几何不变理论和K-稳定性

田刚
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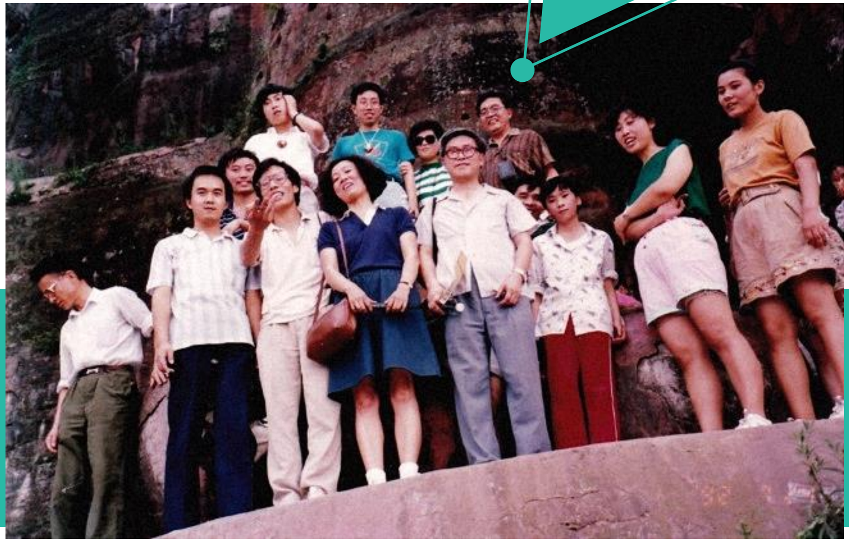


经典不变理论

它具有悠久的历史，主要问题可表述如下：

设**G是一个代数群，如所有行列式为1的**2×2复矩阵构成的$SL(2,\mathbb{C})$，作用在一个向量空间**V上。这作用诱导了G在V**上多项式生成的线性空间$R(\mathbf{V})$上的作用：

$$\sigma\cdot f(v)=f(\sigma^{-1}v),\sigma\in G,v\in V.$$

如果一个多项式f满足:

> 对任意$\sigma \in G$，$\sigma \cdot f = f$，我们称它为在G -作用下是不变的。
> 这样的f称为不变多项式。
>
> 这些不变多项式形成一个交换代数 $A = R(V)^G$。

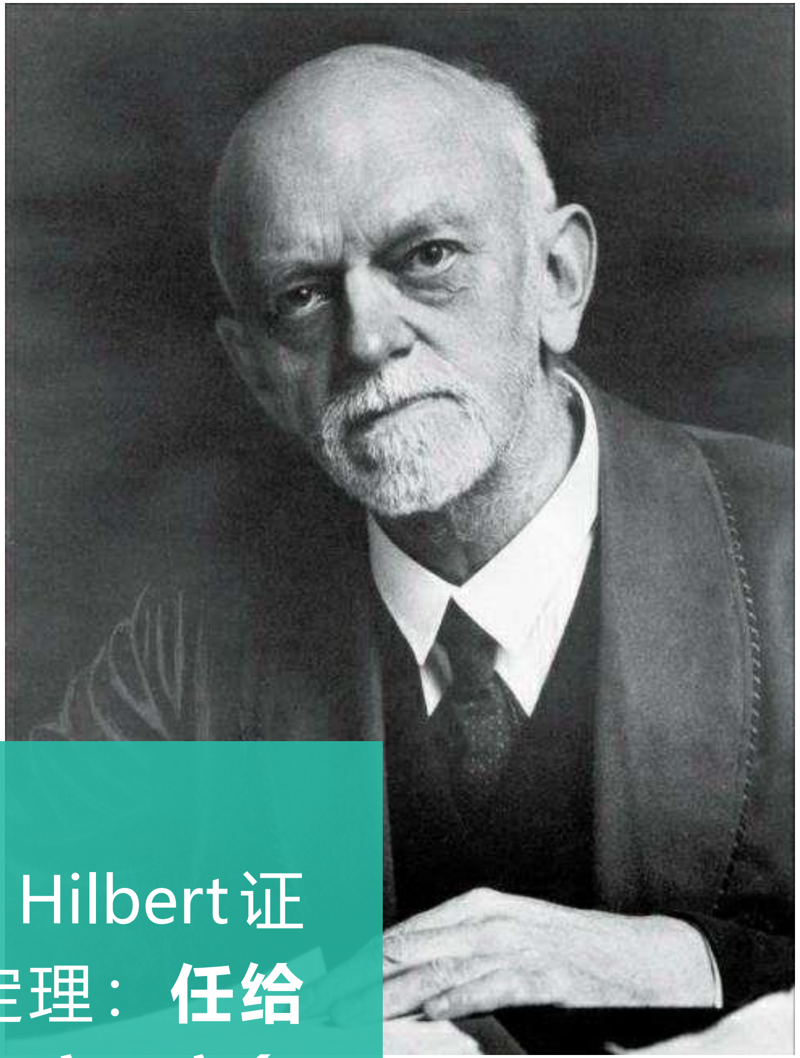
问题:
这个代数是否是有限生成的?
即：是否存在有限个不变多项式$f_1, \cdots, f_k$，使得它们生成的代数就是A?

