defined by

$$D_\lambda^\sigma \varphi(x) = \lambda^{\sigma/2} \varphi(\lambda^{1/2} x).$$

Theorem

For any fixed $\lambda > 1$, the map

$$F_{\lambda}^{\sigma} : S(1)B_M^{\sigma,+} \mapsto S(1)B_M^{\sigma,+},$$

is chaotic.

Periodic Solutions

Consider the special periodic problem

$$\frac{\partial u}{\partial t} = \Delta u, \quad x \in \Omega, \quad t > 0,$$

$$u(x, t) = 0, \quad x \in \partial\Omega,$$

$$u(x, t + T) = u(x, t).$$

This problem has only trivial solution $u(x, t) \equiv 0$.

$$\frac{\partial u}{\partial t} = \Delta u.$$

In fact, multiplying the equation by u and integrating over $Q = \Omega \times (0, T)$, we have

$$\frac{1}{2} \iint_Q \frac{\partial}{\partial t} (u^2) dx dt + \iint_Q |\nabla u|^2 dx dt = 0.$$

Due to the periodicity, the first term is zero. Therefore

$$\iint_Q |\nabla u|^2 dx dt = 0,$$

which implies that $u(x, t) \equiv 0$.

Consider the special periodic problem

$$\frac{\partial u}{\partial t} = \Delta u + m(t)u^p, \quad x \in \Omega, \quad t > 0,$$

$$u(x, t) = 0, \quad x \in \partial\Omega,$$

$$u(x, t + T) = u(x, t).$$

This problem has only a trivial solution $u(x, t) \equiv 0$.

THE LINEAR CASE (1)

T. I. Seidman, Periodic solutions of a non-linear parabolic equation, **J. Differential Equations**, 19(1975), 242–257.

Seidman studied the related special case

$$\frac{\partial u}{\partial t} = \Delta u + m(t), \quad (x, t) \in \Omega \times \mathbb{R},$$

$$u(x, t) = 0, \quad x \in \partial\Omega,$$

$$u(x, t + T) = u(x, t).$$

He established the existence of nontrivial periodic solutions for any $m(t) \not\equiv 0$.

THE LINEAR CASE (2)

A. Beltramo, P. Hess, On the principal eigenvalue of a periodic-parabolic operator, **Comm. Partial Differential Equations**, 9(1984), 919–941.

Beltramo and Hess considered the linear source case

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u + m(t)u, & (x, t) &\in \Omega \times \mathbb{R}, \\ u(x, t) &= 0, & x &\in \partial\Omega, \\ u(x, t + T) &= u(x, t),\end{aligned}$$

and showed that only for some special $m(t)$ can the equation have nontrivial periodic solutions.

THE NONLINEAR CASE



Maria J. Esteban, from 2015 to 2019, president of International Council for Industrial and Applied Mathematics.

The pioneering works for the nonlinear case are the following

M. J. Esteban, On periodic solutions of superlinear parabolic problems, **Trans. Amer. Math. Soc.**, 293(1986), 171–189.

M. J. Esteban, A remark on the existence of positive periodic solutions of superlinear parabolic problems, **Proc. Amer. Math. Soc.**, 102(1988), 131–136.

THE NONLINEAR CASE (1)

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It was Esteban who first consider the nolinear source case

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u + m(t)u^p, & (x, t) \in \Omega \times \mathbb{R}, \\ u(x, t) &= 0, & x \in \partial\Omega, \\ u(x, t + T) &= u(x, t),\end{aligned}$$

where $1 < p < p_c$ with p_c being the **Fujita exponent** for the equation with \mathbb{R}^N instead of Ω , namely

$$p_c \equiv 1 + \frac{2}{N}.$$

She established the existence of nontrivial periodic solutions.

Main approaches: using the blow-up technique

THE NONLINEAR CASE (1)

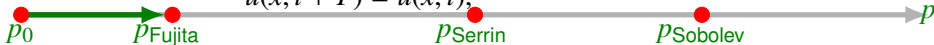
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THE NONLINEAR CASE (2)

M. J. Esteban, A remark on the existence of positive periodic solutions of superlinear parabolic problems, **Proc. Amer. Math. Soc.**, 102(1988), 131–136.

Two years later, Esteban extend the result to

$$1 < p < 2_* \equiv \frac{N}{N-2}.$$

Here 2_* is the well-known **Serrin exponent**.

Main approaches: based on Liouville type results.

THE NONLINEAR CASE (2)

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THE NONLINEAR CASE (3)

S. I. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl., 5(1965), 1408–1411.

Non-existence result for

$$q \geq 2^* - 1 \equiv \frac{N+2}{N-2}$$

with Ω being the star-shape domain, can be proved by the Pohozaev identity. Here 2^* is the well-known **Sobolev exponent**.

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THE NONLINEAR CASE (4)

P. Quittner, Multiple equilibria, periodic solutions and a priori bounds for solutions in superlinear parabolic problems, **Nonlinear Differential Equations Appl.**, 11(2004), 237–258.

About 16 years later, the gap $2_* \leq p < 2^* - 1$ was partially filled by Quittner (2004), in which he proved the existence with some restrictions on the structure of $m(t)$

$$\sup_{t \in (0, \omega)} \frac{(m'(t))^-}{m(t)} < \frac{2N - (N - 2)(q + 1)}{r^2(\Omega)},$$

where $r(\Omega)$ is the radius of Ω .

Main approaches: dynamical method, topological degree argument, . . .

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J. X. Yin, C. H. Jin, Periodic solutions of the evolutionary p -Laplacian with nonlinear sources, J. Math. Anal. Appl., 368(2010), 604–622.

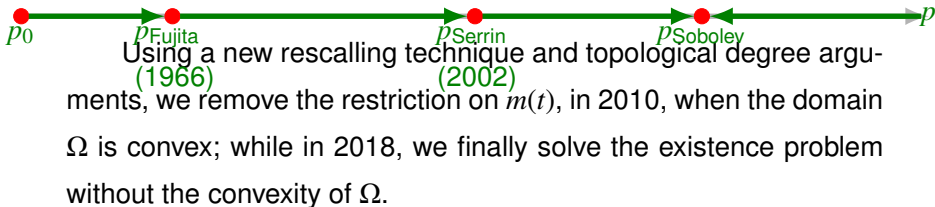
H. C. Wang, J. X. Yin, C. H. Jin, A Note on the Existence of Time Periodic Solution of a Superlinear Heat Equation, Appl. Anal., 2018, Accepted.

Using a new rescaling technique and topological degree arguments, we remove the restriction on $m(t)$, in 2010, when the domain Ω is convex; while in 2018, we finally solve the existence problem without the convexity of Ω .

OUR RESULT

J. X. Yin, C. H. Jin, Periodic solutions of the evolutionary p -Laplacian with nonlinear sources, J. Math. Anal. Appl., 368(2010), 604–622.

H. C. Wang, J. X. Yin, C. H. Jin, A Note on the Existence of Time Periodic Solution of a Superlinear Heat Equation, Appl. Anal., 2018, Accepted.



Small Perturbation

Let us return again to the special case

$$\frac{\partial u}{\partial t} = \Delta u, \quad x \in \mathbb{R}^N, \quad t > 0,$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^N,$$

which has only trivial solution $u(x, t) \equiv 0$. It is interesting to discuss the perturbations, including those for initial data, inner sources, etc.

We already talked about the small perturbation for initial data

$$u_0(x)$$

$$\frac{\partial u}{\partial t} - \Delta u = 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty),$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N,$$

which is stable, since

$$\lim_{t \rightarrow \infty} u(x, t) = 0 \text{ uniformly.}$$

While if we consider the perturbation as inhomogeneity $f(x) \geq 0$ with $f(x) \not\equiv 0$:

$$\frac{\partial u}{\partial t} = \Delta u + f(x), \quad x \in \mathbb{R}^N, \quad t > 0,$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^N,$$

then the solution does definitely not decay to zero, whatever $f(x)$ is small enough.

Now, we turn to the nonlinear perturbation

$$\frac{\partial u}{\partial t} = \Delta u + \lambda u^p, \quad x \in \mathbb{R}^N, \quad t > 0,$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^N,$$

where λ is a small positive constant. Clearly, such a problem admits only trivial solution $u(x, t) \equiv 0$, if $p \geq 1$.

Now, we turn to the nonlinear perturbation

$$\frac{\partial u}{\partial t} = \Delta u + \lambda u^p, \quad x \in \mathbb{R}^N, \quad t > 0,$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^N,$$

where λ is a small positive constant. Clearly, such a problem admits only trivial solution $u(x, t) \equiv 0$, if $p \geq 1$.