例

设$G=\mathbb{Z}_2$，它可作用在$\mathbb{C}^2$如下：

$$(x,y)\longmapsto(-x,-y),$$

则A由x^2,xy,y^2生成, 即G-不变多项式具有形式$f(x^2,xy,y^2)$。



在1893年， Hilbert证明了以下著名定理：**任给一个线性群G，相应不变多项式的代数A是有限生成的。**

在1900年第二届世界数学家大会上，Hilbert提出23个著名的数学问题，其中第14问题问左边结论是否也对一般代数群成立。

在上世纪50年代，日本数学家Nagata证明希尔伯特第14问题是不成立的。

但是，Hilbert第14问题对所有可约代数群是对的，包括经典线性群和有限群。

几何不变理论

几何不变理论(简称GIT) 提供了代数几何中一个构造代数群作用下商空间的方法，它是由D. Mumford于1965年发展的。在发展过程中，他也用到Hilbert在1893年关于经典不变理论的文章中的一些想法 。

自从1970年, 我们发现GIT与辛几何，拓扑以及微分几何有紧密联系。

设G和 V如前。

我们用P (V) 表示V 中所有1维子空间的集合，
也称为V生成的投影空间。
任给V中的非零向量$v \neq 0$，它生成一个子空间 $[v] \in P(V)$。
群G作用在P (V)。

几何不变理论的一个基本问题：

研究P (V)中G轨道组成的集合的结构，如: 是否具有代数结构?

直接处理是行不通的！

如果0不是轨道G(v)的极限点，我们称[v] $\in P$(V)半稳定的，即semi-stable。

定理：

上述半稳定性等价于存在V上一个非常数的G-不变的齐次多项式f使得$f(v) \neq 0$。

如果轨道G(v)是闭的，我们称[v] $\in P$(V)稳定的，即stable。

设$P(V)^{ss}$为所有半稳定1维子空间的集合，
则我们可以有一个$P(V)^{ss}$的商空间Q，
即$P(V)^{ss}$中所有G-轨道的集合。

几何不变理论的一个主要结果：

Q是projective variety。

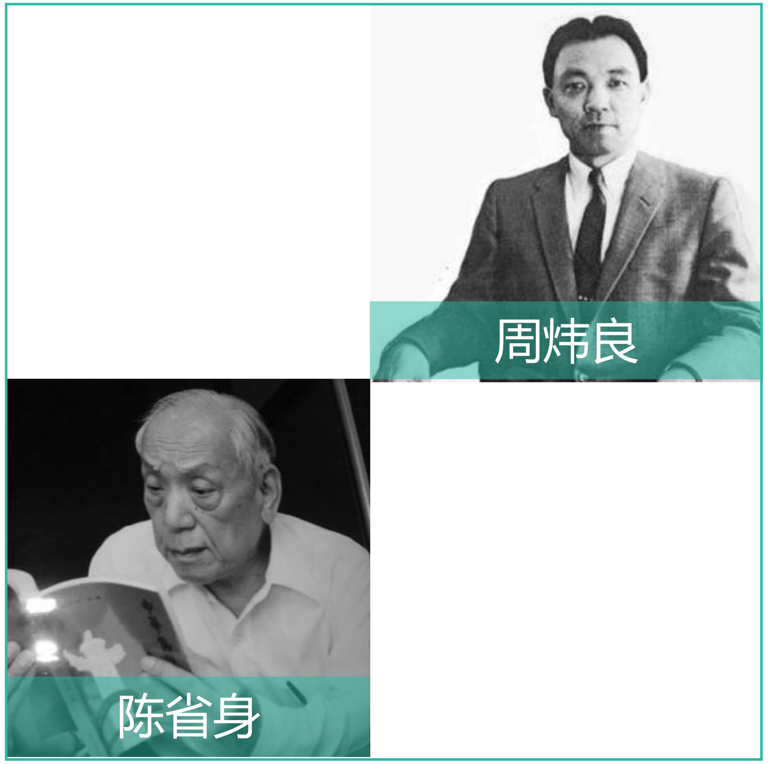
 

与代数流形M的关系?

By the famous Kodaira’s embedding theorem, we can embed M as a subvariety in some complex projective space $\mathbb{C}P^N$ on which the linear group $G = SL(N+1,\mathbb{C})$ acts.

Using a construction of Chow, Mumford associates a nonzero vector R_M, referred as the Chow coordinate, in a vector space $\mathbf{V}$ which has an induced action by $\mathbf{G}$.

M is called Chow-Mumford stable if its Chow coordinate is stable.



Einstein方程: $Ric(g) = \lambda g$

这里$\lambda = -1, 0, 1$ 是Einstein常数。

$Ric(g)$ 表示Ricci曲率，它测量由于空间弯曲而造成的体积元与欧式空间体积元之间的差异。

我们这里关心的是Einstein方程在凯勒流形上的解：

一个凯勒流形M是一个具有凯勒度量ω 的复流形。在局部全纯坐标$z_1, \cdots, z_n$下，凯勒度量ω是一个正定Hermitian矩阵函数$(g_{i\bar{j}})$并满足