However, if $0 < p < 1$, then, in addition to the trivial solution, there exist at least one non-trivial solution. For example, a solution independent of x :

$$u(x, t) = [(1 - p)\lambda t]^{1/(1-p)},$$

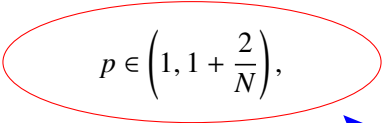
which satisfies

$$\lim_{t \rightarrow \infty} u(x, t) = +\infty.$$

With an additional small perturbation $u_0(x)$, we further consider the problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u + \lambda u^p, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^N.\end{aligned}$$

Then a completely different phenomenon occurs, namely for any


$$p \in \left(1, 1 + \frac{2}{N}\right),$$

all non-trivial solutions blow up in finite time.

Moreover, if we consider the perturbation as inhomogeneity, namely

$$\frac{\partial u}{\partial t} = \Delta u + \lambda u^p + f(x), \quad (x, t) \in \mathbb{R}^N \times (0, \infty),$$

$$u(x, 0) = 0, \quad x \in \mathbb{R}^N.$$

Then another different phenomenon occurs, namely for any

$$p \in \left(1, 1 + \frac{2}{N-2}\right), \quad N \geq 3$$

all non-trivial solutions blow up in finite time.

✿ C. Bandle, H. A. Levine and Q. S. Zhang, Critical exponents of Fujita type for inhomogeneous parabolic equations and systems, *J. Math. Anal. Appl.*, **251**(2000), 624–648.

2005

Viscosity

Nolinear Source

Now, we consider an additional pseudo-parabolic perturbation

$$\frac{\partial u}{\partial t} - k \frac{\partial \Delta u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0.$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N.$$

✿ E. I. Kaikina, P. I. Naumkin and I. A. Shishmarev, The Cauchy problem for a Sobolev type equation with power like non-linearity, *Izv. Math.*, **69**(1)(2005), 59–111.

$p > 1 \Rightarrow$ local existence of mild solutions

$p > 1 + \frac{2}{N} \Rightarrow$ global existence of mild solutions
for small initial data

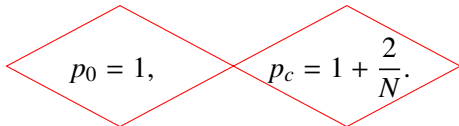
As for the special case $p = 1$, we refer to the early work

✿ Showalter, R. E. and Ting, T. W., Pseudoparabolic partial differential equations, SIAM J. Math. Anal., 1(1)(1970), 1–26.

$p = 1 \implies$ global existence of classical solutions
for any initial data

The case $0 < p < 1$ was not discussed by Kaikina.

With Kaikina's results, it is reasonable to guess that there exist two exponents, the global existence exponent p_0 and Fujita exponent p_c as follows



The diagram consists of two red-outlined diamonds that overlap in the center. The left diamond contains the expression $p_0 = 1,$ and the right diamond contains the expression $p_c = 1 + \frac{2}{N}.$

$$p_0 = 1, \quad p_c = 1 + \frac{2}{N}.$$

In other words, we conclude the following

$0 < p < 1 \implies$ global existence of classical solutions

$1 < p < 1 + 2/N \implies$ Blow-up of weak solutions
for any nontrivial initial data

$p > 1 + 2/N \implies$ Blow-up of weak solutions for large initial data

$p = 1 + 2/N \implies$ Blow-up of weak solutions
for any nontrivial initial data

PERTURBATION

Viscosity

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Another perturbation is

$$\frac{\partial u}{\partial t} - k \frac{\partial \Delta u}{\partial t} = \Delta u + \lambda u^p + f(x), \quad x \in \mathbb{R}^N, \quad t > 0,$$
$$u(x, 0) = 0, \quad x \in \mathbb{R}^N,$$

where $p > 0$, $k > 0$, $\lambda > 0$, $f(x) \geq 0$ is a small nontrivial perturbation with $f(x) \not\equiv 0$.