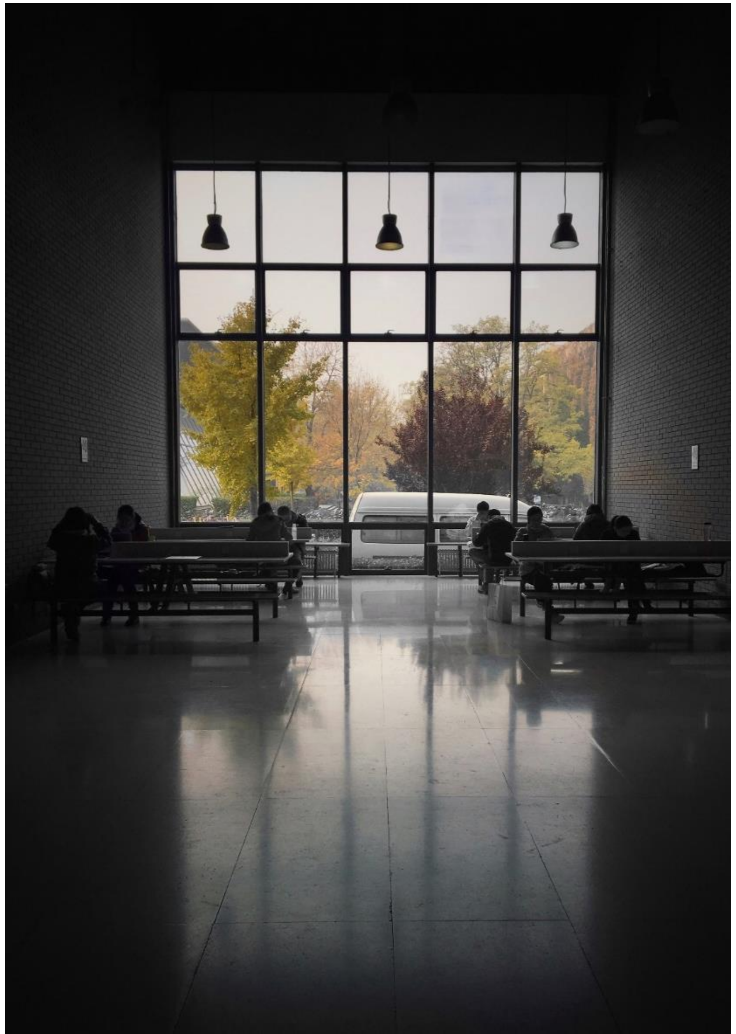
$$d\omega = 0,$$

这里 $\omega = \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$

如果它既是凯勒又满足Einstein方程：

$$\mathrm{Ric}(\omega)=\lambda\omega \quad (\lambda=-1,0,1)\text{，}$$

我们称 ω 为Kahler–Einstein度量。

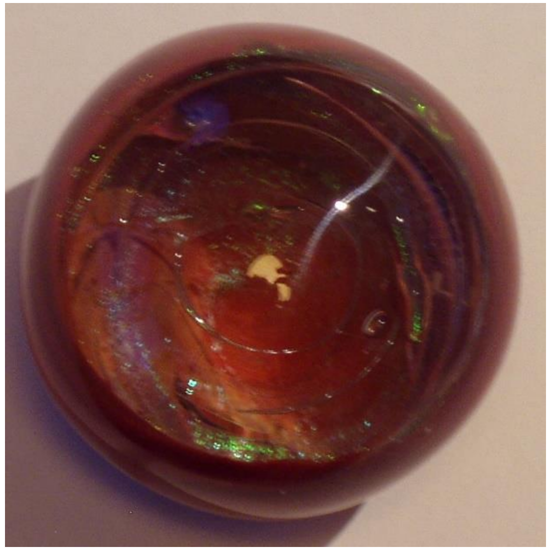


在50年代初, **E. Calabi**启动了Kahler-Einstein 度量存在性问题的研究。

经过20多年的努力，在1976年，两个非常重要情形获得解决，它们分别是：

- Yau when $\lambda = 0$
- Aubin and Yau independently when $\lambda = -1$

当$\lambda = 1$的情形更加困难。
此时M是一个Fano流形。

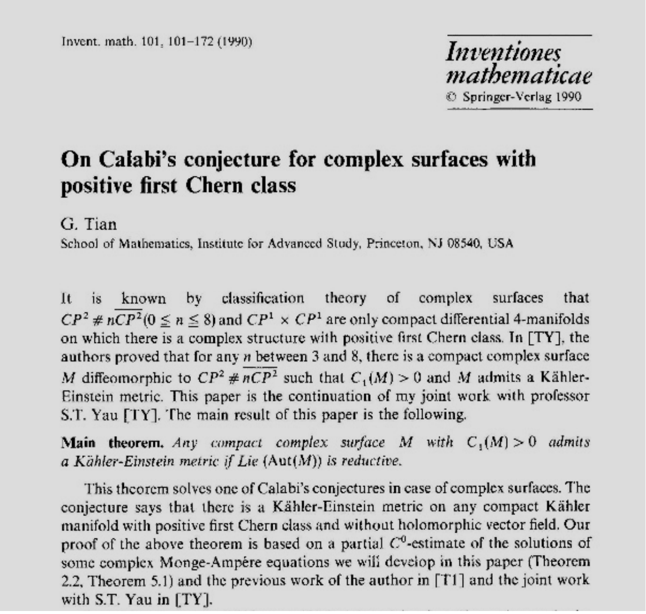


早已知道：

存在没有Kahler-Einstein度量的Fano流形。

因为有对Kahler-Einstein度量存在的障碍：

1. Matsushima（1957）
2. Futaki（1983）
3. Tian （1996）



Invent. math. 101, 101–172 (1990)
Inventiones mathematicae

On Calabi's conjecture for complex surfaces with positive first Chern class

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It is known by classification theory of complex surfaces that $CP^2 \# n\overline{CP^2} (0 \le n \le 8)$ and $CP^1 \times CP^1$ are only compact differential 4-manifolds on which there is a complex structure with positive first Chern class. In [TY], the authors proved that for any n between 3 and 8, there is a compact complex surface M diffeomorphic to $CP^2 \# n\overline{CP^2}$ such that $C_1(M) > 0$ and M admits a Kähler-Einstein metric. This paper is the continuation of my joint work with professor S.T. Yau [TY]. The main result of this paper is the following.

Main theorem. *Any compact complex surface M with $C_1(M) > 0$ admits a Kähler-Einstein metric if Lie* (Aut(M)) *is reductive.*

This theorem solves one of Calabi's conjectures in case of complex surfaces. The conjecture says that there is a Kähler-Einstein metric on any compact Kähler manifold with positive first Chern class and without holomorphic vector field. Our proof of the above theorem is based on a partial C^0-estimate of the solutions of some complex Monge-Ampère equations we will develop in this paper (Theorem 2.2, Theorem 5.1) and the previous work of the author in [T1] and the joint work with S.T. Yau in [TY].

Invent. math. 137, 1–37 (1997)
Inventiones mathematicae

Kähler-Einstein metrics with positive scalar curvature

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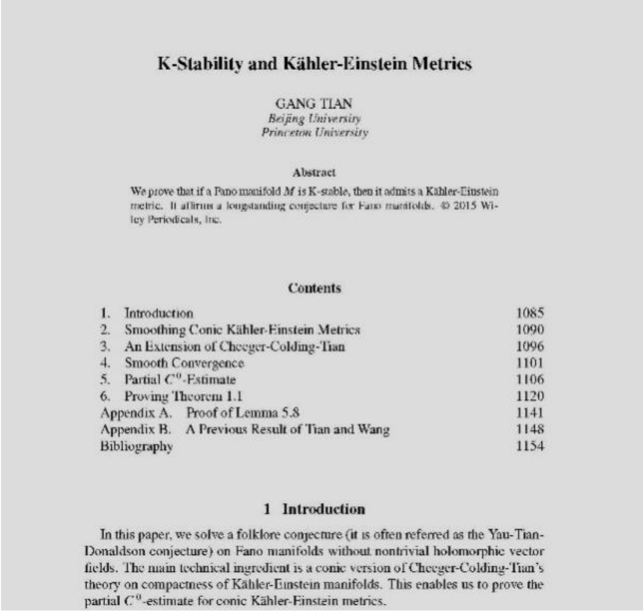
Oblatum 12-IV-1996 & 8-XI-1996

Abstract. In this paper, we prove that the existence of Kähler-Einstein metrics implies the stability of the underlying Kähler manifold in a suitable sense. In particular, this disproves a long-standing conjecture that a compact Kähler manifold admits Kähler-Einstein metrics if it has positive first Chern class and no nontrivial holomorphic vector fields. We will also establish an analytic criterion for the existence of Kähler-Einstein metrics. Our arguments also yield that the analytic criterion is satisfied on stable Kähler manifolds, provided that the partial C^0-estimate posed in [T6] is true.



K-Stability and Kähler-Einstein Metrics

GANG TIAN
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Abstract

We prove that if a Fano manifold M is K-stable, then it admits a Kähler-Einstein metric. It affirms a longstanding conjecture for Fano manifolds. © 2015 Wiley Periodicals, Inc.

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1 Introduction

In this paper, we solve a folklore conjecture (it is often referred as the Yau-Tian-Donaldson conjecture) on Fano manifolds without nontrivial holomorphic vector fields. The main technical ingredient is a conic version of Cheeger-Colding-Tian's theory on compactness of Kähler-Einstein manifolds. This enables us to prove the partial C^0-estimate for conic Kähler-Einstein metrics.

1989年

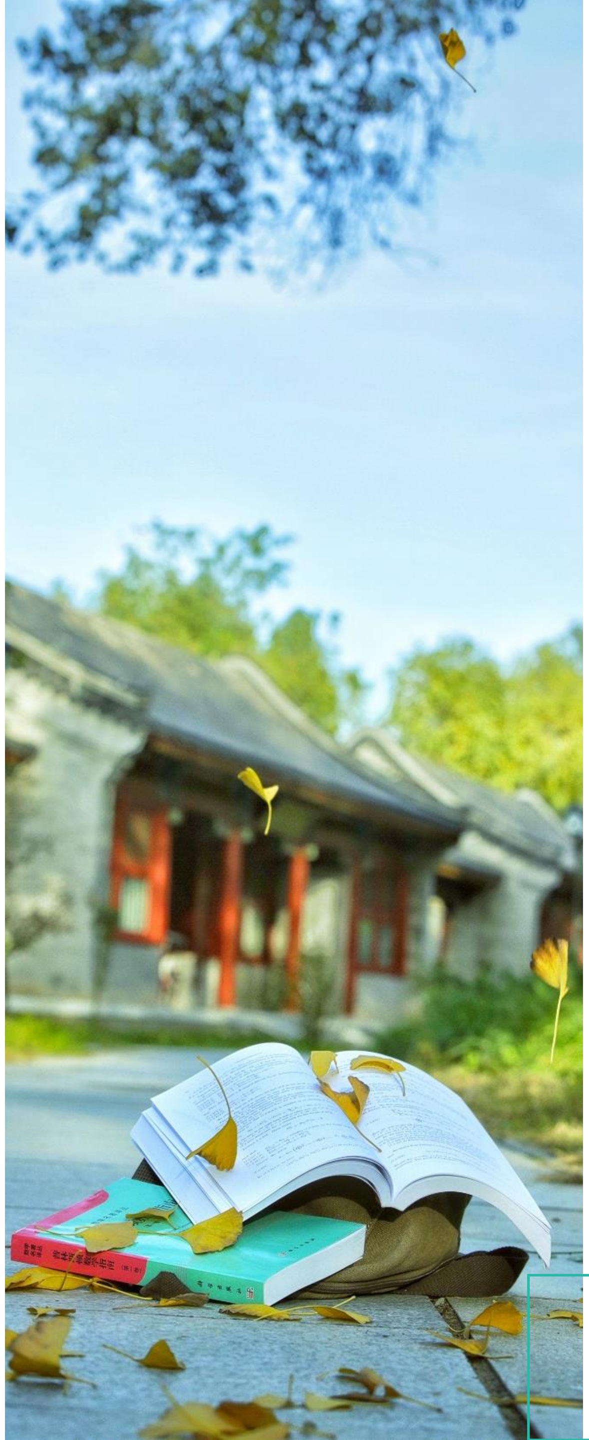
一个 Fano 曲面 M 有 Kahler-Einstein 度量当仅当Futaki不变量等于零。