Given Source

We conclude the following

$0 < p < 1 \implies$ global existence of classical solutions

$1 < p < 1 + 2/(N - 2) \implies$ Blow-up of weak solutions
for any nontrivial initial data

$p > 1 + 2/(N - 2) \implies$ Blow-up of weak solutions
for large initial data

$p = 1 + 2/(N - 2) \implies$ Blow-up of weak solutions
for any nontrivial initial data

Life Span

THE TRIVIAL CASE

For spatial homogeneous case, namely

$$\begin{cases} \frac{\partial u}{\partial t} = |u|^{p-1}u, & (x, t) \in \mathbb{R}^N \times (0, +\infty), \\ u(x, 0) = \lambda, & x \in \mathbb{R}^N, \end{cases} \quad (2)$$

the solution can be given explicitly

$$u(x, t) = \begin{cases} \left(\lambda^{1-p} - (p-1)t \right)^{1/(p-1)}, & p \neq 1, \\ \lambda e^t, & p = 1. \end{cases} \quad (3)$$

So, the solution blows up in finite time if and only if $p > 1$, and the blow-up time is given by

$$T_\lambda = \frac{\lambda^{1-p}}{p-1}.$$

One of the topics in discussing the life span is to scale the initial datum with a parameter, namely by considering the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u, & (x, t) \in \mathbb{R}^N \times (0, +\infty), \\ u(x, 0) = \lambda u_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (4)$$

Let us denote the solution as $u(x, t; \lambda)$ and the blow-up time of $u(x, t; \lambda)$ as T_λ . The values of u_0 near ∞ have a closed relation to the blow-up time, which can be viewed from several works.



Tzong-Yow Lee and Wei-Ming Ni, Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem, Trans. Amer. Math. Soc. **333** (1992), no. 1, 365–378.

Lee and Ni showed that, if there exist $C_1, C_2 > 0$ and $R > 0$ such that

$$C_1 \leq u_0(x) \leq C_2, \quad |x| > R,$$

then there exist $C_3, C_4 > 0$ and $\Lambda > 0$ such that

$$C_3 \leq \lambda^{p-1} T_\lambda \leq C_4 \quad \text{for } \lambda < \Lambda. \quad (5)$$



Changfeng Gui and Xuefeng Wang, Life spans of solutions of the Cauchy problem for a semilinear heat equation, J. Differential Equations **115** (1995), no. 1, 166–172.

Further, Gui and Wang proved that if

$$\lim_{|x| \rightarrow \infty} u_0(x) = A > 0,$$

then

$$\lim_{\lambda \rightarrow 0} \lambda^{p-1} T_\lambda = \frac{1}{p-1} A^{1-p}. \quad (6)$$

LOWER BOUND ESTIMATE ON T^*



Giga, Yoshikazu and Umeda, Noriaki, Blow-up directions at space infinity for solutions of semilinear heat equations, Bol. Soc. Parana. Mat., (3) 23(1-2)(2005), 9–28.

It is shown that, by Giga and Umeda, for any solution u , with initial datum $u_0 \in L^\infty(\mathbb{R}^N)$,

$$T^* \geq \frac{1}{p-1} \|u_0\|_{L^\infty(\mathbb{R}^N)}^{1-p},$$

which can not be improved, since in the trivial case $u_0(x) \equiv \lambda$,

$$T^* = \frac{\lambda^{1-p}}{p-1}.$$

UPPER BOUND ESTIMATE ON T^*

Since the lower bound estimate has already been established by Giga and Umeda, many papers are devoted to the upper bound estimate on T^* .

✿ Masaki Yamaguchi and Yusuke Yamauchi, Life span of positive solutions for a semilinear heat equation with non-decaying initial data, *Differential Integral Equations* **23** (2010), no. 11-12, 1151–1157.

Yamaguchi and Yamauchi established a upper bound estimate for the life span for $N = 1$, namely, if $\lim_{x \rightarrow \pm\infty} u_0(x) = A_{\pm}$, then

$$T^* \leq \frac{1}{p-1} \left(\frac{A_+ + A_-}{2} \right)^{1-p}. \quad (7)$$



Tohru Ozawa and Yusuke Yamauchi, Life span of positive solutions for a semilinear heat equation with general non-decaying initial data, J. Math. Anal. Appl. **379** (2011), no. 2, 518–523.

Ozawa and Yamauchi considered the multi-dimensional case. Let $N \geq 2$. Assume that $\liminf_{r \rightarrow \infty} u_0(rx) = u_\infty(x)$ with $u_\infty \in L^\infty(\mathbb{S}^{N-1})$ for any $x \in \mathbb{S}^{N-1}$ and that $\int_{\mathbb{S}^{N-1}} u_\infty(x') d\sigma(x') > 0$, where σ is the surface measure on the unit sphere. Then

$$T^* \leq \frac{1}{p-1} \left(\frac{1}{\sigma(\mathbb{S}^{N-1})} \int_{\mathbb{S}^{N-1}} u_\infty(x') d\sigma(x') \right)^{1-p}. \quad (8)$$



Yusuke Yamauchi, Life span of solutions for a semilinear heat equation with initial data having positive limit inferior at infinity, *Nonlinear Anal.* **74** (2011), no. 15, 5008–5014.

For $\xi' \in \mathbb{S}^{N-1}$ and $\delta \in (0, \sqrt{2})$, we set conic neighbourhood $\Gamma_{\xi'}(\delta)$:

$$\Gamma_{\xi'}(\delta) = \{\eta \in \mathbb{R}^N \setminus \{0\} \mid \left| \xi' - \frac{\eta}{|\eta|} \right| < \delta\},$$

and set

$$S_{\xi'}(\delta) = \Gamma_{\xi'}(\delta) \cap \mathbb{S}^{N-1}. \quad (9)$$

Define

$$u_\infty(x') = \liminf_{r \rightarrow +\infty} u(rx') \quad \text{for } x' \in \mathbb{S}^{N-1}.$$

UPPER BOUND ESTIMATE ON T^*

Let $N \geq 2$. Assume that there exist $\xi' \in \mathbb{S}^{N-1}$ and $\delta > 0$ such that $\operatorname{ess\,inf}_{x' \in S_{\xi'}(\delta)} u_\infty(x') > 0$. Then the classical solution for (??) blows up in finite time, and the blow-up time is estimated as

$$T^* \leq \frac{1}{p-1} \left(\operatorname{ess\,inf}_{x' \in S_{\xi'}(\delta)} u_\infty(x') \right)^{1-p}. \quad (10)$$

Let $N = 1$. Assume that $\max\{\liminf_{x \rightarrow +\infty} u_0(x), \liminf_{x \rightarrow -\infty} u_0(x)\} > 0$. Then the classical solution for (??) blows up in finite time, and the blow-up time is estimated as

$$T^* \leq \frac{1}{p-1} \left(\max\{\liminf_{x \rightarrow +\infty} u_0(x), \liminf_{x \rightarrow -\infty} u_0(x)\} \right)^{1-p}. \quad (11)$$

RAREFACTION POINT

A point x_0 is called to be a *rarefaction point* of the set E if

$$\lim_{r \rightarrow 0} \frac{\text{mes}(B(x_0, r) \cap E)}{\text{mes}(B(x_0, r))} = 0,$$

where $\text{mes}(F)$ is the Lebesgue measure of the set F , and $B(x, r)$ is the ball centered at x with radius r . It is reasonable to say that ∞ is a *rarefaction point* of the set E if

$$\lim_{r \rightarrow +\infty} \frac{\text{mes}(B(0, r) \cap E)}{\text{mes}(B(0, r))} = 0.$$

Alternatively, ∞ is called to be a *rarefaction point* of a non-negative function $\varphi(x)$ if for any $\alpha > 0$ it is a point of rarefaction of the set $\{x \mid \varphi(x) \geq \alpha\}$.

Theorem 1

If ∞ is not a rarefaction point of the initial datum $u_0(x)$, then the solution blows up in finite time T^* with

$$T^* \leq \frac{1}{p-1} \inf_{\alpha > 0} (\alpha D(\alpha))^{1-p}, \quad (12)$$

where

$$D(\alpha) \equiv \limsup_{r \rightarrow +\infty} \frac{\text{mes}(\{x \mid u_0(x) \geq \alpha\} \cap B(0, r))}{\text{mes}(B(0, r))}. \quad (13)$$

⇒ GUI AND WANG'S RESULT

If $\lim_{|x| \rightarrow \infty} u_0(x) = A > 0$, then

$$D(\alpha) = \begin{cases} 1, & \alpha < A, \\ 0, & \alpha > A, \end{cases}$$

$$\sup \alpha D(\alpha) = A$$

which, together with the lower bound estimate, implies that

$$\lim_{\lambda \rightarrow 0} \lambda^{p-1} T_\lambda = \frac{1}{p-1} A^{1-p}, \quad (14)$$

namely, the same result as did by Gui and Wang.