1996年

我引进了K-稳定性并证明： 如果 M 有 Kahler-Einstein 度量，则M必须是K-稳定的。

2012年

我率先给出其逆的一个证明：一个K-稳定的 Fano 流形 M 有 Kahler-Einstein度量。



1983年, Futaki 引进了一个不变量，它是由M上全纯向量场的李代数$\eta(M)$的一个特征f_M。

Futaki： 如果M有Kahler-Einstein 度量，则不变量f_M平凡。

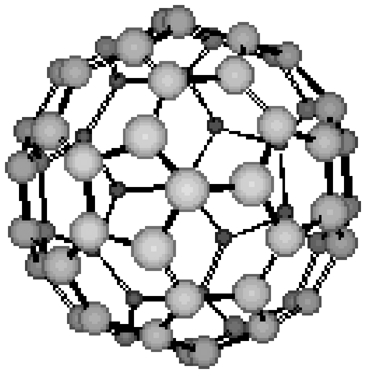
很长时间，仅有的障碍都是由李代数$\eta(M)$来定义的。

因此有一个著名猜想：

如果一个Fano流形M没有非零全纯向量场，则它有Kahler-Einstein度量。

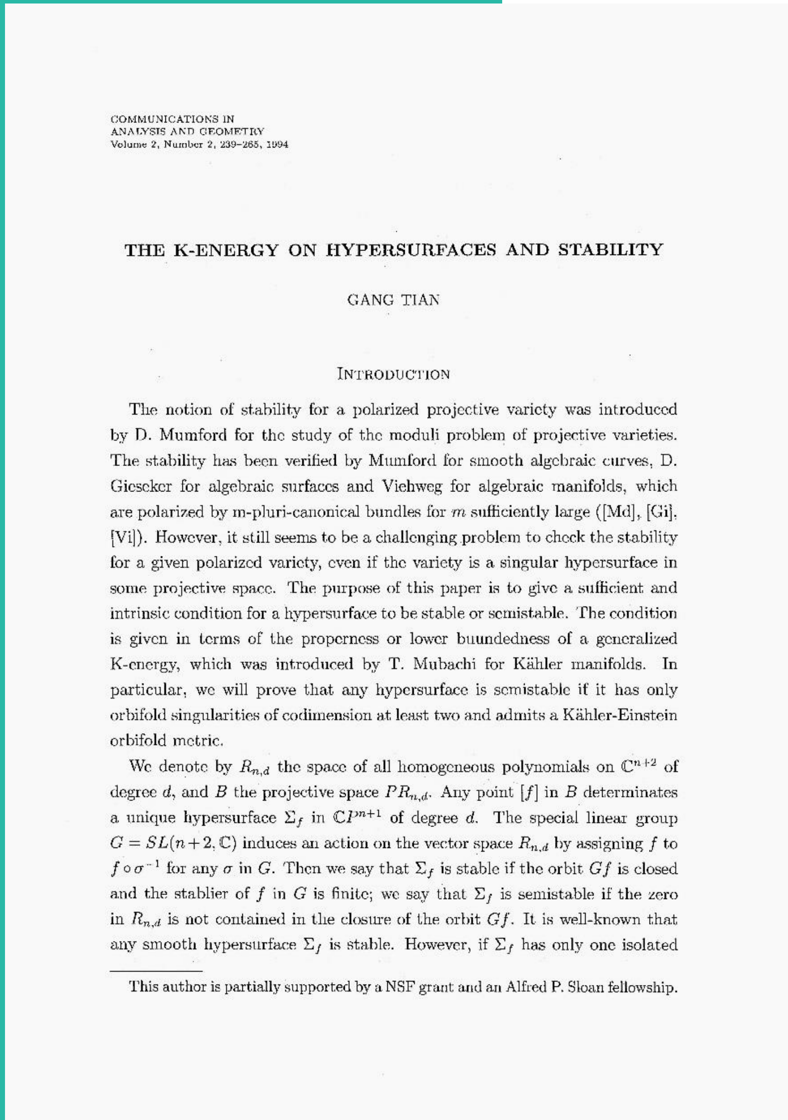
在1996年，我找到一个反例，其证明用到K-稳定性。

1990年夏，赴京都国际数学家大会做邀请报告之前，我参加一个在东京举办的卫星会议。期间我听了S. Mukai的一个报告，他介绍一类3维Fano流形的新构造，它们与**二十面体群**有关。



我当时即意识到这类流形可产生反例，但一直不知如何证。

The main building of Ookayama Campus of Tokyo Institute of Technology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THE K-ENERGY ON HYPERSURFACES AND STABILITY

GANG TIAN

INTRODUCTION

The notion of stability for a polarized projective variety was introduced by D. Mumford for the study of the moduli problem of projective varieties. The stability has been verified by Mumford for smooth algebraic curves, D. Gieseker for algebraic surfaces and Viehweg for algebraic manifolds, which are polarized by m-pluri-canonical bundles for m sufficiently large ([Md], [Gi], [Vi]). However, it still seems to be a challenging problem to check the stability for a given polarized variety, even if the variety is a singular hypersurface in some projective space. The purpose of this paper is to give a sufficient and intrinsic condition for a hypersurface to be stable or semistable. The condition is given in terms of the properness or lower buundedness of a generalized K-energy, which was introduced by T. Mubachi for Kähler manifolds. In particular, we will prove that any hypersurface is semistable if it has only orbifold singularities of codimension at least two and admits a Kähler-Einstein orbifold metric.