Changfeng Gui and Xuefeng Wang, Life spans of solutions of the Cauchy problem for a semilinear heat equation, J. Differential Equations **115** (1995), no. 1, 166–172.

⇒ LEE AND NI'S RESULT

Since $D(\alpha)$ is related to the initial datum $u_0(x)$, let us write $D(\alpha) = D(\alpha; u_0)$. Clearly


$$D(\alpha; \lambda u_0) \equiv D(\alpha/\lambda; u_0).$$

So, if the initial datum is replaced by $\lambda u_0(x)$, then

$$\sup_{\alpha>0} \alpha D(\alpha; \lambda u_0) = \lambda \sup_{\alpha>0} \frac{\alpha}{\lambda} D(\alpha/\lambda; u_0) = \lambda \sup_{\beta>0} \beta D(\beta; u_0),$$

which implies Lee and Ni's result

$$C_3 \leq \lambda^{p-1} T_\lambda \leq C_4.$$

 Tzong-Yow Lee and Wei-Ming Ni, Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem, Trans. Amer. Math. Soc. **333** (1992), no. 1, 365–378.

UPPER BOUND ESTIMATE ON T^*

We do not want to give a proof for **Theorem 1**, but prefer to state a much general version. For this purpose, let $\alpha, r > 0$ and denote

$$D(\alpha; r) \equiv \sup_{x \in \mathbb{R}^N} \frac{\text{mes}(B(x, r) \cap \{y \mid u_0(y) \geq \alpha\})}{\text{mes}(B(x, r))},$$

and define

$$\overline{D}(\alpha) := \limsup_{r \rightarrow +\infty} D(\alpha; r). \quad (15)$$

Theorem 2

Suppose that there exists $\alpha > 0$ such that $\overline{D}(\alpha) > 0$. Then the solution blows up in finite time T^* with

$$T^* \leq \frac{1}{p-1} \inf_{\alpha>0} \left(\alpha \overline{D}(\alpha) \right)^{1-p}. \quad (16)$$


⇒ YAMAUCHI'S RESULT

Our theorem implies the results by Yamauchi 2011. First, let us look at the one-dimensional case.

Theorem (Theorem 2 of Yamauchi 2011)

Let $N = 1$. Assume that $\max\{\liminf_{x \rightarrow +\infty} u_0(x), \liminf_{x \rightarrow -\infty} u_0(x)\} > 0$. Then the classical solution blows up in finite time, and the blow-up time is estimated as

$$T^* \leq \frac{1}{p-1} \left(\max\{\liminf_{x \rightarrow +\infty} u_0(x), \liminf_{x \rightarrow -\infty} u_0(x)\} \right)^{1-p}. \quad (17)$$

 Tohru Ozawa and Yusuke Yamauchi, Life span of positive solutions for a semilinear heat equation with general non-decaying initial data, J. Math. Anal. Appl. **379** (2011), no. 2, 518–523.

\Rightarrow YAMAUCHI'S RESULT

In fact, let us assume that

$$A = \liminf_{x \rightarrow +\infty} u_0(x) \geq \liminf_{x \rightarrow -\infty} u_0(x).$$

Then for any $\varepsilon > 0$ there exists $R > 0$ such that $u_0(x) \geq A - \varepsilon$ for $x > R$. Hence $\bar{D}(A - \varepsilon) = 1$. By Theorem 2 we obtain

$$T^* \leq \frac{1}{p-1} (A - \varepsilon)^{1-p}.$$

Since $\varepsilon > 0$ was arbitrary, it follows $T^* \leq \frac{1}{p-1} A^{1-p}$. The proof for $N = 1$ is finished.

⇒ YAMAUCHI'S RESULT

Theorem (Theorem 1 of Yamauchi 2011)

Let $N \geq 2$. Assume that there exist $\xi' \in \mathbb{S}^{N-1}$ and $\delta > 0$ such that $\operatorname{ess\,inf}_{x' \in S_{\xi'}(\delta)} u_{\infty}(x') > 0$. Then the classical solution for (??) blows up in finite time, and the blow-up time is estimated as

$$T^* \leq \frac{1}{p-1} \left(\operatorname{ess\,inf}_{x' \in S_{\xi'}(\delta)} u_{\infty}(x') \right)^{1-p}. \quad (18)$$