We denote by $R_{n,d}$ the space of all homogeneous polynomials on $\mathbb{C}^{n+2}$ of degree d, and B the projective space $PR_{n,d}$. Any point $[f]$ in B determines a unique hypersurface Σ_f in $\mathbb{C}P^{n+1}$ of degree d. The special linear group $G = SL(n+2, \mathbb{C})$ induces an action on the vector space $R_{n,d}$ by assigning f to $f \circ \sigma^{-1}$ for any σ in G. Then we say that Σ_f is stable if the orbit Gf is closed and the stablier of f in G is finite; we say that Σ_f is semistable if the zero in $R_{n,d}$ is not contained in the closure of the orbit Gf. It is well-known that any smooth hypersurface Σ_f is stable. However, if Σ_f has only one isolated

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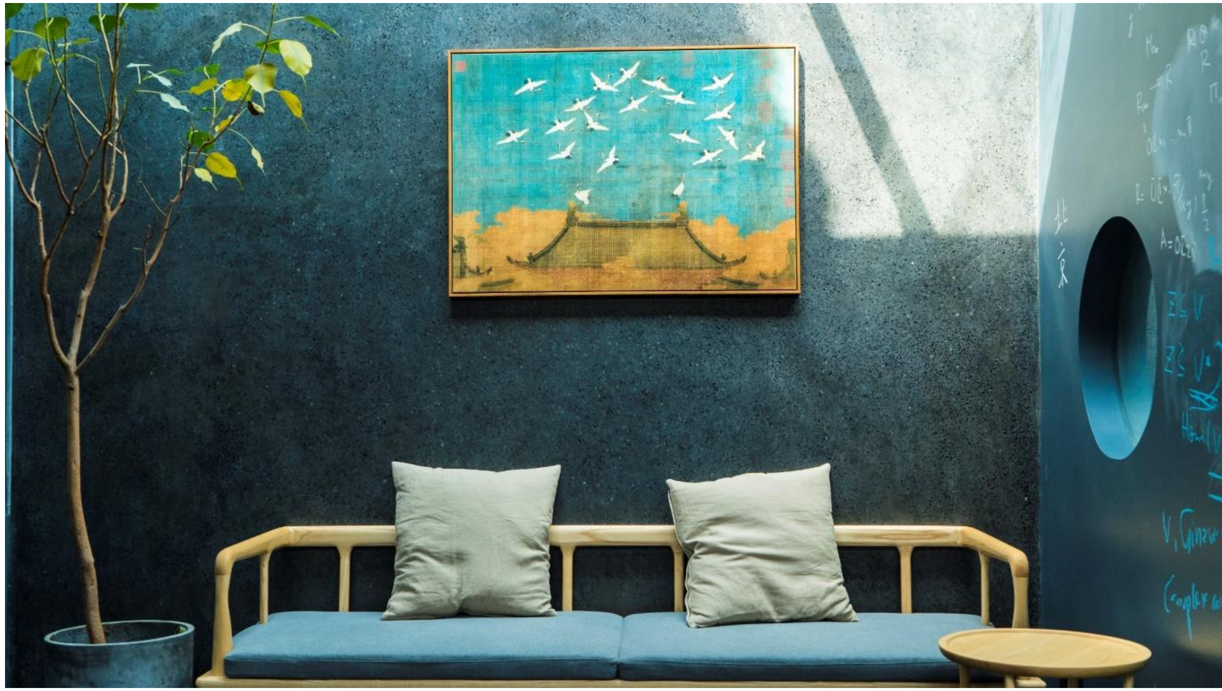
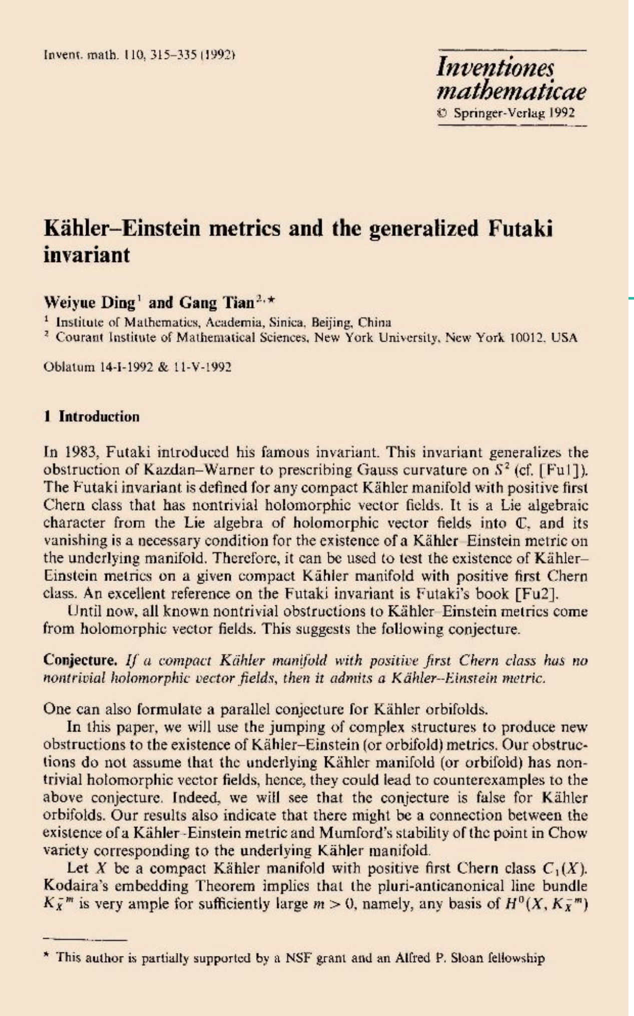
有很多年，我都在尝试证明Kahler-Einstein度量与Chow-Mumford稳定性是紧密相关的，即使我已知怎样定义K-稳定性。

现在我们知道K-稳定性才是正确的条件。进一步，K-稳定性的研究启发了几何不变理论的推广。

什么是K-稳定性?





Invent. math. 110, 315–335 (1992)

Inventiones mathematicae
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Kähler–Einstein metrics and the generalized Futaki invariant

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Oblatum 14-I-1992 & 11-V-1992

1 Introduction

In 1983, Futaki introduced his famous invariant. This invariant generalizes the obstruction of Kazdan–Warner to prescribing Gauss curvature on S^2 (cf. [Fu1]). The Futaki invariant is defined for any compact Kähler manifold with positive first Chern class that has nontrivial holomorphic vector fields. It is a Lie algebraic character from the Lie algebra of holomorphic vector fields into $\mathbb{C}$, and its vanishing is a necessary condition for the existence of a Kähler–Einstein metric on the underlying manifold. Therefore, it can be used to test the existence of Kähler–Einstein metrics on a given compact Kähler manifold with positive first Chern class. An excellent reference on the Futaki invariant is Futaki's book [Fu2].

Until now, all known nontrivial obstructions to Kähler–Einstein metrics come from holomorphic vector fields. This suggests the following conjecture.

Conjecture. *If a compact Kähler manifold with positive first Chern class has no nontrivial holomorphic vector fields, then it admits a Kähler–Einstein metric.*

One can also formulate a parallel conjecture for Kähler orbifolds.

In this paper, we will use the jumping of complex structures to produce new obstructions to the existence of Kähler–Einstein (or orbifold) metrics. Our obstructions do not assume that the underlying Kähler manifold (or orbifold) has nontrivial holomorphic vector fields, hence, they could lead to counterexamples to the above conjecture. Indeed, we will see that the conjecture is false for Kähler orbifolds. Our results also indicate that there might be a connection between the existence of a Kähler–Einstein metric and Mumford's stability of the point in Chow variety corresponding to the underlying Kähler manifold.

Let X be a compact Kähler manifold with positive first Chern class $C_1(X)$. Kodaira's embedding Theorem implies that the pluri-anticanonical line bundle K_X^{-m} is very ample for sufficiently large $m > 0$, namely, any basis of $H^0(X, K_X^{-m})$

* This author is partially supported by a NSF grant and an Alfred P. Sloan fellowship

为了引进K-稳定性，我们首先需要在奇异簇(singular variety)上定义Futaki型不变量。

在1992年，我和丁伟岳将Futaki不变量推广到奇异簇上。我们的构造想法类似于Futaki光滑情形的，但分析要困难很多。我们的构造在解决Fano情形的YTD猜想中发挥了关键性作用。

Other generalizations of Futaki invariant:

- In 2002, Donaldson gave an algebraic definition of generalized Futaki invariant which works for any polarized varieties. If M is smooth, it follows from the equivariant index theorem that this definition coincides with the Futaki invariant.

- In 2008, S. Paul gave another algebraic formula of generalized Futaki invariant in terms of Chow coordinate and hyperdiscriminant.

- In 1996, I also introduced the notion of CM line bundle and CM weight which is the first Chern class of the CM line bundle over certain compactification of the subgroup.

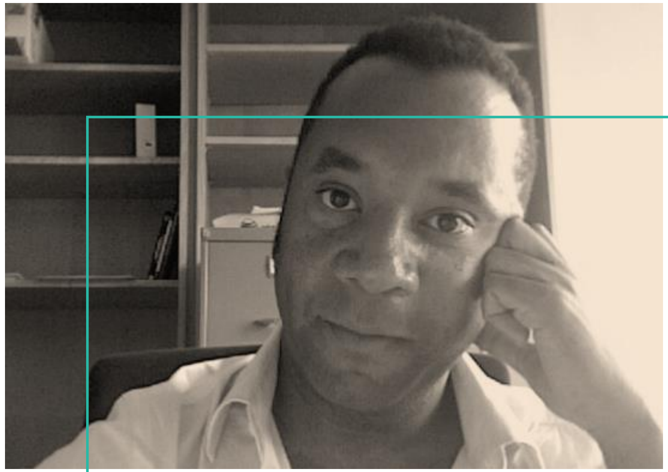
The CM weight turns out to be more useful in algebraic geometry as Li and Xu manifested in 2011.

K-稳定性: By Kodaira, we can embed $M \longmapsto \mathbb{C}P^N$ as a subvariety.

Set $G = SL(N+1, \mathbb{C})$. For any algebraic subgroup $G_0 = \{\sigma(t)\}_{t \in \mathbb{C}^*}$ of G, we can associate a CM-weight $w(G_0)$ which is also equal to the generalized Futaki invariant defined by Ding-Tian or Donaldson etc..