\Rightarrow YAMAUCHI'S RESULT

Let $A = \operatorname{ess\,inf}_{x' \in \mathcal{S}_{\xi'}(\delta)} u_{\infty}(x')$. It is sufficient to show that, for any $\bar{R} > 0$,

$0 < \tau < 1$ and $0 < \varepsilon < A$, there exists a ball $B(x, r)$ with $r > \bar{R}$ such that

$$\frac{\operatorname{mes}(\{y \mid u_0(y) \geq A - \varepsilon\} \cap B(x, r))}{\omega_N r^N} > 1 - \tau. \quad (19)$$

⇒ YAMAUCHI'S RESULT

First, we show that there exists $R_1 > 0$ such that

$$\frac{\sigma^{N-1}\left(\{x' \in \mathbb{S}^{N-1} \mid u_0(rx') \geq A - \varepsilon \text{ for } r > R_1\} \cap S_{\xi'}(\delta)\right)}{\sigma^{N-1}(S_{\xi'}(\delta))} > 1 - \tau, \quad (20)$$

where $\sigma^{N-1}(M)$ is the spherical measure for the measurable set $M \subset \mathbb{S}^{N-1}$ and $S_{\xi'}(\delta)$ is defined by (9).

Next, we show that there exists a ball $B(x_0, r_0)$ such that

$$B(x_0, r_0) \subset \{x \in \mathbb{R}^N \mid R_1 < |x| < R_1 + 1 \text{ and } \frac{x}{|x|} \in S_{\xi'}(\delta)\}, \quad (21)$$

and

$$\frac{\text{mes}(\{x \in B(x_0, r_0) \mid \frac{x}{|x|} \in S(R_1)\})}{\text{mes}(B(x_0, r_0))} \geq 1 - \tau. \quad (22)$$

\Rightarrow YAMAUCHI'S RESULT

Finally, we complete the proof by scaling the ball $B(x_0, r_0)$. We claim that for $\lambda > 1$, it follows that

$$\begin{aligned} & \frac{\text{mes}(\{y \in \mathbb{R}^N \mid u_0(y) \geq A - \varepsilon\} \cap B(\lambda x_0, \lambda r_0))}{\text{mes}(B(\lambda x_0, \lambda r_0))} \\ & \geq \frac{\text{mes}(\{x \in B(x_0, r_0) \mid \frac{x}{|x|} \in S(R_1)\})}{\text{mes}(B(x_0, r_0))}. \end{aligned}$$

The upper bound in our result is optimal. For this purpose, let us consider an example. Let $a_k = k!$, $0 < \varepsilon < 1$ and set

$$u_0(x) = \begin{cases} \varepsilon, & |x| \in [a_{2k-1} + \frac{1}{4}, a_{2k} - \frac{1}{4}], \\ 1, & |x| \in [a_{2k}, a_{2k+1}]. \end{cases}$$

By the definition of $D(\alpha)$, we have

$$\begin{aligned} D(1) &\geq \limsup_{k \rightarrow +\infty} \frac{\text{mes}(\{x \mid u_0(x) \geq 1\} \cap B(0, a_{2k+1}))}{\text{mes}(B(0, a_{2k+1}))} \\ &\geq \limsup_{k \rightarrow +\infty} \frac{a_{2k+1}^N - a_{2k}^N}{a_{2k+1}^N} = 1. \end{aligned}$$

So, the life span T^* of the solution u can be estimated by

$$T^* \leq \frac{1}{p-1} (1D(1))^{1-p} = \frac{1}{p-1}.$$

Since $v(x, t) = (1 - (p-1)t)^{\frac{1}{p-1}}$, which blows up at $T = \frac{1}{p-1}$, is an upper solution of (??) with $v(x, 0) \geq u(x, 0)$, by the comparison principle it follows that $T^* \geq T = \frac{1}{p-1}$. Thus

$$T^* = \frac{1}{p-1},$$

which shows that the minimal time blow-up occurs for such initial datum $u_0(x)$.

Remark

The above example shows that the upper bound estimate is optimal, since

$$T^* = \frac{1}{p-1}.$$

On the other hand, applying Yamauchi's result to the above example, the estimate is reduced to

$$T^* \leq \frac{1}{p-1} \varepsilon^{1-p}.$$

In particular, Yamauchi's result is inapplicable to the case with $\varepsilon = 0$.

Thank You!