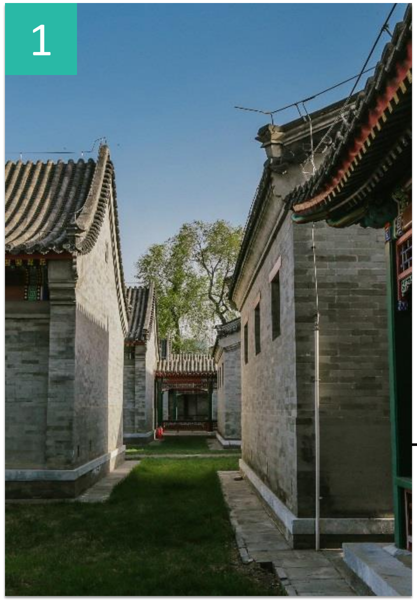
对给定M，如果对任何 $\mathbf{G}_0 \subset \mathbf{G}$，$\mathbf{w}(\mathbf{G}_0) \geq 0$，称它是K-半稳定的（K-semistable）。

如果它是K-半稳定的并且除非$\mathbf{G}_0$保持M，$\mathbf{w}(\mathbf{G}_0)>0$，称它是K-稳定的。



K-稳定性与几何不变理论：

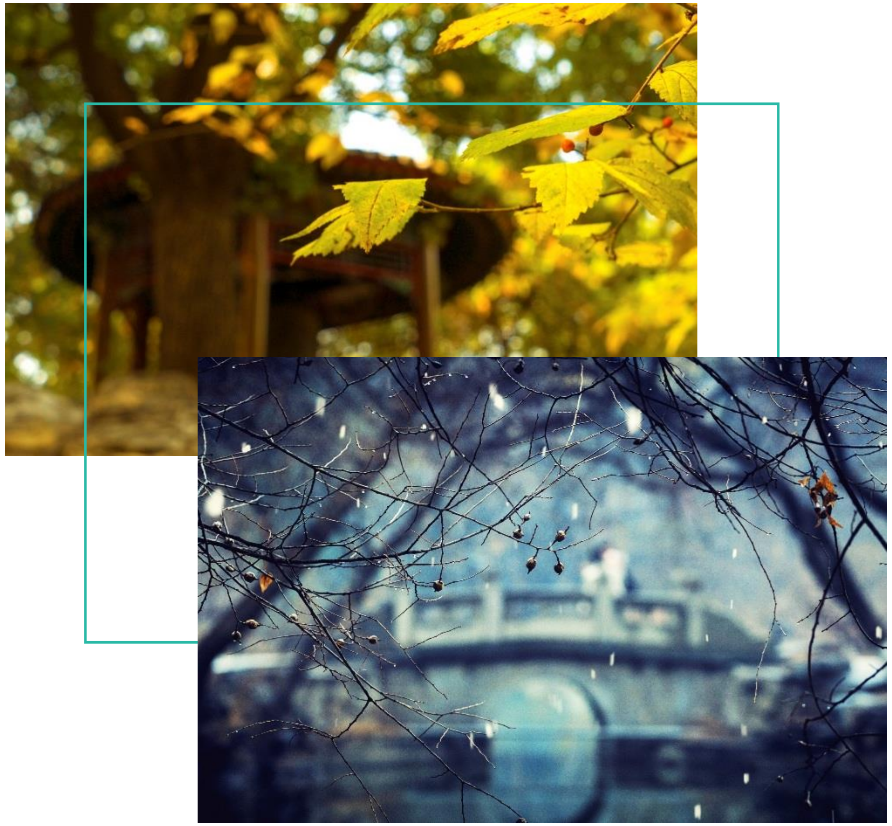
与Chow-Mumford稳定性不同，K-稳定性不在几何不变理论的适用范围内。几何不变理论只涉及G的一个表示，而从S. Paul 的工作看，K-稳定性的正确设置应包含G的两个表示。这样，K-稳定性的研究诱导了几何不变理论（GIT）的推广，我们称之为EGIT。



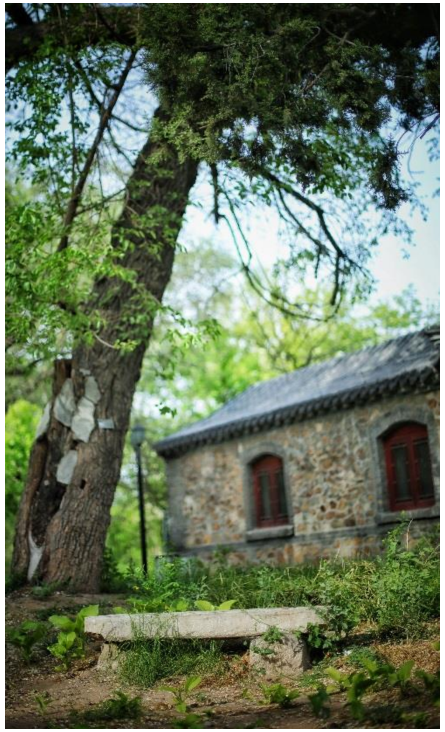
Extending Geometric Invariant Theory:

Let $\mathbf{V}$ and $\mathbf{W}$ be two representations of $\mathbf{G}$. Given a pair $v \in V/\{0\}$ and $w \in W/\{0\}$,we say the pair (v, w) is semistable if

$$\mathbf{G}[v, w] \cap \mathbf{G}[0, w] = \varnothing \text{ in } P(\mathbf{V} \oplus \mathbf{W}).$$



If $\mathbf{W} = \mathbb{C}$, $w = 1$ be the trivial 1-dimensional representation of $\mathbf{G}$. Then $(v, 1)$ is semistable if and only if 0 is not in the closure of the orbit $\mathbf{G}v$. In other words, v is semistable in the usual sense of Geometric Invariant Theory.



K-stability fits well in the frame of the extended GIT:

For each M embedded in $\mathbb{C}P^N$, S. Paul associates the hyperdiscriminant Δ_M and the Chow coordinate R_M. They lie in two vector spaces $\mathbf{V}$ and $\mathbf{W}$ on which $\mathbf{G}$ acts naturally.

Paul showed that M being K-semistable is equivalent to the semistability of the pair (Δ_M, R_M).

In some sense, as vector bundles verses K-theory, stability of pairs corresponds to the GIT for the representation of $\mathbf{G}$ on the difference $\mathbf{V}-\mathbf{W}$.

平畴交远风，良苗亦怀新。

——陶渊明

谢谢大家